ARITHMETIC DIFFERENTIAL GEOMETRY

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1. Introduction

The aim of these notes is to explain past work and proposed research aimed at developing an arithmetic analogue of classical differential geometry. In the theory we are going to describe the ring of integers $\mathbb{Z}$ will play the role of a ring of functions on an infinite dimensional manifold. The role of coordinate functions on this manifold will be played by the prime numbers $p \in \mathbb{Z}$. The role of the partial derivative of an integer $n \in \mathbb{Z}$ with respect to a prime p will be played (in the spirit of [16, 28]) by the *Fermat quotient* $\delta_p n := \frac{n-n^p}{p} \in \mathbb{Z}$. The role of metrics/2-forms will be played by symmetric/antisymmetric matrices with entries in $\mathbb{Z}$. The role of connection/curvature attached to metrics or 2-forms will be played by certain adelic/global objects attached to integral matrices. The resulting geometry will be referred to as *arithmetic differential geometry*. One of the somewhat surprising conclusions of our theory is that (the space whose ring of functions is) $\mathbb{Z}$ appears as "intrinsically curved."

A starting point for this circle of ideas can be found in our paper [16] where we showed how one can replace derivation operators by Fermat quotient operators in the construction of the classical jet spaces of Lie and Cartan; the new objects thus obtained were referred to as *arithmetic jet spaces*. With these objects at hand it was natural to try to develop an arithmetic analogue of *differential calculus*, in particular of *differential equations* and *differential geometry*. A theory of *arithmetic differential equations* was developed in a series of papers [16]-[35], [4] and was partly summarized in our monograph [28] (cf. also the survey papers [36, 73]); this theory led to a series of applications to invariant theory [20, 4, 21, 28], congruences between modular forms [20, 30, 31], and Diophantine geometry of Abelian and Shimura varieties [17, 29]. On the other hand an *arithmetic differential geometry* was developed in a series of papers [35]-[40], [5] and in the book [42]. The present notes follow closely the Introduction to [42].

We should note that our book [28] on *arithmetic differential equations* and the book [42] on *arithmetic differential geometry*, although based on the same ideology introduced in [16], are concerned with rather different objects. In particular the two books are independent of each other and the overlap between them is minimal. Indeed the book [28] was mainly focussed on the arithmetic differential equations naturally occurring in the context of Abelian and Shimura varieties. By contrast, the book [42] naturally concentrates on the arithmetic differential equations related to the classical groups GL_n, SL_n, SO_n, Sp_n, and their corresponding symmetric spaces. Of course, the world of Abelian and Shimura varieties on the one hand and the world of classical groups on the other, although not directly related within *abstract algebraic geometry*, are closely related through *analytic* concepts such as *uniformization* and *representation theory*. The prototypical example of this relation is the identification $M_1(\mathbb{C}) \simeq SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$ where M_1 is the moduli space of Abelian varieties, $\mathbb{C}$/lattice, of dimension one; the curve M_1 is, of course, one the simplest Shimura varieties. It is conceivable that the analytic relation between the two worlds referred to above be realized via certain *arithmetic differential equations/correspondences*; in particular it is conceivable that the book [42] and our previous book [28] be related in ways that go beyond what can be seen at this point. A suggestion for such a possible relation comes from the theory of *δ-Hodge structures* developed in [15] and from a possible arithmetic analogue of the latter that could emerge from [4, 28].

Needless to say *arithmetic differential geometry* is still in its infancy. However, its foundations, which we present here, seem to form a solid platform upon which one could further build. Indeed, the main differential geometric concepts of this theory turn out to be related to classical number theoretic concepts (e.g., Christoffel symbols are related to Legendre symbols); existence and uniqueness results for the main objects (such as the arithmetic analogues of Ehresmann, Chern, Levi-Civita, and Lax connections) are being proved; the non-trivial problem of defining curvature is solved in some important cases (via our method of analytic continuation between primes and, alternatively, via algebraization by correspondences); and some basic vanishing/non-vanishing theorems are being proved for various types of curvature. It is hoped that all of the above will create a momentum for further investigation and further discovery.

As a final general comment let us consider the question: *what is the position of our theory among more established mathematical theories?* The answer we would like to suggest is that our curvature of $\mathbb{Z}$, suitably encoded into a Lie $\mathbb{Q}$-algebra $\mathfrak{hol}_{\mathbb{Q}}$ that we refer to as the *holonomy algebra* of $\mathbb{Q}$, could be viewed as an infinitesimal counterpart of the absolute Galois group $\Gamma_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$: our Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$ should be to the absolute Galois group $\Gamma_{\mathbb{Q}}$ what the identity component Hol^0 of the holonomy group Hol is to the monodromy group Hol/Hol^0 in classical differential geometry. As such our $\mathfrak{hol}_{\mathbb{Q}}$ could be viewed as an object of study in its own right, as fundamental, perhaps, as the absolute Galois group $\Gamma_{\mathbb{Q}}$ itself. A "unification" of $\mathfrak{hol}_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ may be expected in the same way in which Hol^0 and Hol/Hol^0 are "unified" by Hol.

Here is the plan of these notes. Section 2 will be devoted to an outline of our *arithmetic differential geometry*; this will be done by presenting the theory in parallel with *classical differential geometry*. In order to stress the analogies we will revisit the main concepts of classical differential geometry from a somewhat non-standard angle. Section 3 will be devoted to comparing our theory with other "geometric theories of the discrete."

2. Outline of the theory

2.1. **Classical versus arithmetic setting.** In classical differential geometry one starts with an m-dimensional smooth manifold M and its ring of smooth real valued functions $C^\infty(M, \mathbb{R})$. For our purposes here it is enough to think of M as being the Euclidean space $M = \mathbb{R}^m$. In these notes the arithmetic analogue of the manifold $\mathbb{R}^m$ will be the scheme *Spec* $\mathbb{Z}$. Next, in order to stress our analogy between functions and numbers it is convenient to fix a subring

$$A \subset C^\infty(\mathbb{R}^m, \mathbb{R}) \tag{1}$$

that is stable under partial differentiation and to frame the classical differential geometric definitions in terms of this ring. The arithmetic analogue of A in (1) will then be the ring of integers $\mathbb{Z}$ or, more generally, rings of fractions A of rings of integers in an abelian extension of $\mathbb{Q}$. A prototypical example of such a ring is the ring

$$A = \mathbb{Z}[1/M, \zeta_N], \tag{2}$$

where M is an even integer, N is a positive integer, and ζ_N is a primitive N-th root of unity. Inverting M allows, as usual, to discard a set of "bad primes" (the divisors of M); adjoining ζ_N amounts to adjoining "new constants" to $\mathbb{Z}$.

Most of the times the ring A in (1) can be thought of as equal to $C^\infty(\mathbb{R}^m, \mathbb{R})$. But it will be sometimes useful to think of the ring A in (1) as consisting of analytic functions; indeed we will sometimes view analytic/algebraic functions as corresponding, in our theory, to global arithmetic objects in which case C^∞ objects will correspond, in our theory, to adelic objects. We will adopt these various viewpoints according to the various situations at hand.

The analogue of the set of coordinate functions

$$\mathcal{U} = \{\xi_1, \xi_2, ..., \xi_m\} \subset C^\infty(\mathbb{R}^m, \mathbb{R}) \tag{3}$$

on $\mathbb{R}^m$ will be a (possibly infinite) set of primes,

$$\mathcal{V} = \{p_1, p_2, p_3, ...\} \subset \mathbb{Z} \tag{4}$$

not dividing MN. We denote by $m = |\mathcal{V}| \leq \infty$ the cardinality of $\mathcal{V}$.

Next one considers the partial derivative operators on $A \subset C^\infty(\mathbb{R}^m, \mathbb{R})$,

$$\delta_i := \delta_i^A : A \to A, \quad \delta_i f := \frac{\partial f}{\partial \xi_i}, \quad i \in \{1, ..., m\}. \tag{5}$$

Following [16] we propose to take, as an analogue of the operators (5), the operators δ_p on $A = \mathbb{Z}[1/M, \zeta_N]$ defined by

$$\delta_p := \delta_p^A : A \to A, \quad \delta_p(a) = \frac{\phi_p(a) - a^p}{p}, \quad p \in \mathcal{V}, \tag{6}$$

where $\phi_p := \phi_p^A : A \to A$ is the unique ring automorphism sending ζ_N into ζ_N^p.

More generally recall that a *derivation* on a ring B is an additive map $B \to B$ that satisfies the Leibniz rule. This concept has, as an arithmetic analogue, the concept of *p-derivation* defined as follows; cf. [16, 28].

Definition 1. Assume B is a ring and assume, for simplicity, that p is a non-zero divisor in B; then a *p-derivation* on B is a set theoretic map

$$\delta_p := \delta_p^B : B \to B \tag{7}$$

with the property that the map

$$\phi_p := \phi_p^B : B \to B \tag{8}$$

defined by

$$\phi_p(b) := b^p + p\delta_p b \tag{9}$$

is a ring homomorphism.

We will always denote by ϕ_p the ring homomorphism (8) attached to a p-derivation δ_p as in (7) via the formula (9) and we shall refer to ϕ_p as the *Frobenius lift* attached to δ_p; note that ϕ_p induces the p-power Frobenius on B/pB.

