RED FOR UGRADUATES

INTRODUCTION TO A COURSE ON ZETA'S

(0) (Euler's reform of FTA) $\sum_{n\geq 1} n^{-s} = \prod_{p \text{ pr.}} (1-p^{-s})^{-1}$ 319 , $s \in \mathbb{R}, s>1$
(Euclid)

(1) Th $\exists \infty$ many primes 319

162 Pf (Euler) Assume fin many pr's. Then $\underbrace{\infty > \prod (1-p^{-1})^{-1}}_{\text{mk } s\to 1. \text{ Get}} = \sum_{n\geq 1} n^{-1} = \infty$. $\lim_{s\to 1} \sum n^{-s}$ abs
~~(fixable, easily)~~

(2) Th (Euler) $\sum_{n\geq 1} n^k = (1-2^{k+1})^{-1}\left[\left(T\frac{d}{dT}\right)^k\left(\frac{T}{T+1}\right)\right]_{T=1} \in \mathbb{Q}$ for $k \geq 1$
(OUTRAGEOUS, BUT SEE QUANTUM PHYSICS! RENORMALIZATION).

Pf (fixable, not easily; otherwise - nonsense!) — give "meaning" (assign values) to divergent series

CRAZY 162. $\ni \Big\{ (1-2^{k+1})\sum_{n\geq 1} n^k = \sum_{n\geq 1} n^k - 2\sum_{n\geq 1}(2n)^k = \sum_{n \text{ odd}} n^k - \sum_{n \text{ even}} n^k = -\sum_{n\geq 1}(-1)^n n^k =$

WOULD GET EULER AN F IN 162 $\quad = -\sum_{n\geq 1}(-1)^n n^k T^n \Big|_{T=1} = \left(T\frac{d}{dT}\right)^k \left[\sum_{n\geq 1}(-1)^{n+1}T^n\right]_{T=1} = \left(T\frac{d}{dT}\right)^k\left(\frac{T}{T+1}\right)$

(3) Th (Euler) $\sum_{n\geq 1} n^k = 0$ for $k \geq 2$, k even (called triv. zeroes) $\quad T/1+T = 1/1+U$

Pf (outrageous) Do $T \mapsto 1/T$ in prev th. $U = T^{-1} \Rightarrow dU = -T^{-2}dT \Rightarrow U\frac{d}{dU} = -T\frac{d}{dT}$

k even $\Rightarrow \left(U\frac{d}{dU}\right)^k = \left(T\frac{d}{dT}\right)^k \Rightarrow \sum_{n\geq 1} n^k = (1-2^{k+1})^{-1}\left[\left(U\frac{d}{dU}\right)^k\left(\frac{1}{1+U}\right)\right]_{U=1}$ as above

$= (1-2^{k+1})^{-1}\left[\left(U\frac{d}{dU}\right)^k\left(-\frac{U}{1+U}\right)\right]_{U=1} = -\sum_{n\geq 1} n^k$

b/c $\frac{U}{1+U} = 1 - \frac{1}{1+U}$

(4) Th (Euler) $r \geq 1 \Rightarrow \sum_{n\geq 1} n^{2r-1} \Big/ \sum_{n\geq 1} n^{-2r} = (-1)^r\, 2^{1-2r}\, \pi^{-2r}\,(2r-1)!$; so $\sum_{n\geq 1} n^{-2r} \in \mathbb{Q}\cdot\pi^{2r}$

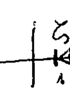
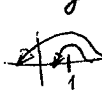
Pf (maybe later)

Moral so far : SOME ACTION TAKES PLACE AFTER "RENORMALIZATION" — a sort of renormalization

(Riemann) 313 561

(5) Th $\zeta(s) = \sum_{n\geq 1} n^{-s}$, $\mathrm{Re}\, s > 1$, can be ~~mero~~ analytically continued to $\mathbb{C}^\times$ w/ $s=1$ as only pole (define an cont along a path, a ζ on)

Moreover Euler's Thms true for $\sum n^k$ replaced by $\zeta(-k)$ if $k \geq 0$.
& ! replaced by Γ. (More precisely, if $\xi(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$

