

Hilbert, Turing, Matijasevich

"closure axioms" (1)

- (Data) A seq $\mathcal{T} = \{T_1, T_2, T_3, \dots\}$, $T_n: \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ (Assume some ∞ output if ∞)
- (D) T_n called n -th (Turing machine); $m \in \mathbb{N}$ ~~input~~ ^{input} $T_n(m)$ called ∞ if $T_n(m) = \infty$ say "computation doesn't stop for input m & machine T_n ".
- (D) A sequence $\{a_1, a_2, \dots\}$, $a_i \in \mathbb{N}$ is called computable if $\exists n$ s.t. $T_n(i) = a_i$
- (D) A set $S \subset \mathbb{N}$ is listable if $\exists n$, $\text{Im } T_n = S$
- (D) A set $S \subset \mathbb{N}$ is decidable if $\exists n$ s.t. $\text{Im } T_n = \{0, 1\}$ & $\forall x \in \mathbb{N}, T_n(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$
- (T) S decidable $\Leftrightarrow S$ & $\mathbb{N} \setminus S$ listable

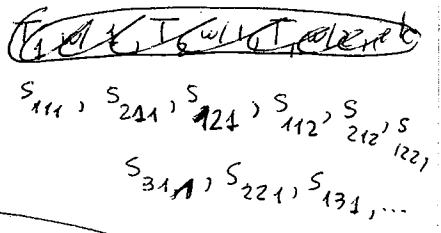
Pf " \Leftarrow " Let T_p & T_q give S & $\mathbb{N} \setminus S$, Mix them "run alternatively" i.e. consider $T_p(1), T_q(1), T_p(2), T_q(2), \dots$. If x shows up in even place define $T(x) = 1$ if i odd, 0 if $T(x) = 0$.
By (A) this T must be a T_r .

" \Rightarrow " Say T_n "decides" S , consider $T_n(1), T_n(2), \dots$; if $T_n(i) = 1$ put i in S if $= 0$ put it in $\mathbb{N} \setminus S$.

Table $\mathbb{N}^2 \approx \mathbb{N}$ given by $\{(1,1), (1,2), (2,1), (3,1), (2,2), (1,3), \dots\}$ & view $T_n: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$

(T) (Turing) ~~...~~
The set $S \subset \mathbb{N} \times \mathbb{N}$, $S = \{(m,n) \mid T_n(m) \neq \infty\}$ is (listable (enumerable) & non-decidable)

Pf. Let T_1, T_2, T_3, \dots run w/ inputs $1, 2, 3, \dots$ as follows ~~(one clock at a time)~~



As soon as a comp stops list (m,n) .
So S enumerable. Assume decidable.
Then \exists machine that can compute table

	m
n	$T_n(m)$

Now replace all ∞ by 0 & add 1 on diag. This table is computable (while doing this)
so the diagonal is computable so the diag must be one of the rows of (which is not ∞).

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$\forall m \exists \tilde{T}_m: \mathbb{N} \rightarrow \{(s_{11}, s_{21}, \dots) \mid s_i \in \mathbb{N} \cup \{\text{STOP}\} \text{ s.t. if } s_j = \text{STOP} \Rightarrow s_{j+1} = \text{STOP}\}$, $T_n(m) = s_j$

time 1
time 2
↓
↓

internal states.

$\tilde{T}_m(m) = (s_{11}, \dots, s_j, \text{STOP}, \dots)$

$\tilde{T}_m(m) = (s_{m1}, s_{m2}, \dots)$

or enumerable

- (D) $S \subset \mathbb{N}^m$ called Diophantine if \exists polns $\checkmark F(x,y)$ in $m+n$ vars s.t. $a \in S \Leftrightarrow F(x,a) = 0$
has sol in \mathbb{Z}^m .
- (P) S Diophantine $\Rightarrow S$ listable [Take (x,a) one by one \checkmark & list a if $F(x,a) = 0$]
- (T) (Matijasevich) S listable $\Rightarrow S$ Diophantine.
- (C) \exists Dioph $S \subset \mathbb{N}$ which is not decidable.