Math 321
Linear Algebra

March 3, 2006 Quiz 5 Spring 2006

Name: Key – Yellow

1. Show that the map $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $F(x, y, z) = (2x - y, x + y - 2z)$ is linear.

Show $\qquad v + v' = (x + x', y + y', z + z')$

i) $F(v + v') = F(v) + F(v')$

$F(v+v') = F(x+x', y+y', z+z') = (2(x+x') - (y+y'), x + x' + y + y' - 2(z+z'))$

$= (2x - y, x + y - 2z) + (2x' - y', x' + y' - 2z') = F(v) + F(v')$

ii) $F(cv) = F(cx, cy, cz) = (2cx - cy, cx + cy - 2cz) = c(2x - y, x + y - 2z)$

$= cF(v).$

2. Prove that the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

has rank 3.

The rank will $= 3$ if the 3 columns are linearly independent. So

$$a\begin{pmatrix}1\\0\\2\\0\end{pmatrix} + b\begin{pmatrix}2\\1\\0\\-1\end{pmatrix} + c\begin{pmatrix}1\\-1\\1\\2\end{pmatrix} = 0 \Rightarrow \begin{aligned} a + 2b + c &= 0 \\ b - c &= 0 \\ 2a + c &= 0 \\ -b + 2c &= 0 \end{aligned}$$

$\left.\begin{aligned} b - c &= 0 \\ 2a + c &= 0 \\ -b + 2c &= 0 \end{aligned}\right] \Rightarrow c = 0$

$\Rightarrow b = c = 0$ and $2a = -c = 0 \Rightarrow a = b = c = 0$

$\Rightarrow$ rank $= 3.$

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1. 2. Prove that the matrix

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

has rank 3. There are many ways to do this. Here is one: We know rank ≤ 3 since there are only 3 columns. The rank will be 3, if these columns are linearly independent, so

$$a\begin{pmatrix}2\\0\\1\\0\end{pmatrix} + b\begin{pmatrix}2\\1\\-1\\-1\end{pmatrix} + c\begin{pmatrix}1\\-1\\0\\2\end{pmatrix} = 0 \Rightarrow \begin{aligned} 2a+2b+c &= 0 \\ b - c &= 0 \\ a - b &= 0 \\ -b+2c &= 0 \end{aligned} \Bigg] \Rightarrow c = 0$$

So $b = c = 0$ and $a = b = 0 \Rightarrow a = b = c = 0$

$\Rightarrow$ rank $= 3$.

Show that the map $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $F(x, y, z) = (x - y + z, 2x - z)$ is linear.

i) Show $F(v+v') = F(v) + F(v')$ $\quad v+v' = (x+x', y+y', z+z')$

Then

$F(v+v') = F(x+x', y+y', z+z') = (x+x'-(y+y') + z+z', 2(x+x') - (z+z'))$

$= (x - y + z, 2x - z) + (x' - y' + z', 2x' - z') = F(v) + F(v')$

(ii) $F(cv) = F(cx, cy, cz) = (cx - cy + cz, 2cx - cz) = c(x - y + z, 2x - z)$

$= cF(v)$.

$\Rightarrow F$ is linear.