 ζ explodes to ∞  ζ doesn't explode to ∞ then $\xi(s) = \xi(1-s)$)

(6) Th (Riemann-Mangoldt) (Main motiv. for Riemann ; not for us) Let $\pi(x) = \#\{p \mid p \leq$
$F(x) = \sum_{n\geq 1} \frac{1}{n}\pi(x^{1/n})$. Then $\pi(x) = \sum_{n\geq 1} \frac{\mu(n)}{n} F(x^{1/n})$ &
$\lim_{\varepsilon\to 0} \frac{F(x+\varepsilon)+F(x-\varepsilon)}{2} = \sum li(x^\rho) + \int_0^\infty \frac{du}{(u^2-1)u\log u} - \log 2$, $li(e^w) = \int_{-\infty}^{w} \frac{e^z}{z}dz$

(congr $x \equiv y \mod I$ if $x-y \in I$; $A/M = A/\equiv$. Ex $\mathbb{Z}/5\mathbb{Z} = \dots$)

7 (Riemann Hyp.) $\zeta(s) = 0 \Rightarrow \operatorname{Re} s = \frac{1}{2}$. ($\Rightarrow$ conseq. for π)
$s \notin \{-2,-4,\dots\}$

Riem was trying to pr Gauss' conj (PNT) $\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$ [pved by Hadamard – de la Vallée P.]

8 Th (Dirichlet) Let $m \in \mathbb{Z}$, $a \in \mathbb{Z}$, $(m,a)=1$, $Y = \{q \mid q \text{ prime}\}$, $Y_a = \{q \in Y \mid q \equiv a \mod m\}$
(generalizing 1)

Then
$$\lim_{\substack{s\in\mathbb{R}\\ s\downarrow 1}} \sum_{q\in Y_a} q^{-s} \Big/ \sum_{q\in Y} q^{-s} = \frac{1}{\phi(m)}, \text{ So } \# Y_a = \infty.$$

322 Recall: ring = a set where $+, \cdot$, 0,1 defined w/ "usual" rules. Ideal. Max id. Ex $\mathbb{Z}$, $4\mathbb{Z}$, $5\mathbb{Z}$ (or $A = \mathbb{Z}[x]/(f)$)

9 Def (Dedekind) Let $A = \underbrace{\mathbb{Z}[\alpha]}_{\text{smallest ring, e.g.}\dots}$, ~~$\alpha_1,\dots,\alpha_n$~~, $\alpha \in \mathbb{C}$ integral $/\mathbb{Z}$ (recall). (root of pol w/ $\mathbb{Z}$-coeff eq $\sqrt{\ }$ 1)

$$\zeta_A(s) := \prod_{P \text{ max in } A} (1 - N(P)^{-s})^{-1}$$ where $N(P) = \#(A/P)$. Note $\zeta = \zeta_{\mathbb{Z}}$

Ded. extended Riemann's theory for ζ_A.

10 (Class field th. dream). Given # fields $K \overset{\text{Gal abelian}}{\subset} L$ ($L \subset \mathbb{C}$) (i.e. $[L:\mathbb{Q}] < \infty$)
describe "explicitely" $G(L|K)$ in terms of "arithmetic" of K (i.e of prime id's of $\mathcal{O}_K := \{\alpha \in K \mid \alpha \text{ int}/\mathbb{Z}\}$. (Generalis. of Gauss. rec. law as we shall see).

Artin realized this dream using 9

11 Def (conj) (Artin) Let $A = \overset{\text{ring}}{F[x,y]}/\overset{\text{ideal of mult of } f}{(f)}$, F fin "field" (e.g $\mathbb{Z}/p\mathbb{Z}$), x, y var's, f pol.
Define ζ_A as in 9. Artin conjectured an analogue for ζ_A of fcnal equ 6 & Riem Hyp 7.

Lef fixed pt th (3??)

