

Projective modules that are not free

(L1) R ring, $I, J \subset R$ ideals, $I+J=R$. Then $I \cap J = IJ$, $\frac{R}{IJ} \cong \frac{R}{I} \times \frac{R}{J}$,
 and $I \oplus J \cong IJ \oplus R$. So if IJ principal & R dom $\Rightarrow I, J$ projective

Pf $I \cap J = (I+J)(I \cap J) \subset IJ \subset I \cap J$. * is Chin Rem Th

Also $0 \rightarrow I \cap J \rightarrow I \oplus J \rightarrow R \rightarrow 0$ split seq

(L2) Let $R = \mathbb{Z}[\sqrt{D}]$, $D \in \mathbb{Z}$ sq free. $\sigma: R \rightarrow R, \sigma(a+b\sqrt{D}) = a-b\sqrt{D}$.
 $N\theta = \theta\theta^\sigma$. Then $\theta \in R^\times \Leftrightarrow N\theta = \pm 1$

Pf. Clear (N mult &)

(L3) R as in (L2). Let $\theta \in R$ best. $N\theta = \theta\theta^\sigma = pq$ w/ $p, q \in \mathbb{Z}$ distinct primes*. Then

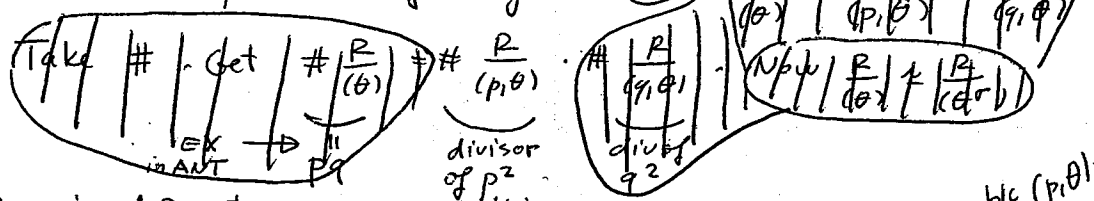
1) $(\theta) = (p, \theta)(q, \theta)$ [Pf RHS = $(\hat{p}q, p\theta, q\theta, \theta^2) \subset (\theta)$ b/c $(p, q) = (1)$
so $\theta = ap + bq$]

2) $(p, \theta) + (q, \theta) = (1)$ [Pf: b/c $(p, q) = (1)$]

3) (p, θ) is projective [Pf. by (L1)]

4) (p, θ) is not free i.e. not princ [Pf. If $(p, \theta) = (a) \Rightarrow$
 $\Rightarrow Na | Np \ \& \ Na | N\theta \Rightarrow Na | p^2 \ \& \ Na | pq \Rightarrow$
 $\Rightarrow Na = \pm 1 \Rightarrow (a) = (1) \Rightarrow bp + c\theta = 1 \Rightarrow bpq + cq\theta = q$
 $\Rightarrow \theta | q \Rightarrow N\theta | Nq \Rightarrow pq | q^2 \rightarrow \leftarrow$ $\theta\theta^\sigma$]

5) (p, θ) is prime [Pf. By 1) & (L1)]



6) p irred & not prime in R

& ass no $a \in R$ has $Na = p$

* Eg $D = -5, p = 2, q = 3, \theta = 1 + \sqrt{-5}$
 $D = 10, p = 2, q = 3, \theta = 4 + \sqrt{10}$