2.2. **Classical connections and curvature revisited.** As our next step, in classical differential geometry, we consider the frame bundle $P \to M$ of a rank n vector bundle over the manifold $M = \mathbb{R}^m$. Then P is a principal homogeneous space for the group GL_n; and if the vector bundle is trivial (which we assume from now on) then P is identified with $M \times GL_n$. (Note that the rank n of the vector bundle and the dimension m of M in this picture are unrelated.) We want to review the classical concept of connection in P; we shall do it in a somewhat non-standard way so that the transition to arithmetic becomes more transparent. Indeed consider an $n \times n$ matrix $x = (x_{ij})$ of indeterminates and consider the ring of polynomials in these n^2 indeterminates over some $A \subset C^\infty(\mathbb{R}^m, \mathbb{R})$ as in (1), with the determinant inverted,

$$B = A[x, \det(x)^{-1}]. \tag{10}$$

Note that B is naturally a subring of the ring $C^\infty(M \times GL_n, \mathbb{R})$. Denote by $\mathrm{End}(B)$ the Lie ring of $\mathbb{Z}$-module endomorphisms of B.

Definition 2. A *connection* on $P = M \times GL_n$ is a tuple $\delta = (\delta_i)$ of derivations

$$\delta_i := \delta_i^B : B \to B, \quad i \in \{1, ..., m\}, \tag{11}$$

extending the derivations (5). The *curvature* of δ is the matrix (φ_{ij}) of commutators $\varphi_{ij} \in \mathrm{End}(B)$,

$$\varphi_{ij} := [\delta_i, \delta_j] : B \to B, \quad i, j \in \{1, ..., m\}. \tag{12}$$

The *holonomy ring* of δ is the $\mathbb{Z}$-span $\mathfrak{hol}$ in $\mathrm{End}(B)$ of the commutators

$$[\delta_{i_1}, [\delta_{i_2}, [...., [\delta_{i_{s-1}}, \delta_{i_s}]...]]], \tag{13}$$

where $s \geq 2$; it is a Lie subring of $\mathrm{End}(B)$.

In particular one can consider the *trivial connection* $\delta_0 = (\delta_{0i})$ defined by

$$\delta_{0i} x = 0. \tag{14}$$

Here $\delta_{0i} x = (\delta_{0i} x_{kl})$ is the matrix with entries $\delta_{0i} x_{kl}$; this, and similar, notation will be constantly used in the sequel. A connection is called *flat* if its curvature vanishes: $\varphi_{ij} = 0$ for all $i, j = 1, ..., m$. For instance δ_0 is flat. For a flat δ the holonomy ring $\mathfrak{hol}$ vanishes: $\mathfrak{hol} = 0$.

More generally let us define a *connection* on an arbitrary A-algebra B as a tuple of derivations (11) extending the derivations (5); one can then define the *curvature* by the same formula (12).

Note that, in classical differential geometry, a prominent role is played by connections on vector bundles; the framework of vector bundles is, however, too "linear" to have a useful arithmetic analogue: our arithmetic theory will be essentially a "non-linear" theory in which vector bundles (more generally modules) need to be replaced by principal bundles (more generally by rings).

There are various types of connections that we shall be interested in and for which we will seek arithmetic analogues; they will be referred to as:

(1) Ehresmann connections,
(2) Chern connections,
(3) Levi-Cività connections,
(4) Fedosov connections,
(5) Lax connections,
(6) Hamiltonian connections,

(7) Cartan connections,
(8) Riccati connections,
(9) Weierstrass connections,
(10) Painlevé connections.

In some cases our terminology above is not entirely standard. For instance what we call *Fedosov connection* is usually called *symplectic connection* [51]; we chose a different name in order to avoid confusion with the symplectic paradigm underlying the Hamiltonian case; our choice of name is based on the appearance of these connections in [51]. Also there are many connections in classical differential geometry which are known under the name of *Cartan connection*; the Cartan connections that we will be considering are usually introduced as *Cartan distributions* [1], p. 133, and "live" on jet bundles; they are unrelated, for instance, to the Maurer-Cartan connections.

In what follows we discuss the connections (1) through (7) in some detail. For these connections we assume A as in (1); for the connections (1) through (5) we assume, in addition, that B is as in (10); for connections (6) and (7) B will be defined when we get to discuss these connections. Connections (8), (9), (10) will not be discussed here but will appear in [42]; these are connections on curves (for (8), (9)) or surfaces (for (10)) appearing in relation to the classical theory of differential equations with no movable singularities. The Painlevé equations attached to (10) are closely related to the Hamiltonian connections (6). Both (9) and (10) lead to elliptic curves.

Definition 3. A connection (δ_i) on B in (10) is an *Ehresmann connection* if it satisfies one of the following two equivalent conditions:

1a) There exist $n \times n$ matrices A_i with coefficients in A such that

$$\delta_i x = A_i x \tag{15}$$

1b) The following diagrams are commutative:

$$\begin{array}{ccc} B & \stackrel{\mu}{\longrightarrow} & B \otimes_A B \\ \delta_i \downarrow & & \downarrow \delta_i \otimes 1 + 1 \otimes \delta_{0i} \\ B & \stackrel{\mu}{\longrightarrow} & B \otimes B \end{array} \tag{16}$$

Here μ is the comultiplication. Condition 1a can be referred to as saying that (δ_i) is *right linear*. Condition 1b can be referred to as saying that δ is *right invariant*. (As we shall see the arithmetic analogues of conditions 1a and 1b will cease to be equivalent.) If (δ_i) is an Ehresmann connection the curvature satisfies $\varphi_{ij}(x) = F_{ij}x$ where F_{ij} is the matrix given by the classical formula

$$F_{ij} := \delta_i A_j - \delta_j A_i - [A_i, A_j]. \tag{17}$$

Also there is a Galois theory attached to flat Ehresmann connections, the Picard-Vessiot theory. Indeed, for a flat Ehresmann connection $\delta = (\delta_i)$ consider the *logarithmic derivative map*, $l\delta : GL_n(A) \to \mathfrak{gl}_n(A)^m$, with coordinates

$$l\delta_i(u) = \delta_i u \cdot u^{-1}, \tag{18}$$

where $\mathfrak{gl}_n$ is the Lie algebra of GL_n. The fibers of the map $l\delta$ are solution sets of systems of linear equations

$$\delta_i u = A_i \cdot u. \tag{19}$$

And if one replaces A by a ring of complex analytic functions then Galois groups can be classically attached to such systems; these groups are algebraic subgroups of $GL_n(\mathbb{C})$ measuring the algebraic relations among the solutions to the corresponding systems.

We next discuss Chern connections. Let us consider again the rings A as in (1) and B as in (10), and let $q \in GL_n(A)$ be an $n \times n$ invertible matrix with coefficients in A which is either symmetric or antisymmetric,

$$q^t = \pm q; \tag{20}$$

so the t superscript means here *transposition*. Of course, a symmetric q as above is viewed as defining a (semi-Riemannian) metric on the trivial vector bundle

$$M \times \mathbb{R}^n \to M, \tag{21}$$

on $M = \mathbb{R}^m$ while an antisymmetric q is viewed as defining a 2-form on (21); here metrics/2-forms on vector bundles mean smoothly varying non-degenerate symmetric/antisymmetric bilinear forms on the fibers of the bundle. Set $G = GL_n$ and consider the maps of schemes over A,

$$\mathcal{H}_q : G \to G, \quad \mathcal{B}_q : G \times G \to G \tag{22}$$

defined by

$$\mathcal{H}_q(x) = x^t q x, \quad \mathcal{B}_q(x, y) = x^t q y. \tag{23}$$

We continue to denote by the same letters the corresponding maps of rings $B \to B$ and $B \to B \otimes_A B$. Consider the *trivial* connection $\delta_0 = (\delta_{0i})$ on G defined by $\delta_{0i} x = 0$. Then one has the following trivial:

Theorem 4. *There is a unique connection $\delta = (\delta_i)$ on G such that the following diagrams are commutative:*

$$\begin{array}{ccc} B & \stackrel{\delta_i}{\longleftarrow} & B \\ \mathcal{H}_q \uparrow & & \uparrow \mathcal{H}_q \\ B & \stackrel{\delta_{0i}}{\longleftarrow} & B \end{array} \qquad \begin{array}{ccc} B & \stackrel{\delta_i \otimes 1 + 1 \otimes \delta_{0i}}{\longleftarrow} & B \otimes_A B \\ \delta_{0i} \otimes 1 + 1 \otimes \delta_i \uparrow & & \uparrow \mathcal{B}_q \\ B \otimes_A B & \stackrel{\mathcal{B}_q}{\longleftarrow} & B \end{array} \tag{24}$$

For an argument and an explicit formula for (δ_i) (that are useful to compare with the arithmetic case) see (29) below.

Definition 5. We say that a connection δ is $\mathcal{H}_q$*-horizontal* (respectively $\mathcal{B}_q$*-symmetric*) with respect to δ_0 if the left (respectively right) diagram in (24) is commutative. The unique connection which is $\mathcal{H}_q$*-horizontal* and $\mathcal{B}_q$*-symmetric* with respect to δ_0 (cf. Theorem 4) is called the *Chern connection* attached to q.