12 Th (Weil) Artin's conj. true. [Pf: "Alg. topology" on $Z(f) \subset \overline{F}^2$]

13 Conj (Weil) Let $A = F[x_1,\dots,x_n]/(f_1,\dots,f_m)$ & def ζ_A as in 9
Then a fcnal equ 6 & Riem Hyp. 7 hold.

14 Th (Deligne – Fields medal) 13 OK. [Pf. "Alg top" on $Z(f_1,\dots,f_m) \subset \overline{F}$

15 (Denniger's program) Prove Riem Hyp. 7 doing "alg top" on an analogue of $Z(f)$. This analogue is missing. More gen:
16 do it for ζ_A where $A = \mathbb{Z}[x_1,\dots,x_n]/(f_1,\dots,f_m)$.

17 (back to Kubota, Leopold, Serre, Katz) p-adic interp of ζ's.

Hasse – Weil – Serre – Deligne – Wiles

Remarks on (12): computing ζ_A for $A = F[x_1,\dots,x_n]/(f_1,\dots,f_m)$.

Set $p = \mathrm{char}\, F$, $q = \# F$, $n_k = \#\{P \in \mathrm{Max}(A) \mid N(P) = q^k\} = \#\{P \mid \deg(P) = k\}$ (where $\deg(P) := [A/P : F]$)

(30) $X = \{\alpha \in \overline{\mathbb{F}}_p^{\,n} \mid f_1(\alpha) = \cdots = f_m(\alpha) = 0\}$, $X(\mathbb{F}_{q^\ell}) = X \cap \mathbb{F}_{q^\ell}^{\,n}$

$N_\ell = \# X(\mathbb{F}_{q^\ell})$. Note $N_\ell = \#$ fixed pts of ϕ^ℓ in X. ($\phi(\alpha) = \alpha^q$)

(L) n_k, ν_ℓ are finite & $N_\ell = \sum_{k \mid \ell} k n_k$ [Pf. Any $\alpha \in X(\mathbb{F}_{q^\ell})$ gives an F-alg homo $A \xrightarrow{f} \mathbb{F}_{q^\ell}$ so a pair (P, i), $P = \ker f \in \mathrm{Max}(A)$, $\underbrace{i : A/P \to \mathbb{F}_{q^\ell}}_{\text{(\& viceversa)}}$. So $k \mid \ell$

$A/P \cong \mathbb{F}_{q^k}$, j^*

For each $k \mid \ell$ $\exists$ exactly k embeddings $\mathbb{F}_{q^k} \to \mathbb{F}_{q^\ell}$ (given by $id, \phi, \phi^2, \dots, \phi^k$, $\phi(x) = x^q$).]

(Def) $Z(t) = \prod_{P \in \mathrm{Max}(A)} (1 - t^{\deg(P)})^{-1}$ (so $\zeta_A(s) = Z(q^{-s})$)

where $\deg(P) = [\frac{A}{P} : F]$, so $N(P) = q^{\deg(P)}$

(Prop) $Z(t) = \exp\left[\sum_{\ell=1}^{\infty} N_\ell \frac{t^\ell}{\ell}\right]$. ← link b/t ζ & fixed pts of powers of a map likely to be handled by ALG TOP

Pf. $\log Z(t) = \log \prod_P (1 - t^{\deg(P)})^{-1} = \log \prod_{k=1}^{\infty} (1 - t^k)^{-n_k}$

$= -\sum_{k \geq 1} n_k \log(1 - t^k) = \sum_{k \geq 1} \sum_{m \geq 1} n_k \frac{t^{km}}{m}$

$\underset{\ell = km}{=} \sum_{\ell \geq 1} \Big(\sum_{\substack{k \geq 1 \\ k \mid \ell}} n_k \frac{k}{\ell} \Big) t^\ell$

$= \sum_{\ell \geq 1} \frac{N_\ell}{\ell} t^\ell$ $\big(= \log \det(I - tA)\big)$

~~take $t\frac{d}{dt}$ Get $t\frac{Z'(t)}{Z(t)} = \sum \nu$~~

if, by any chance $\nu_\ell = \mathrm{Tr}(A^\ell)$ for some $A \in M_n(\mathbb{C})$

~~This amounts to $\zeta(s)$ having zeros w/ $\mathrm{Re}\, s = i/2$, $0 \leq i \leq 2d$~~

(Rem) RH for this case (Weil conj proved by Dwork - Groth - Del) is

$Z(t)$ rational $= \prod_{i=0}^{2d} P_i(t)^{(-1)^{i+1}}$ (analogue of analytic cont) are alg int.

all recipr roots of the P_i's have abs values $q^{i/2}$.

 or any field of char 0, K.

