Hilbert, Turing, Matijasevich

(Data) A seq $\mathcal{T} = \{T_1, T_2, T_3, \dots\}$, $T_n : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ (Assume some "closure axioms" (A) (**))

(D) T_n called n-th (Turing machine); $m \in \mathbb{N}$ input $T_n(m)$ called output if $\in \mathbb{N}$

If $T_n(m) = \infty$ say "computation doesn't stop for input m & machine T_n".

(D) A sequence $\{a_1, a_2, \dots\}$, $a_i \in \mathbb{N}$ is called <u>computable</u> if $\exists n$ s.t $T_n(i) = a_i$

(D) A set $S \subset \mathbb{N}$ is <u>listable</u>* if $\exists n$, $\operatorname{Im} T_n = S$

(D) A set $S \subset \mathbb{N}$ is <u>decidable</u> if $\exists n$ s.t. $\operatorname{Im} T_n = \{0,1\}$ & $\forall x \in \mathbb{N}$, $T_n(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$

(T) S decidable $\iff$ S & $\mathbb{N} \setminus S$ listable

Pf "$\Leftarrow$" Let T_p & T_q give S & $\mathbb{N} \setminus S$, Mk them "run alternatively" i.e consider $T_p(1), T_q(1), T_p(2), T_q(2), \dots$. If x shows up in even place define $T(x) = 1$ if in odd pl $T(x) = 0$

By (A) this T must be a T_r.

"$\Rightarrow$" Say T_n "decides" S. Consider $T_n(1), T_n(2), \dots$; if $T_n(i) = 1$ put i in S if $= 0$ put it in $\mathbb{N} \setminus S$.

Take $\mathbb{N}^2 \simeq \mathbb{N}$ given by $\{(1,1), (1,2), (2,1), (3,1), (2,2), (1,3), \dots\}$ & view $T_n : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \cup \{\infty\}$

(T) (Turing) ~~[illegible]~~

The set $S \subset \mathbb{N} \times \mathbb{N}$, $S = \{(m,n) \mid T_n(m) \neq \infty\}$ is listable (enumerable) & non-decidable

Pf. Let $T_1, T_2, T_3, \dots$ run w/ inputs $1, 2, 3,$ (one click @ a time) as follows ~~[illegible]~~

$s_{111}, s_{211}, s_{121}, s_{112}, s_{212}, s_{122}, s_{311}, s_{221}, s_{131}, \dots$

As soon as a comp stops list (m, n).

So S enumerable. Assume decidable.

Then $\exists$ machine that can compute table

$$\begin{array}{c|c} & m \\ \hline n & T_n(m) \end{array}$$

Now replace all ∞ by 0 (while doing this) & add 1 on diag. This table is computable so the diagonal is computable so the diag must be one of the rows of (which is not so).

(**) $\forall n$ $\exists \tilde{T}_n : \mathbb{N} \to \{(s_1, s_2, \dots) \mid s_i \in \mathbb{N} \cup \{STOP\}$ s.t if $s_j = STOP \Rightarrow s_{j+1} = STOP\}$, $T_n(m) = s_j$,

(time 1, time 2 over s_1, s_2; s_i: internal states.)

$\tilde{T}_n(m) = (s_1, \dots, s_j, STOP, \dots)$

$\tilde{T}_n(m) = (s_{nm1}, s_{nm2}, \dots)$ ($\neq$ STOP)

(*) or <u>enumerable</u>

(D) $S \subset \mathbb{N}^n$ called Diophantine if $\exists$ polns $F(x,y)$ in $m+n$ vars s.t $a \in S \iff F(x,a)=0$ has sol in $\mathbb{Z}^m$.

(P) S Diophantine $\Rightarrow$ S listable [Take (x,a) one by one $\mathbb{Z}$ & list a if $F(x,a)=0$]

(T) (Matijasevich) S listable $\Rightarrow$ S Diophantine.

(C) $\exists$ Dioph $S \subset \mathbb{N}$ which is not decidable.