Chern connections turn out to be automatically Ehresmann connections; the analogue of this in arithmetic will cease to be true.

The definition just given may look non-standard. It turns out that the Chern connection we just defined is a "real" analogue of the Chern connection on a hermitian vector bundle in differential geometry [54, 66] (in which δ_0 is an analogue of a complex structure). A special case of this real analogue is the connection introduced in [50] which we shall refer to as the *Duistermaat connection*.

To see the analogy of our Chern connection with the Chern connection in [54, 66] and to see an argument plus an explicit formula for δ in Theorem 4 we introduce more notation as follows.

For any connection (δ_i) on B in (10) set

$$\begin{array}{rcl} A_i = A_i(x) & := & \delta_i x \cdot x^{-1} \in \mathfrak{gl}_n(B), \\ \Gamma^k_{ij} = -A_{ikj} & = & (k,j) \text{ entry of } (-A_i). \end{array} \tag{25}$$

The quantities Γ^k_{ij} will be referred to as *Christoffel symbols of the* 2*nd kind* of the connection. Also, for q as in (20), set

$$\Gamma_{ijk} := \Gamma^l_{ij} q_{lk}, \tag{26}$$

which we refer to as the *Christoffel symbols of the* 1*st kind*. Here we use Einstein summation.

It is trivial to check that δ is $\mathcal{H}_q$-horizontal if and only if the followng condition is satisfied:

$$\delta_i q_{jk} = \Gamma_{ijk} \pm \Gamma_{ikj}. \tag{27}$$

Classically if (27) holds one says that q is *parallel* with respect to δ.

Similarly it is trivial to check that δ is $\mathcal{B}_q$-symmetric if and only if the following condition is satisfied:

$$\Gamma_{ijk} = \pm \Gamma_{ikj}. \tag{28}$$

This is, as we shall see, a condition different from the classical condition of *symmetry* for a connection on the tangent bundle.

In (27) and (28) the upper (respectively lower) sign correspond to the upper (respectively lower) sign in (20).

Using (27) and (28) it is trivial to see that the Chern connection attached to q exists and is unique, being given by

$$\Gamma_{ijk} = \frac{1}{2} \delta_i q_{jk}. \tag{29}$$

This proves Theorem 4 and gives the explicit formula for δ which is analogous to the classical formulas for the Chern and Duistermaat connections. Note that, as promised earlier, the Chern connection is an Ehresmann connection, i.e. A_i belong to $\mathfrak{gl}_n(A)$ rather than $\mathfrak{gl}_n(B)$.

By the way the Chern connection has the following compatibility with the special linear group SL_n: if $\delta_i(\det(q)) = 0$ then the Chern connection $\delta_i : B \to B$ attached to q sends the ideal of SL_n into itself and hence induces a "connection on SL_n." (This compatibility with SL_n will fail to hold in the arithmetic case.)

We next discuss Levi-Cività connections. Assume A as in (1), B as in (10), and assume that $n = m$. (Note we also implicitly assume here that a bijection is given between the set indexing the derivations δ_i and the set indexing the rows and columns of the matrix x_{ij}; such a bijection plays the role of what is classically called a *soldering*.)

Definition 6. The connection (δ_i) is *symmetric* or *torsion free* if

$$\Gamma_{ijk} = \Gamma_{jik}. \tag{30}$$

Note the difference between the condition (30) defining *symmetry* and the condition (28) defining $\mathcal{B}_q$*-symmetry*: the two types of symmetry involve different pairs of indices. To avoid any confusion we will use the term *torsion free* rather than *symmetric* in what follows. The fundamental theorem of Riemannian geometry is, in this setting, the following (completely elementary) statement:

Theorem 7. *Let $q \in GL_n(A)$, $q^t = q$. Then there is a unique connection δ that is $\mathcal{H}_q$-horizontal (i.e., satisfies equation (27) with the + sign) and is torsion free (i.e., it satisfies equation (30)); it is given by*

$$\Gamma_{kij} = \frac{1}{2}\left(\delta_k q_{ij} + \delta_i q_{jk} - \delta_j q_{ki}\right) \quad (31)$$

Definition 8. The connection in Theorem 7 is called the *Levi-Cività* connection attached to q.

Note that, in particular, the Levi-Cività connection is an Ehresmann connection, i.e. A_i belong to $\mathfrak{gl}_n(A)$ rather than $\mathfrak{gl}_n(B)$. (The latter will fail to hold in the arithmetic case.) The Levi-Cività connection is, in a precise sense to be discussed later, "dual" to the Chern connection and is generally different from the Chern connection; it coincides with the Chern connection if and only if

$$\delta_i q_{jk} = \delta_j q_{ik} \quad (32)$$

in which case q is called *Hessian* (the "real" analogue of *Kähler*).

For (F_{ij}) the curvature of the Levi-Cività connection attached to a metric $q = (q_{ij})$ we set:

$$F_{ij} = (F_{ijkl}), \quad R^k_{lij} := -F_{ijkl}, \quad R_{ijkl} = q_{im} R^m_{jkl}. \quad (33)$$

One refers to R_{ijkl} as the *Riemann curvature tensor*; the latter has following symmetries:

$$R_{ijkl} = -R_{jikl}, \quad R_{ijkl} = -R_{ijlk}, \quad R_{ijkl} + R_{iklj} + R_{ijlk} = 0, \quad R_{ijkl} = R_{klij}. \quad (34)$$

We next discuss Fedosov connections. To explain these we start, again, with A as in (1) and B as in (10).

Definition 9. Let $q \in GL_n(A)$, $q^t = -q$, so n is even. A *Fedosov connection* relative to q is a connection δ that is $\mathcal{H}_q$-horizontal (i.e., satisfies equation (27) with the $-$ sign) and is torsion free (i.e. it satisfies (30)).

One trivially checks that a Fedosov connection relative to q exists if and only if q is *symplectic* in the sense that it satisfies

$$\delta_i q_{jk} + \delta_j q_{ki} + \delta_k q_{ij} = 0. \quad (35)$$

Fedosov connections are not necessarily Ehresmann. And, for a given symplectic matrix q, Fedosov connections that are Ehresmann exist but are not unique; one Fedosov connection which is an Ehresmann connection is given by

$$\Gamma_{ijk} = \frac{1}{3}\left(\delta_i q_{jk} + \delta_j q_{ik}\right). \quad (36)$$

We next discuss Lax connections. Let A be as in (1) and B as in (10).

Definition 10. A connection (δ_i) on B is called a *Lax connection* if it satisfies

$$\delta_i x = [A_i(x), x] := A_i(x)x - xA_i(x) \quad (37)$$

for some $n \times n$ matrix $A_i(x)$ with coefficients in B.

Note that, unlike Chern and Levi-Cività connections, Lax connections are *not* a subclass of the Ehresmann connections. For a Lax connection, the following diagrams are commutative:

$$\begin{array}{ccc} B & \stackrel{\delta_i}{\longleftarrow} & B \\ \mathcal{P}\uparrow & & \uparrow\mathcal{P} \\ A[z] & \stackrel{\delta_{0i}}{\longleftarrow} & A[z] \end{array} \tag{38}$$

where $A[z] = A[z_1, ..., z_n]$ is a ring of polynomials in the variables z_j, δ_{0i} are the unique derivations extending the corresponding derivations on A with $\delta_{0i} z_j = 0$, and $\mathcal{P}$ is the A-algebra homomorphism with $\mathcal{P}(z_j) = \mathcal{P}_j(x)$,

$$\det(s \cdot 1 - x) = \sum_{j=0}^{n} (-1)^j \mathcal{P}_j(x) s^{n-j}. \tag{39}$$

The commutativity of (38) expresses the fact that the Lax connections describe "isospectral flows" on GL_n.

By the way the "real" theory summarized above has a "complex" analogue (and hence a "$(1,1)$"-analogue) for which we refer to [42]. Suffices to say here that, for the complex theory, one may start with $M = \mathbb{C}^m$ and with a subring

$$A \subset C^{\infty}(M, \mathbb{C}) \tag{40}$$

of the ring of smooth complex valued functions on M which is stable under the derivations

$$\delta_i := \frac{\partial}{\partial z_i}, \quad \delta_{\overline{i}} := \frac{\partial}{\partial \overline{z}_i}, \quad i = 1, ..., m, \tag{41}$$

where $z_1, ..., z_m$ are the complex coordinates on M. A connection on $G = Spec\ B$, $B = A[x, \det(x)^{-1}]$, is then an m-tuple of derivations $\delta_i : B \to B$ extending the derivations $\delta_i : A \to A$. Consider the unique derivations $\delta_{\overline{i}} : B \to B$, extending the derivations $\delta_{\overline{i}} : A \to A$, such that $\delta_{\overline{i}} x = 0$. Then one defines the $(1,1)$-curvature of $\delta = (\delta_i)$ as the matrix $(\varphi_{i\overline{j}})$ with entries the A-derivations

$$\varphi_{i\overline{j}} := [\delta_i, \delta_{\overline{j}}] : B \to B. \tag{42}$$

The theory proceeds from here.