(L) (520) Let $A \in M_n(\mathbb{C})$ be a matrix w/ eig. values $\lambda_1, \dots, \lambda_n$ (not nec. distinct)

Then

32λ →
$$\exp\left(\sum_{\ell=1}^{\infty} \mathrm{Tr}(A^\ell)\frac{t^\ell}{\ell}\right) \underset{①}{=} \det(I - tA)^{-1} \underset{②}{=} \prod_{i=1}^{n}(1-\lambda_i t)^{-1}$$

Pf. May assume A in Jordan form. Now ② clear by def. Now

$$\mathrm{LHS} = \exp\left(\sum_{\ell=1}^{\infty}\sum_{i=1}^{n} \lambda_i^\ell t^\ell/\ell\right) = \prod_{i=1}^{n} \exp\left(\sum_{\ell=1}^{\infty} \frac{(\lambda_i t)^\ell}{\ell}\right) = \prod_{i=1}^{n} \exp(-\log(1-\lambda_i t)) \quad \square$$

(Th) (Lefschetz fixed pt formula) M compact manif.; $\phi: M \to M$ continuous endo; $N_1 := $ "$\#\{x \in M \mid \phi(x) = x\}$" (where " " means w/ certain multiplicities)

Then $N = \sum_{i=0}^{\dim M} (-1)^i \,\mathrm{Tr}(\nu_* : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q}))$.

(Weil's Conj)(*) $\exists$ K-vect. spaces $H^i(X)$ $0 \le i \le 2d$, $d = $ "$\dim X$" and endomorphisms A_i of these s.t

1) "Lef tr for" $N_\ell = \sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}(A_i^\ell)$ & 2) all eigenvalues λ_{ij} of A_i have abs val $q^{i/2}$

3) fcnal equ relating $\zeta(s)$ & $\zeta(d-s)$

(Rem) (Weil) His conj implies that

$$Z(t) = \exp\left[\sum_{\ell=1}^{\infty}\sum_{i=0}^{2d}(-1)^i \mathrm{Tr}(A_i^\ell)\frac{t^\ell}{\ell}\right] = \prod_{i=0}^{2d} \exp\left[\mathrm{Tr}(A_i^\ell)\frac{t^\ell}{\ell}\right]^{(-1)^i}$$

$$= \prod_{i=0}^{2d} \det(I - tA_i)^{(-1)^{i+1}} = \prod_{i=0}^{2d}\prod_{j=1}^{d_i}(1-\lambda_{i,j}t)^{(-1)^{i+1}}, \quad d_i = \dim H^i$$

So zeroes & poles of $\zeta(s) = Z(q^{-s})$ are given by $\lambda_{ij} q^{-s} = 1 \Rightarrow$

$$= |\lambda_{ij}|\, q^{-\mathrm{Re}\, s} = 1 \Rightarrow q^{i/2} q^{-\mathrm{Re}\, s} = 1 \Rightarrow \boxed{\mathrm{Re}\, s = i/2}$$

(Note) RH reduced to a spectral property of a hypothetical op. A_i Riem Hyp.

(Note) Fcnal equ: expression of a "Poincaré duality".

(Denninger's) program: for $A = \mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_m)$ (e.g $A = \mathbb{Z}$) find "H^i" & A_i. (probably ∞ dimensional) + Lef fixed pt formula + Poinc. duality.

(*) proved by Groth- Deligne. $H^i = H^i_{\acute{e}t}$

Aspects of $\zeta(s)$ & its relatives

- behaviour at $s=1$
  - density theorems → class field theory
  - class # formulas
- functional eqn
  - mero continuation
  - link w/ modular forms, alg. geo & Langlands prog.
- Special values at integers
  - (@ even + #'s) Bernoulli #'s → p-adic interp. (Iwasawa
  - (@ odd + #'s) K-theory, Beilinson conj's
- zeroes
  - PNT
  - RH
  - Weil conjectures