Here I am using that p tot ramifies in $\mathbb{Q}(\zeta_{p^n})$

$\boxed{\text{The totally ramified direction}}$ (as a geometric direction)

Let $\Lambda = \mathbb{Z}_p[[T]]$

Let $K_n = \mathbb{Q}_p(\zeta_{p^{n+1}})$, U_n = units in $\mathbb{Z}_p[\zeta_{p^{n+1}}] = \mathcal{O}_{K_n}$, $U_n^1 = \{u \in U_n \mid u \equiv 1 \bmod \pi_n\}$

$\pi_n = \zeta_{p^{n+1}} - 1$, $\zeta_{p^{n+1}}^p = \zeta_{p^n}$, $N_{n,n-1} : K_n \to K_{n-1}$ the norm. Then (π_n) = max in $\mathcal{O}_{K_n}$

$G(K_n/K_0) = \{a \in (\mathbb{Z}/p^{n+1})^\times \mid a \equiv 1 \bmod p\}$, $G(K_n/K_{n-1}) = \{a \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times \mid a \equiv 1 \bmod p^n\}$

$\{$conjugates of $\zeta_{p^{n+1}}$ in $K_n/K_{n-1}\} = \{\zeta_{p^{n+1}}^a \mid a \in \mathcal{G}\} = \{\zeta\zeta_{p^{n+1}} \mid \zeta^p = 1\}$

For $f \in \Lambda$, $N_{n,n-1} f(\pi_n) = \prod_{\zeta^p=1} f(\zeta\zeta_{p^{n+1}} - 1) = \prod_{\zeta^p=1} f(\zeta(\pi_n+1)-1)$; so $N_{n,n-1}\zeta_{p^{n+1}} = \zeta_{p^n}$

(D) $U = \varprojlim \left(U_0^1 \xleftarrow{N_{10}} U_1^1 \xleftarrow{N_{21}} U_2^1 \xleftarrow{N_{32}} U_3^1 \leftarrow \cdots \right) \ni u = (u_n)$

(D) $\tilde{\Lambda} = \{ f \in \Lambda^\times \mid f(0) \equiv 1 \bmod p,\ f((1+T)^p - 1) = \prod_{\zeta^p=1} f(\zeta(1+T)-1) \} < \Lambda^\times$

(T) (Kummer-Coates-Wiles-Coleman) $\exists!$ group iso $U \xrightarrow{\sim} \tilde{\Lambda}$, $u \mapsto f_u$ s.t.
$f_u(\pi_n) = u_n$, $n \geq 0$.

Pf. Define homo $\tilde{\Lambda} \to U$, $f \mapsto (f(\pi_n))$

1) Well defined b/c $1 \underset{(p)}{\equiv} f(0) \underset{(\pi_0)}{\equiv} f(\pi_0)$ so $f(\pi_0) \in U_0^1$ &

$$N_{n,n-1} f(\pi_n) = \prod_{\zeta^p=1} f(\zeta(\pi_n+1)-1) = f(\underbrace{(1+\pi_n)}_{\zeta_{p^{n+1}}}{}^p - 1) = f(\zeta_{p^n} - 1) = f(\pi_{n-1}).$$

2) Inj b/c of (L) Any $f \in \mathbb{Z}_p[[T]] \setminus 0$ has only fin many roots in $\{x \in \mathbb{C}_p \mid |x| < 1\}$.
(This is a Corollary of "pWei prep th": any $f \in \mathbb{Z}_p[[T]]$ is $f = p^\nu \cdot$ pol $\cdot$ unit)

3) Surj (This is tricky & skip)

Then let $u \mapsto f_u$ be the inverse of $\tilde{\Lambda} \to U$ □

(D) (Kummer-Coates-Wiles) For $k \geq 1$, $\delta_k : U \to \mathbb{Z}_p$, $\delta_k(u) := \left(D^k \log f_u(T) \right)\Big|_{T=0}$.
(where $D = (1+T)\frac{d}{dT}$)

(R) U is a Λ-module as follows: Let $\Gamma_n = G(K_n/K_0)$, $\Gamma = \varprojlim \Gamma_n$, $\mathbb{Z}_p[[\Gamma]] = \varprojlim \mathbb{Z}_p[\Gamma_n]$

Then (T) (see my notes on meas th) $\mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[T]] = \Lambda$ via $\gamma - 1 \mapsfrom T$ where $\gamma = (\gamma_n)$
$\langle \gamma_n \rangle = \Gamma_n$. But U is clearly a $\mathbb{Z}_p[[\Gamma]]$-module. [so mod str dep on γ!]

(T) (see Washington p 312) Let γ be as in $\mathcal{G}$ viewed as an elt of $1 + p\mathbb{Z}_p$ (i.e. as a Gal elt does $\zeta_{p^n} \mapsto \zeta_{p^n}^\gamma$). Let $h(T) \in \Lambda$, $u \in U$. Then

$$\delta_k(h(T)u) = h(\gamma^k - 1)\delta_k(u)$$