So far we discussed connections on the ring B in (10). In what follows we informally discuss Hamiltonian and Cartan connections; for a precise discussion we refer to the main text. These are connections on rings B other than (10).

To discuss Hamiltonian connections consider the ring

$$B = A[x], \tag{43}$$

with A as in (1) and $x = \{x_1, ..., x_d\}$ a d-tuple of variables and consider a non-degenerate "vertical" 2-form

$$\omega := \omega^{ij} \cdot dx_i \wedge dx_j, \tag{44}$$

$\omega^{ij} \in B$, with ω *symplectic* (i.e. closed) "in the vertical directions"

$$\partial^i := \partial / \partial x_i$$

in the sense that

$$\partial^k \omega^{ij} + \partial^j \omega^{ki} + \partial^i \omega^{jk} = 0. \tag{45}$$

Definition 11. A *Hamiltonian* connection with respect to ω is a connection on a B as in (43) for which the corresponding "Lie derivatives" annihilate ω:

$$0 = \delta_k \omega := (\delta_k \omega^{ij}) \cdot dx_i \wedge dx_j + \omega^{ij} \cdot d(\delta_k x_i) \wedge dx_j + \omega^{ij} \cdot dx_i \wedge d(\delta_k x_j). \tag{46}$$

Note the contrast with the case of Fedosov connections which are attached to a matrix q that is symplectic "in the horizontal directions"

$$\delta_j = \partial / \partial \xi_j;$$

cf. (35). Hamiltonian connections with respect to symplectic forms naturally appear, by the way, in the background of some of the basic differential equations of mathematical physics, in particular in the background of the Painlevé VI equations. This kind of Hamiltonian connections turn out to have an arithmetic analogue [35] which will be discussed in [42].

On the other hand one can consider *Hamiltonian connections* on a B as in (43) with respect to a *Poisson structure*. The two concepts of Hamiltonian connections (relative to symplectic forms and relative to Poisson structures) are related in case

$$\delta_k \omega^{ij} = 0.$$

An example of Poisson structure is provided by *Lie-Poisson structures* on Lie algebras in which case the corresponding Hamiltonian connections typically lead to Lax connections. A classical example of equations arising from a Lie-Poisson structure are the *Euler equations* for the rigid body; the Euler equations are related to Lax equations in at least two different ways. Euler equations have an arithmetic analogue [41], although this arithmetic analogue is not be a priori related to our arithmetic analogues of Lax equations.

Finally we mention *Cartan connections*. In the same way we considered principal bundles and attached to them algebras B as in (10) one can consider "infinite jet bundles" (which we do not define here) and attach to them polynomial algebras,

$$B = A[x_j^{(\alpha)};\ \alpha \in \mathbb{Z}^m_{\geq 0}, j = 1, ..., d], \tag{47}$$

where A is as in (1) and $x_j^{(\alpha)}$ are indeterminates; these algebras come equipped with a natural flat connection

$$\delta_i := \frac{\partial}{\partial \xi_i} + \sum_j \sum_\alpha x_j^{(\alpha + e_i)} \frac{\partial}{\partial x_j^{(\alpha)}}, \tag{48}$$

where e_i is the canonical basis of $\mathbb{Z}^m$.

Definition 12. The flat connection (δ_i) on the ring B in (47) defined by (48) is called the *Cartan connection*.

Other names for (δ_i) above are: the *total derivative* or the *Cartan distribution*. The Cartan connections have an arithmetic analogue which was thoroughly studied in [28, 27] and plays a central role in the whole theory.

2.3. **Arithmetic connections and curvature.** We would like to introduce now our arithmetic analogues of connection and curvature and to state some of our main results about them. We start with the analogue of the real case. The first step is to consider the ring

$$B = A[x, \det(x)^{-1}]$$

defined as in (10), with $x = (x_{ij})$ an $n \times n$ matrix of indeterminates, but where A is given now by

$$A = \mathbb{Z}[1/M, \zeta_N]$$

as in (2). We again consider the group scheme over A,

$$G = GL_n := Spec\ B.$$

A first attempt to define arithmetic analogues of connections would be to consider families of p-derivations

$$\delta_p : B \to B, \quad p \in \mathcal{V},$$

extending the p-derivations (6); one would then proceed by considering their commutators on B (or, if necessary, expressions derived from these commutators). But the point is that the examples of "arithmetic analogues of connections" we will encounter in practice will almost never lead to p-derivations $B \to B$! What we shall be led to is, rather, an adelic concept we next introduce. (Our guiding "principle" here is that, as mentioned before, C^∞ geometric objects should correspond to adelic objects in arithmetic while analytic/algebraic geometric objects correspond to global objects in arithmetic.)

To introduce our adelic concept let us consider, for each $p \in \mathcal{V}$, the p-adic completion of B:

$$B^{\widehat{p}} := \lim_{\leftarrow} B/p^n B. \tag{49}$$

Then we make the following:

Definition 13. An *adelic connection* on $G = GL_n$ is a family (δ_p) of p-derivations

$$\delta_p := \delta_p^B : B^{\widehat{p}} \to B^{\widehat{p}}, \quad p \in \mathcal{V}, \tag{50}$$

extending the p-derivations in (6).

If $\phi_p : B^{\widehat{p}} \to B^{\widehat{p}}$ are the Frobenius lifts attached to δ_p and $G^{\widehat{p}} = Spf\ B^{\widehat{p}}$ is the p-adic completion of $G = GL_n = Spec\ B$ then we still denote by $\phi_p : G^{\widehat{p}} \to G^{\widehat{p}}$ the induced morphisms of p-adic formal schemes.

Next we will explore analogues of the various types of connections encountered in classical differential geometry: Ehresmann, Chern, Levi-Cività, Fedosov, and Lax. (The stories of Hamiltonian connections and Cartan connections will be discussed in [42] and will be skipped here.)

In what follows we need an analogue of the trivial connection in (14). It will be given by the adelic connection (δ_{0p}) defined by

$$\delta_{0p} x = 0. \tag{51}$$

The associated Frobenius lifts will be denoted by (ϕ_{0p}); they satisfy

$$\phi_{0p}(x) = x^{(p)} \tag{52}$$

where $x^{(p)}$ is the matrix (x_{ij}^p). We call $\delta_0 = (\delta_{0p})$ the *trivial* adelic connection.

To introduce arithmetic analogues of Ehresmann connections one starts by noting that, for $n \geq 2$, there are *no* adelic connections $\delta = (\delta_p)$ whose attached Frobenius lifts (ϕ_p) make the following diagrams commute:

$$\begin{array}{ccc} G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times G^{\widehat{p}} & \xrightarrow{\mu} & G^{\widehat{p}} \end{array} \tag{53}$$

Since (53) is an analogue of (16) one can view this as saying that there are no adelic connections that are analogues of right invariant connections. This is an elementary observation; one can in fact prove a less elementary result:

Theorem 14. [42] *For $n \geq 2$ and $p \not| n$ there are no adelic connections (δ_p) and (δ_{1p}) whose attached Frobenius lifts (ϕ_p) and (ϕ_{1p}) make the following diagrams commute:*

$$\begin{array}{ccc} G^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \\ \phi_p \times \phi_{1p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \end{array} \tag{54}$$

There is a useful property, weaker than the commutativity of (53), namely an invariance property with respect to the action on G by right translation of the group

$$N := (\text{normalizer of the diagonal maximal torus } T \text{ of } G). \tag{55}$$

Indeed we will say that an adelic connection (δ_p) with associated Frobenius lifts (ϕ_p) is *right invariant with respect to N* if the following diagrams are commutative:

$$\begin{array}{ccc} G^{\widehat{p}} \times N^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \phi_p \\ G^{\widehat{p}} \times N^{\widehat{p}} & \stackrel{\mu}{\longrightarrow} & G^{\widehat{p}} \end{array} \tag{56}$$

This latter property has its own merits but is too weak to function appropriately as a defining property of Ehresmann connections in arithmetic. Instead, we will consider an appropriate analogue of "linearity," (15). What one can do [38] is to replace the Lie algebra $\mathfrak{gl}_n$ by an arithmetic analogue of it, $\mathfrak{gl}_{n,\delta_p}$, and then introduce an arithmetic analogue of the logarithmic derivative. This new framework naturally leads to the following:

Definition 15. An adelic connection (δ_p) is an *Ehresmann connection* if

$$\delta_p x = \alpha_p \cdot x^{(p)}, \tag{57}$$

where α_p are matrices with coefficients in A.

By the way, clearly, Ehresmann connections are right invariant with respect to N. One can attach Galois groups to such Ehresmann connections [39] and develop the basics of their theory. A natural expectation is that these Galois groups belong to the group $N(A)^{\delta}$ of all matrices in $N(A)$ whose entries are roots of unity or 0. This expectation is not always realized but one can prove that something close to it is realized for (α_p) "sufficiently general;" cf. [39]. The above expectation is justified by the fact that, according to the general philosophy of the field with one element $\mathbb{F}_1$, the union of the $N(A)^{\delta}$'s, as A varies, plays the role of "$GL_n(\mathbb{F}_1^a)$," where $\mathbb{F}_1^a$ is the "algebraic closure of $\mathbb{F}_1$."

