Projective modules that are not free

(L1) R ring, $I, J \subset R$ ideals, $I+J=R$. Then $I \cap J = IJ$, $\frac{R}{IJ} \overset{*}{\simeq} \frac{R}{I} \times \frac{R}{J}$,
and $I \oplus J \simeq IJ \oplus R$. So if IJ principal & R dom $\Rightarrow I, J$ projective

Pf $I \cap J = (I+J)(I\cap J) \subset IJ \subset I \cap J$. $*$ is Chin Rem Th
Also $0 \to I \cap J \to I \oplus J \to R \to 0$ split qed

(L2) Let $R = \mathbb{Z}[\sqrt{D}]$, $D \in \mathbb{Z}$ sq free. $\sigma: R \to R$, $\sigma(\overbrace{a+b\sqrt{D}}^{\theta}) = \overbrace{a-b\sqrt{D}}^{\theta^\sigma}$.
$N\theta = \theta\theta^\sigma$. Then $\underbrace{\theta \in R^\times}_{N \text{ mult } \&} \Leftrightarrow N\theta = \pm 1$
Pf. Clear

(L3) R as in (L2). Let $\theta \in R$ be s.t. $N\theta = \theta\theta^\sigma = pq$ w/
$p, q \in \mathbb{Z}$ distinct primes(*). Then

1) $(\theta) = (p,\theta)(q,\theta)$ [Pf RHS $= (\overbrace{pq}^{\theta\theta^\sigma}, p\theta, q\theta, \theta^2) \overset{\supset}{\subset} (\theta)$ b/c $(p,q)=(1)$ so $\theta = ap\theta + bq\theta$

2) $(p,\theta) + (q,\theta) = (1)$ [Pf: b/c $(p,q)=1$]

3) (p,θ) is projective [Pf. by (L1)]

4) (p,θ) is not free i.e not princ [Pf. If $(p,\theta) = (a) \Rightarrow$
$\Rightarrow Na \mid Np$ & $Na \mid N\theta \Rightarrow Na \mid p^2$ & $Na \mid pq \Rightarrow$
$\Rightarrow Na = \pm 1 \Rightarrow (a) = (1) \Rightarrow bp + c\theta = 1 \Rightarrow b\underbrace{pq}_{\theta\theta^\sigma} + cq\theta = q$
$\Rightarrow \theta \mid q \Rightarrow N\theta \mid Nq \Rightarrow pq \mid q^2 \to\leftarrow$.

5) (p,θ) is prime [Pf. By 1) & (L1) ~~$\frac{R}{(\theta)} \simeq \frac{R}{(p,\theta)} \times \frac{R}{(q,\theta)}$~~
~~Take # . Get $\# \underbrace{\frac{R}{(\theta)}}_{\overset{\text{EX in ANT}}{\longrightarrow} pq}$ =~~ $\# \underbrace{\frac{R}{(p,\theta)}}_{\text{divisor of } p^2 \text{ b/c this ring is a factor of } R/pR = \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}} \text{ as grs}}$ ~~$\cdot \# \underbrace{\frac{R}{(q,\theta)}}_{\text{div of } q^2}$. $Np\psi$ $\frac{R}{(\theta)} \simeq \frac{R}{(\theta^\sigma)}$~~
. It can't be 1 b/c $(p,\theta) \neq (1)$
& can't be p^2 b/c $(p,\theta) \neq (p)$

6) p irred & not prime in R

& ass no $a \in R$ has $Na = p$

(*) Eg $D = -5$, $p=2, q=3$, $\theta = 1+\sqrt{-5}$
$D = 10$, $p = 2, q = 3$, $\theta = 4 + \sqrt{10}$