Next we explain our arithmetic analogue of Chern connections. Let $q \in GL_n(A)$ with $q^t = \pm q$. Attached to q we have, again, maps

$$\mathcal{H}_q : G \to G, \quad \mathcal{B}_q : G \times G \to G; \quad \mathcal{H}_q(x) = x^t q x, \quad \mathcal{B}_q(x,y) = x^t q y.$$

We continue to denote by $\mathcal{H}_q, \mathcal{B}_q$ the maps induced on the p-adic completions $G^{\widehat{p}}$ and $G^{\widehat{p}} \times G^{\widehat{p}}$. Consider again the trivial adelic connection $\delta_0 = (\delta_{0p})$ on G (so $\delta_{0p} x = 0$) and denote by (ϕ_{0p}) the attached Frobenius lifts (so $\phi_{0p}(x) = x^{(p)}$). Then one can prove the following:

Theorem 16. [38] *For any $q \in GL_n(A)$ with $q^t = \pm q$ there exists a unique adelic connection $\delta = (\delta_p)$ whose attached Frobenius lifts (ϕ_p) make the following diagrams commutative:*

$$\begin{array}{rcl} G^{\widehat{p}} & \stackrel{\phi_p}{\longrightarrow} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \stackrel{\phi_{0p}}{\longrightarrow} & G^{\widehat{p}} \end{array} \qquad \begin{array}{rcl} G^{\widehat{p}} & \stackrel{\phi_{0p}\times\phi_p}{\longrightarrow} & G^{\widehat{p}} \times G^{\widehat{p}} \\ \phi_p \times \phi_{0p} \downarrow & & \downarrow \mathcal{B}_q \\ G^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\mathcal{B}_q}{\longrightarrow} & G^{\widehat{p}} \end{array} \tag{58}$$

Definition 17. We say that an adelic connection $\delta = (\delta_p)$ is *$\mathcal{H}_q$-horizontal* (respectively *$\mathcal{B}_q$-symmetric*) with respect to $\delta_0 = (\delta_{0p})$ if the left (respectively right) diagram in (58) is commutative. The unique connection which is *$\mathcal{H}_q$-horizontal* and *$\mathcal{B}_q$-symmetric* with respect to δ_0 (cf. Theorem 16) is called the *Chern connection* attached to q.

Unlike in the case of classical differential geometry our adelic Chern connections are *not* be special cases of Ehresmann connections (although they are right invariant with respect to N).

Note the following relation between the "Christoffel symbols" defining our Chern connection and the Legendre symbol. We explain this in a special case. Let

$$q \in GL_1(A) = A^\times, \qquad A = \mathbb{Z}[1/M],$$

and let $\delta = (\delta_p)$ be the Chern connection associated to q. Then it turns out that $\phi_p : G^{\widehat{p}} \to G^{\widehat{p}}$ is defined by $\phi_p : \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}} \to \mathbb{Z}_p[x, x^{-1}]^{\widehat{p}}$,

$$\phi_p(x) = q^{(p-1)/2} \left(\frac{q}{p}\right) x^p, \tag{59}$$

where $\left(\frac{q}{p}\right)$ is the Legendre symbol of $q \in A^\times \subset \mathbb{Z}_{(p)}$.

We can also introduce adelic connections that are analogues of Levi-Cività connections. They are already relevant in case $\mathcal{V}$ consists of one prime only. So assume, in the theorem below, that $\mathcal{V} = \{p\}$ consists of one prime p.

Theorem 18. [42] *For any symmetric $q \in GL_n(A)$ with $q^t = q$ there is a unique n-tuple $(\delta_{1p}, ..., \delta_{np})$ of adelic connections on $G = GL_n$ with attached Frobenius lifts $(\phi_{1p}, ..., \phi_{np})$, such that the following diagrams are commutative for $i = 1, ..., n$,*

$$\begin{array}{rcl} G^{\widehat{p}} & \stackrel{\phi_{ip}}{\longrightarrow} & G^{\widehat{p}} \\ \mathcal{H}_q \downarrow & & \downarrow \mathcal{H}_q \\ G^{\widehat{p}} & \stackrel{\phi_{0p}}{\longrightarrow} & G^{\widehat{p}} \end{array} \tag{60}$$

and such that, for all $i, j = 1, ..., n$, we have:

$$\delta_{ip} x_{kj} = \delta_{jp} x_{ki}. \tag{61}$$

Definition 19. The tuple $(\delta_{1p}, ..., \delta_{np})$ in Theorem 18 is called the *Levi-Cività connection* attached to q,

The condition (60) says of course that for each i, (δ_{ip}) is $\mathcal{H}_q$-horizontal with respect to (δ_{0p}) and hence is analogous to the condition of parallelism (27) (with the + sign). The condition (61) is analogous to the condition of torsion freeness (30). This justifies our Definition 19 of the Levi-Cività connection. But note that, unlike in the case of classical differential geometry, our adelic Levi-Cività connections are *not* special cases of Ehresmann connections.

The adelic Levi-Cività connection and the adelic Chern connection attached to q are related by certain congruences mod p that are reminiscent of the relation between the two connections in classical differential geometry.

The one prime paradigm of the Levi-Cività connection above can be viewed as corresponding to the classical Levi-Cività connections of *central* metrics, by which we understand metrics $g = \sum g_{ij} d\xi_i d\xi_j$ satisfying

$$\delta_k g_{ij} = \delta_l g_{ij}. \tag{62}$$

Continuing to assume $\mathcal{V} = \{p\}$ one can also attempt to develop an arithmetic analogue of Fedosov connections as follows. Consider an antisymmetric $q \in GL_n(A)$, $q^t = -q$. Let us say that an n-tuple of $(\delta_{1p}, ..., \delta_{np})$ of adelic connections on $G = GL_n$ is a *Fedosov connection* relative to q if the attached Frobenius lifts $(\phi_{1p}, ..., \phi_{np})$ make the diagrams (60) commutative and, in addition, the equalities (61) hold. One can prove that for $n = 2$ and any antisymmetric q Fedosov connections relative to q exist. However, in contrast with the Levi-Cività story, one can prove that for $n \geq 4$ there is no Fedosov connection relative to the split q, for instance.

Finally there are adelic connections that are analogous to Lax connections. In fact there are two such analogues which we call *isospectral* and *isocharacteristic* Lax connections. They offer two rather different arithmetic analogues of isospectral flows in the space of matrices. Indeed the isocharacteristic property essentially says that a certain characteristic polynomial has "δ-constant" coefficients whereas the isospectrality property essentially says that the characteristic polynomial has "δ-constant" roots. (Here *δ-constant* means "killed by all δ_p" which amounts to "being a root of unity or 0".) In usual calculus the two properties are equivalent but in our arithmetic calculus these two properties are quite different. By the way isospectral and isocharacteristic Lax connections are not defined on the whole of G but rather on certain Zariski open sets G^* and G^{**} of G respectively. Let us give some details of this in what follows. Let $T^* \subset T$ be the open set of *regular* matrices in the diagonal maximal torus T; *regular* means here "with distinct diagonal entries." One can prove:

Theorem 20. [38] *There exists a unique adelic connection $\delta = (\delta_p)$ with Frobenius lifts (ϕ_p) such that each ϕ_p makes the following diagram commute:*

$$\begin{array}{ccc} (T^*)^{\widehat{p}} \times G^{\widehat{p}} & \stackrel{\phi_{0p} \times \phi_{0p}}{\longrightarrow} & (T^*)^{\widehat{p}} \times G^{\widehat{p}} \\ \mathcal{C} \downarrow & & \downarrow \mathcal{C} \\ (G^*)^{\widehat{p}} & \stackrel{\phi_p}{\longrightarrow} & (G^*)^{\widehat{p}} \end{array} \tag{63}$$

where $T^* := G^* \cap T$ and $\mathcal{C}(t, x) := x^{-1} t x$.

Definition 21. The adelic connection δ in Theorem 20 is called the *canonical isospectral Lax connection.*

On the other hand isocharacteristic Lax connections make commutative the following diagrams that are analogous to (38):

$$\begin{array}{ccc} (G^{**})^{\widehat{p}} & \stackrel{\phi_p}{\longrightarrow} & (G^{**})^{\widehat{p}} \\ \mathcal{P} \downarrow & & \downarrow \mathcal{P} \\ (\mathbb{A}^n)^{\widehat{p}} & \stackrel{\phi_{0p}}{\longrightarrow} & (\mathbb{A}^n)^{\widehat{p}}, \end{array} \tag{64}$$

where $\mathbb{A}^n = Spec\ A[z]$. Among isocharacteristic Lax connections there is a *canonical* one.

Note that the canonical Frobenius lifts ϕ_p in the two diagrams (63) and (64) do *not* coincide on the intersection $G^* \cap G^{**}$; so the isospectral and isocharacteristic stories are really different.

Next we would like to explore the curvature of adelic connections. Consider first the case of Ehresmann connections, (57). Since $\alpha_p \in A$ for all p our Frobenius lifts $\phi_p : B^{\widehat{p}} \to B^{\widehat{p}}$ induce Frobenius lifts $\phi_p : A[x] \to A[x]$ and hence one can consider the "divided" commutators

$$\varphi_{pp'} := \frac{1}{pp'}[\phi_p, \phi_{p'}] : A[x] \to A[x], \quad p, p' \in \mathcal{V}. \tag{65}$$

The family $(\varphi_{pp'})$ can be referred to as the *curvature* of the adelic connection (δ_p).

The situation for general adelic connections (including the cases of Chern and Lax connections) is quite different. Indeed, in defining curvature we face the following dilemma: our p-derivations δ_p in (50) do not act on the same ring, so there is no a priori way of considering their commutators and, hence, it does not seem possible to define, in this way, the notion of curvature. It turns out, however, that some of our adelic connections satisfy a remarkable property which we call *globality along the identity* (more generally along various subvarieties); this property allows us to define curvature via commutators.

Definition 22. Consider the matrix $T = x - 1$, where 1 is the identity matrix. An adelic connection $\delta = (\delta_p)$ on GL_n, with attached family of Frobenius lifts (ϕ_p), is *global along* 1 if, for all p, $\phi_p : B^{\widehat{p}} \to B^{\widehat{p}}$ sends the ideal of 1 into itself and, moreover, the induced homomorphism $\phi_p : A^{\widehat{p}}[[T]] \to A^{\widehat{p}}[[T]]$ sends the ring $A[[T]]$ into itself. If the above holds then the *curvature* of (δ_p) is defined as the family of "divided" commutators $(\varphi_{pp'})$,

$$\varphi_{pp'} := \frac{1}{pp'}[\phi_p, \phi_{p'}] : A[[T]] \to A[[T]], \tag{66}$$

where $p, p' \in \mathcal{V}$. Let $\text{End}(A[[T]])$ denote the Lie ring of $\mathbb{Z}$-module endomorphisms of $A[[T]]$. Then define the *holonomy ring* $\mathfrak{hol}$ of δ as the $\mathbb{Z}$-linear span in $\text{End}(A[[T]])$ of all the *Lie monomials*

$$[\phi_{p_1}, [\phi_{p_2}, ..., [\phi_{s-1}, \phi_{p_s}]...]] : A[[T]] \to A[[T]]$$

where $s \geq 2$, $p_i \in \mathcal{V}$. Similarly define the *holonomy* $\mathbb{Q}$-*algebra* $\mathfrak{hol}_{\mathbb{Q}}$ of δ as the $\mathbb{Q}$-linear span of $\mathfrak{hol}$ in $\text{End}(A[[T]]) \otimes \mathbb{Q}$. Finally define the *completed holonomy ring*,

$$\widehat{\mathfrak{hol}} = \lim_{\leftarrow} \mathfrak{hol}_n,$$

where $\mathfrak{hol}_n$ is the image of the map

$$\mathfrak{hol} \to \text{End}(A[[T]]/(T)^n). \tag{67}$$

The various maps referred to above can be traced on the following diagram:

$$\begin{array}{ccccccccc}
B^{\widehat{p}} & \subset & A^{\widehat{p}}[[T]] & \supset & A[[T]] & \subset & A^{\widehat{p'}}[[T]] & \supset & B^{\widehat{p'}} \\
\phi_p \downarrow & & \phi_p \downarrow & & \phi_p \downarrow\downarrow \phi_{p'} & & \downarrow \phi_{p'} & & \downarrow \phi_{p'} \\
B^{\widehat{p}} & \subset & A^{\widehat{p}}[[T]] & \supset & A[[T]] & \subset & A^{\widehat{p'}}[[T]] & \supset & B^{\widehat{p'}}
\end{array}$$

The idea of comparing p-adic phenomena for different p's by "moving along the identity section" was introduced in [27] where it was referred to as *analytic continuation between primes*. Analytic continuation is taken here in the sense of Zariski [**?**], Preface, pp. xii-xiii, who was the first to use completions of varieties along subvarieties as a substitute for classical analytic continuation over the complex numbers. The technique of analytic continuation is also used, in the form of *formal patching*, in inverse Galois theory [56]. Note that in our context we are patching data defined on "tubular neighborhoods"

$$Spf\ B^{\widehat{p}} \quad \text{and} \quad Spf\ A[[T]]$$

of two *closed* subsets

$$Spec\ B/pB \quad \text{and} \quad Spec\ B/(T)$$

of the scheme *Spec* B; the data are required to coincide on the "tubular neighborhood"

$$Spf\ A^{\widehat{p}}[[T]]$$

of the intersection

$$Spec\ B/(p,T) = (Spec\ B/pB) \cap (Spec\ B/(T)).$$

This is in contrast with the use of patching in Galois theory [56] where one patches data defined on two *open* sets covering a formal scheme.

Of course, the trivial adelic connection $\delta_0 = (\delta_{0p})$, $\delta_{0p}x = 0$, is global along 1 so it induces ring endomorphisms $\phi_{0p} : A[[T]] \to A[[T]]$,

$$\phi_{0p}(T) = (1+T)^{(p)} - 1.$$

We may morally view δ_0 as an analogue of a flat connection in real geometry (where $A \subset C^{\infty}(\mathbb{R}^m, \mathbb{R})$). Alternatively we may view δ_0 as an arithmetic analogue of the derivations $\delta_{\overline{i}} = \partial/\partial \overline{z}_i$ on $A[x, \det(x)^{-1}]$ which kill x, where $A \subset C^{\infty}(\mathbb{C}^m, \mathbb{C})$. Following this second analogy we may consider an arbitrary adelic connection $\delta = (\delta_p)$, with attached Frobenius lifts (ϕ_p), and introduce the following:

Definition 23. The $(1,1)$-*curvature* of δ is the matrix of "divided commutators" $(\varphi_{p\overline{p}'})$

$$\varphi_{p\overline{p}'} := \frac{1}{pp'}[\phi_p, \phi_{0p'}] : A[[T]] \to A[[T]], \quad p \neq p', \tag{68}$$

$$\varphi_{p\overline{p}} := \frac{1}{p}[\phi_p, \phi_{0p}] : A[[T]] \to A[[T]]. \tag{69}$$

Going back to our discussion of curvature for Chern connections one can prove:

Theorem 24. [5] *Let $q \in GL_n(A)$ with $q^t = \pm q$. If all the entries of q are roots of unity or* 0 *then the Chern connection δ attached to q is global along* 1*; in particular δ has a well defined curvature and* $(1,1)$*-curvature.*

So we may address the question of computing the curvature and $(1,1)$-curvature of Chern connections for various q's whose entries are 0 or roots of unity. A special case of such q's is given by the following:

Definition 25. A matrix $q \in GL_n(A)$ is *split* if it is one of the following:

$$\left(\begin{array}{cc} 0 & 1_r \\ -1_r & 0 \end{array}\right), \quad \left(\begin{array}{cc} 0 & 1_r \\ 1_r & 0 \end{array}\right), \quad \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1_r \\ 0 & 1_r & 0 \end{array}\right), \tag{70}$$

where 1_r is the $r \times r$ identity matrix and $n = 2r, 2r, 2r+1$ respectively.

One can prove:

Theorem 26. [5] *Let q be split and let $(\varphi_{pp'})$ and $(\varphi_{p\overline{p}'})$ be the curvature and the $(1,1)$-curvature of the Chern connection on G attached to q. Then the following hold:*

1) Assume $n \geq 4$. Then for all $p \neq p'$ we have $\varphi_{pp'} \neq 0$.

2) Assume $n = 2r \geq 2$. Then for all p, p' we have $\varphi_{pp'}(T) \equiv 0$ mod $(T)^3$.

3) Assume $n = 2$ and $q^t = -q$. Then for all p, p' we have $\varphi_{pp'} = 0$.

4) Assume $n \geq 2$. Then for all p, p' we have $\varphi_{p\overline{p}'} \neq 0$.

5) Assume $n = 1$. Then for all p, p' we have $\varphi_{pp'} = \varphi_{p\overline{p}'} = 0$.

Assertion 1 morally says that *Spec* $\mathbb{Z}$ is "curved," while assertion 2 morally says that *Spec* $\mathbb{Z}$ is only "mildly curved." Assertions 1 and 2 imply, in particular, assertion 1 in the following:

Theorem 27. [42] *Assume q split and $n \geq 4$ is even. Then the following hold:*

1) $\widehat{\mathfrak{hol}}$ is non-zero and pronilpotent.

2) $\mathfrak{hol}_{\mathbb{Q}}$ is not spanned over $\mathbb{Q}$ by the components of the curvature.

Assertion 1 is in stark contrast with the fact that *holonomy Lie algebras* arising from Galois theory are never nilpotent unless they vanish. Assertion 2 should be viewed as a statement suggesting that the flavor of our arithmetic situation is rather different from that of classical *locally symmetric* spaces; indeed, for the latter, the Lie algebra of holonomy *is* spanned by the components of the curvature.

Note that the above theorem says nothing about the vanishing of the curvature $\varphi_{pp'}$ in case $n = 2, 3$ and $q^t = q$; our method of proof does not seem to apply to these cases.

By the way if $q \in GL_{2r}$ is symmetric and split then the Chern connection $\delta_p : B^{\widehat{p}} \to B^{\widehat{p}}$ attached to q *does not* send the ideal of SL_n into itself; this is in contrast with the situation encountered in classical differential geometry. To remedy this situation one constructs connections on GL_n that *do* send the ideal of SL_n into itself; there are many such connections and the "simplest" one will be called the *special linear connection.* Cf. [42].

Other curvatures can be introduced and vanishing/non-vanishing results for them can be proved; they can be referred to as 3-*curvature, first Chern form, first Chern* $(1,1)$-*form, mean curvature, scalar curvature*, etc. One can also introduce *Chern connections* attached to hermitian matrices and results can be proved for their curvature. Cf. [42].

Similar results can be proved for the curvature of Lax connections. What happens is that the open sets G^* and G^{**} where isospectral and isocharacteristic Lax connections are defined (cf. (63) and (64)) do not contain the identity of the group $G = GL_n$ hence curvature cannot be defined by analytic continuation along the identity; however these open sets contain certain torsion points of the diagonal maximal torus of G and one can use analytic continuation along such torsion points to define curvature and $(1,1)$-curvature. One can then prove the non-vanishing of the $(1,1)$-curvature of isocharacteristic Lax connection for $n=2$. The (canonical) isospectral Lax connection has, immediately from its definition, a vanishing curvature.

Note that the concept of curvature discussed above was based on what we called *analytic continuation between primes*; this was the key to making Frobenius lifts corresponding to different primes act on a same ring and note that it only works as stated for adelic connections that are global along 1. This restricts the applicability of our method to "metrics" q with components roots of unity or 0. One can generalize our method to include q's with more general entries by replacing the condition of being global along 1 with the condition of "being global along certain tori." However, there is a different approach towards making Frobenius lifts comparable; this approach is based on *algebraizing Frobenius lifts via correspondences* and works for adelic connections that are not necessarily global along 1 (or along a torus). The price to pay for allowing this generality is that endomorphisms (of $A[[T]]$) are replaced by correspondences (on GL_n). Let us explain this alternative road to curvature in what follows.

Definition 28. Let $\phi_p : X^{\widehat{p}} \to X^{\widehat{p}}$ be a Frobenius lift on $X^{\widehat{p}}$ where X is a scheme of finite type over A. An *algebraizing correspondence* for ϕ_p is a triple $\Gamma_p := (Y_p, \pi_p, \varphi_p)$ where $\pi_p : Y_p \to G$ and $\varphi_p : Y_p \to G$ are morphisms of A-schemes of finite type such that:

1) π_p is affine and étale;
2) the p-adic completion of π_p, $\pi_p^{\widehat{p}} : Y_p^{\widehat{p}} \to G^{\widehat{p}}$, is an isomorphism;
3) we have an equality of maps, $\varphi_p^{\widehat{p}} = \phi_p \circ \pi_p^{\widehat{p}} : Y_p^{\widehat{p}} \to G^{\widehat{p}}$.

For the Chern connection one can prove the following:

Theorem 29. [40] *Let $\delta = (\delta_p)$ be the Chern connection on $G = GL_n$ attached to a matrix $q \in GL_n(A)$ with $q^t = \pm q$ and let (ϕ_p) be the attached Frobenius lifts on $G^{\widehat{p}}$. Then for each p there exists an algebraizing correspondence $\Gamma_p = (Y_p, \pi_p, \varphi_p)$ for ϕ_p.*

Similarly for the Levi-Civitá connection we have the following (in which, to simplify, we assume $A = \mathbb{Z}[1/M]$):

Theorem 30. [43] *Let $(\delta_{1p}, ..., \delta_{np})$ be the Levi-Cività connection attached to $q \in GL_n(A)$, $q^t = q$, and let $(\phi_{1p}, ..., \phi_{np})$ be the attached Frobenius lifts on $G^{\widehat{p}}$. Then there exist algebraizing correspondences $\Gamma_{pi} = (Y_p, \pi_p, \varphi_{pi})$ for ϕ_{pi}.*

Definition 31. The family (Γ_p) (respectively (Γ_{pi})) is called a *correspondence structure* for (δ_p) (respectively (δ_{pi})).

This structure is not unique but does have some "uniqueness features."

On the other hand correspondences as above act on the field E of rational functions of $G = GL_n$. Let us explain this in the Chern case; the Levi-Civitá case is similar. The action is by the formula $\Gamma_p^* : E \to E$,

$$\Gamma_p^*(z) := \mathrm{Tr}_{\pi_p}(\varphi_p^*(z)), \quad z \in E, \tag{71}$$

where

$$\mathrm{Tr}_{\pi_p} : F_p \to E$$

is the trace of the extension

$$\pi_p^* : E \to F_p := Y_p \otimes_G E$$

and

$$\varphi_p^* : E \to F_p$$

is induced by φ_p.

Definition 32. The *curvature* of (Γ_p) is the matrix $(\varphi_{pp'}^*)$ with entries the additive homomorphisms

$$\varphi_{pp'}^* := \frac{1}{pp'}[\Gamma_p^*, \Gamma_{p'}^*] : E \to E, \quad p, p' \in \mathcal{V}. \tag{72}$$

Note that, in this way, we have defined a concept of "curvature" for Chern connections attached to arbitrary q's (that do not necessarily have entries zeroes or roots of unity). There is a $(1,1)$-version of the above as follows. Indeed the trivial adelic connection $\delta_0 = (\delta_{0p})$ has a canonical correspondence structure (Γ_{0p}) given by

$$\Gamma_{0p} = (G, \pi_{0p}, \varphi_{0p}),$$

where π_{0p} is the identity, and $\varphi_{0p}(x) = x^{(p)}$.

Definition 33. The $(1,1)$-*curvature* of (Γ_p) is the family $(\varphi_{p\overline{p}'}^*)$ where $\varphi_{p\overline{p}'}^*$ is the additive endomorphism

$$\varphi_{p\overline{p}'}^* := \frac{1}{pp'}[\Gamma_{0p'}^*, \Gamma_p^*] : E \to E \ \text{ for } p \neq p', \tag{73}$$

$$\varphi_{p\overline{p}}^* := \frac{1}{p}[\Gamma_{0p}^*, \Gamma_p^*] : E \to E. \tag{74}$$

Then one can prove the following:

Theorem 34. [40] *Let $q \in GL_2(A)$ be split. Then the following hold:*

1) Assume $q^t = -q$. Then for all p, p' we have $\varphi_{pp'}^* = 0$ and $\varphi_{p\overline{p}'}^* \neq 0$.

2) Assume $q^t = q$. Then for all p, p' we have $\varphi_{p\overline{p}'}^* \neq 0$.

Once again our results say nothing about curvature in case $n = 2$ and $q^t = q$; our method of proof does not seem to apply to this case.

Finally note that one can define curvature for the Levi-Cività connection in case $\mathcal{V} = \{p\}$.

Definition 35. Let $(\delta_{1p}, ..., \delta_{np})$ be the Levi-Cività connection attached to a symmetric $q \in GL_n(A)$. The *curvature* is defined as the family (φ_p^{ij}), indexed by $i, j = 1, ..., n$ given by the divided commutators

$$\varphi_p^{ij} := \frac{1}{p}[\phi_{ip}, \phi_{jp}] : \mathcal{O}(G^{\widehat{p}}) \to \mathcal{O}(G^{\widehat{p}}). \tag{75}$$

This is a "vertical" curvature (indexed by the index set of the columns and rows of x) rather than a "horizontal" curvature, in the style of the previously introduced curvatures (which are indexed by primes). This curvature satisfies congruences mod p that are analogous to the symmetries of the classical Riemann tensor. One can prove non-vanishing results for these curvatures. For instance we have:

Theorem 36. [42] *Let i, j, k, l be fixed indices between 1 and n. Then the following hold:*

1) Assume $\delta_p q_{jk} + \delta_p q_{il} \not\equiv \delta_p q_{ik} + \delta_p q_{jl}$ mod p. Then $\varphi_p^{ij}, \varphi_p^{kl} \not\equiv 0$ mod p.

2) Assume $n = 2r$ and q is split. Then $\varphi_p^{ij} \not\equiv 0$ mod p for $i \neq j$.

We will actually prove a more precise result than assertion 1 of the above Theorem. Indeed set

$$\Phi^{ij} := \Phi^{ij}(x) := \varphi^{ij}(x), \quad \Psi_{ij} := \Psi_{ij}(x) := x^{(p^2)t} q^{(p^2)} \Phi^{ij}(x),$$

and let R_{ijkl} be the (k, l)-entry of the matrix Ψ_{ij}, so

$$\Psi_{ij} = (R_{ijkl}). \tag{76}$$

Then the R_{ijkl} in 76 can be viewed as an arithmetic analogue of the covariant Riemann tensor in classical differential geometry. We will prove that (R_{ijkl}) in 76 satisfies congruences mod p:

$$R_{ijkl} \equiv -R_{jikl}, \quad R_{ijkl} \equiv -R_{ijlk}, \quad R_{ijkl} + R_{iklj} + R_{ijlk} \equiv 0, \quad R_{ijkl} \equiv R_{klij}, \tag{77}$$

which are, of course, an arithmetic analogue of the classical symmetries of the Riemann curvature tensor. In addition we will prove the congruence

$$R_{ijkl} \equiv \frac{1}{2}(\delta q_{jk} + \delta q_{il} - \delta q_{ik} - \delta q_{jl})^p \mod (p, x-1). \tag{78}$$

This can be viewed as an analogue of the expression of curvature at a point, in normal coordinates.

In case the entries of q are roots of unity or 0 one can prove that for each i, (δ_{ip}) is global along 1 so we will be able to define a *mixed curvature* $(\varphi_{pp'}^{ij})$ indexed by $i, j = 1, ..., n$ and $p, p' \in \mathcal{V}$ given by the divided commutators

$$\varphi_{pp'}^{ij} := \frac{1}{pp'}[\phi_{ip}, \phi_{jp'}] : A[[T]] \to A[[T]], \quad p \neq p', \tag{79}$$

$$\varphi_{pp}^{ij} := \frac{1}{p}[\phi_{ip}, \phi_{jp}] : A[[T]] \to A[[T]]. \tag{80}$$

For a fixed p and the Fedosov connection $(\delta_{1p}, \delta_{2p})$ relative to any antisymmetric $q \in GL_2(A)$ the formula (75) defines, again, a *curvature*; one can prove that this curvature does not vanish in general even if q is split.

Finally, using the algebraization by correspondences one can define a *mixed curvature* for the Levi-Civitá connection for $\mathcal{V}$ consisting of more than one prime and for q with entries not necessarily roots of unity or 0 and one can prove various non-vanishing results for this curvature; cf. [43].

3. Comparison with other theories

3.1. **Three perspectives.** A number of analogies between primes and geometric objects have been proposed. Here are three of them:

A) Primes are analogous to points on a Riemann surface.
B) Primes are analogous to knots in a 3-dimensional manifold.
C) Primes are analogous to directions in an infinite dimensional manifold.

The viewpoint A is classical, it has a complex analytic flavor, and goes back to Dedekind, Hilbert, etc. The framework of Grothendieck, Arakelov, etc., also fits into viewpoint A. According to this viewpoint the ring of integers $\mathbb{Z}$, or more generally rings of integers in number fields, can be viewed as analogues of rings of functions on Riemann surfaces or affine algebraic curves; these are objects of complex dimension 1 (or real dimension 2). Genera of number fields are classically defined and finite, as in the case of Riemann surfaces. There is a related viewpoint according to which $\mathbb{Z}$ is the analogue of an algebraic curve of infinite genus; cf. e.g., [49].

The viewpoint B has a topological flavor and originates in suggestions of Mazur, Manin, Kapranov, and others. According to this viewpoint $Spec\,\mathbb{Z}$ should be viewed as an analogue of a 3-dimensional manifold, while the embeddings $Spec\ \mathbb{F}_p \to Spec\ \mathbb{Z}$ should be viewed as analogues of embeddings of circles. The Legendre symbol is then an analogue of linking numbers. This analogy goes rather deep [76].

Our approach here and previous papers by the author adopt the viewpoint C and have a differential geometric flavor.

There is a possibility that our theory has connections with viewpoint B, as shown, for instance, by the presence of the Legendre symbol in our Chern connections. Indeed the underlying Galois theory of reciprocity is an analogue of the monodromy in the 3-dimensional picture; in the same way our arithmetic curvature theory could be an analogue of the identity component of a natural "holonomy" in the 3-dimensional picture.

3.2. **Field with one element.** There are other approaches that adopt the viewpoint C. For instance Haran's theory of the field with one element, $\mathbb{F}_1$, cf. [55], (and previous $\mathbb{F}_1$ flavored work of Kurokawa and others [70]) considers the operators

$$\frac{\partial}{\partial p} : \mathbb{Z} \to \mathbb{Z}, \quad \frac{\partial a}{\partial p} := v_p(a)\frac{a}{p},$$

where $v_p(a)$ is the p-adic valuation of a. These operators have a flavor that is rather different from that of Fermat quotients, though, and it seems unlikely that Haran's theory and ours are directly related. Even more remote from our theory are the $\mathbb{F}_1$ theories of Soulé [80] and Connes-Consani [49] which do not directly provide a way to differentiate integers.

Borger's philosophy of $\mathbb{F}_1$, cf. [8], is, in some sense, perpendicular to the above mentioned approaches to $\mathbb{F}_1$ and, in the "case of one prime" is consistent with our approach: roughly speaking, in the case of one prime, Borger's theory [8, 10] can be viewed as an algebraization of our analytic theory in [16, 28]. In the case of more (all) primes Borger's $\mathbb{F}_1$ theory can also be viewed as a viewpoint consistent with C above: indeed Borger's beautiful suggestion is to take λ-*structures* (in the sense of Grothendieck) as *descent data* from $\mathbb{Z}$ to $\mathbb{F}_1$. Recall that a λ-*structure* on a scheme X flat over $\mathbb{Z}$ is the same as a commuting family (ϕ_p) of Frobenius lifts $\phi_p : X \to X$. So our theory would fit into "λ-geometry" as long as:

1) the Frobenius lifts are defined on the schemes X themselves (rather than on the various p-adic completions $X^{\widehat{p}}$) and

2) the Frobenius lifts commute.

However conditions 1 and 2 are almost never satisfied in our theory: the failure of condition 2 is precisely the origin of our curvature, while finding substitutes for condition 1 requires taking various convoluted paths (such as analytic continuation between primes or algebraization by correspondences). So in practice our approach places us, most of the times, outside the paradigm of λ-geometry.

3.3. **Ihara's differential.** Next we would like to point out what we think is an important difference between our viewpoint here and the viewpoint proposed by Ihara in [61]. Our approach, in its simplest form, proposes to see the operator

$$\delta = \delta_p : \mathbb{Z} \to \mathbb{Z},\ a \mapsto \delta a = \frac{a - a^p}{p},$$

where p is a fixed prime, as an analogue of a derivation with respect to p. In [61] Ihara proposed to see the map

$$d : \mathbb{Z} \to \prod_p \mathbb{F}_p,\ a \mapsto \left(\frac{a - a^p}{p}\ mod\ p \right) \tag{81}$$

as an analogue of differentiation for integers and he proposed a series of conjectures concerning the "zeroes" of the differential of an integer. These conjectures are still completely open; they are in the spirit of the approach A listed above, in the sense that counting zeroes of 1-forms is a Riemann surface concept. But what we see as the main difference between Ihara's viewpoint and ours is that *we do not consider the reduction mod p of the Fermat quotients but the Fermat quotients themselves*. This allows the possibility of considering compositions between our δ_p's which leads to the possibility of considering arithmetic analogues of differential equations, curvature, etc.

3.4. **Fontaine-Colmez calculus.** The Fontaine-Colmez theory of p-adic periods [52] also speaks of a *differential calculus with numbers*. Their calculus is perpendicular to ours in the following precise sense. For a fixed prime p our calculus based on the Fermat quotient operator δ_p should be viewed as a a differential calculus in the "unramified direction, "i.e. the "direction" given by the extension

$$\mathbb{Q} \subset \bigcup_{p \not| N} \mathbb{Q}(\zeta_N),$$

whereas the Fontaine-Colmez calculus should be viewed as a differential calculus in the "totally ramified direction," i.e., roughly, in the "direction" given by the extension

$$\mathbb{Q} \subset \bigcup_n \mathbb{Q}(\zeta_{p^n}).$$

The Fontaine-Colmez theory is based on the *usual* Kähler differentials of totally ramified extensions and hence, unlike ours, it is about *usual* derivations. It is not unlikely, however, that, for a fixed p, a theory unifying the unramified and the totally ramified cases, involving two "perpendicular" directions, could be developed leading to arithmetic partial differential equations in two variables. Signs of possibility of such a theory can be found in our papers [25, 26].

3.5. **Grothendieck's p-curvature.** It is worth pointing out that the study of our *curvature* here resembles the study of the *p-curvature* appearing in the arithmetic theory of differential equations that has been developed around the Grothendieck conjecture (cf., e.g., [65]). Both curvatures measure the lack of commutation of certain operators and both of theories rely on technical matrix computations. However we should also point out that the nature of the above mentioned operators in the two theories is quite different. Indeed our curvature here involves the "p-differentiation" of numbers with respect to primes p (in other words it is about d/dp) whereas the theory in [65] (and related papers) is about usual differentiation d/dt with respect to a variable t of power series in t with arithmetically interesting coefficients. In spite of these differences the two types of curvatures could interact; a model for such an interaction between d/dp and d/dt is in the papers [25, 26]. A similar remark can be made about the difference between our approach and that in [13] which, again, is about usual Kähler differentials and hence about usual derivations.

3.6. **Discrete geometry as Euclidean geometry.** Finally we would like to point out that the theory presented here is a priori unrelated to topics such as *the geometry of numbers* [44] on the one hand and *discrete differential geometry* [7] on the other. Indeed in both these geometries what is being studied are discrete configurations of points in the Euclidean space $\mathbb{R}^m$; in the geometry of numbers the configurations of points typically represent rings of algebraic numbers while in discrete differential geometry the configurations of points approximate smooth submanifolds of the Euclidean space. This framework is, therefore, that of the classical geometry of Euclidean space, based on $\mathbb{R}$-coordinates, and *not* that of an analogue of this geometry, based on "prime coordinates." It may very well happen, however, that (one or both of) the above topics are a natural home for some (yet to be discovered) Archimedian counterpart of our (finite) adelic theory.

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