## 2. Topological Manifolds.

I will first give a definition of a manifold which is common in the literature and then give a more easily understood definition. Actually the two definitions are completely equivalent, although this will not be proved here.

**Definition 2.1**: A manifold M of dimension n is a top space which satisfies the following:

i) M is Hausdorff

ii) M has a countable basis of open sets.

iii) Every point $p \in M$ has a neighborhood U which is homeomorphic to a (or open ball) spherical neighborhood of $R^n$, i.e. M is *locally Euclidean*.

Actually there are more general definitions of a manifold. OF particular interest is the generalization to the infinite dimensional situation (See Lang). We can also drop the Hausdorff requirement i) in the definition and talk about non Hausdorff manifolds. These will appear briefly later on. Unless explicitly stated as not necessarily Hausdorff, a manifold will be assumed to be Hausdorff.

It follows from iii) and the facts that $\mathbb{R}^n$ is locally compact and local compactness is invariant under homeomorphisms, that M is locally compact. But it is known (Hocking and Young) that every locally compact Hausdorff space with a countable basis is metrizable. Thus we can replace definition 2.1 by:

Definition 2.2: An n-dimensional manifold is a separable metric space which is locally Euclidean.

Furthermore, every locally compact Hausdorff space with a countable basis is normal: hence regular (Hocking and Young). Thus manifolds are both normal and regular.

There is actually a problem with these definitions concerning the uniqueness of the dimension n. It is in fact unique, but we must use the fact that $R^n$ is homeomorphic to $R^m$ if and only if $n = m$. This innocent looking fact is very difficult to prove. It follows from a deep theorem, whose proof uses algebraic topology, known as the theorem of the invariance of the domain. (cf. Hocking and Young).

One finds several variations of iii) of definition 2.1 in the literature which are all equivalent. We have

Proposition 2.1: The following are equivalent to iii) of definition 2.1:

1) Every point $p \in M$ has a neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

2) Every point $p \in M$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$.

Proof: That iii) implies 1) is trivial. That 1) implies iii) is easy since there is always a spherical neighborhood contained within any open subset of $\mathbb{R}^n$. That iii) and 2) are equivalent follows by checking that $y:\mathbb{R}^n \longrightarrow S(0,r)$ defined by

$$y^i = \frac{rx^i}{[1+|x|^2]^{1/2}} \quad , \quad |x|^2 = \sum_{i=1}^{n} (x^i)^2 \text{ , is a homeomorphism onto .}$$

Thus we will use iii) of definition 2.1 interchangebly with 1) and 2) above. In fact 1) above is the most convenient.

Another easy consequence of the definition is

Proposition 2.2: An open subset U of a manifold is a manifold.

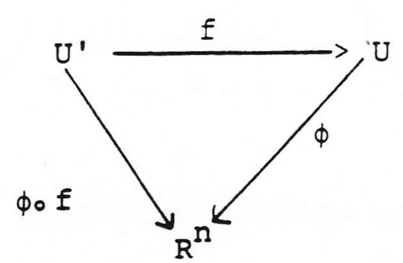
Proof: For $p \varepsilon U \subset M$ there is a V containing p homeomorphic to an open subset of $\mathbb{R}^n$ by $\phi$. Then $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and contains a spherical neighborhood whose inverse image under $\phi$ is open contained in U and contains p.

Such open subsets are called open submanifolds.

Theorem 2.1:

If M' is homeomorphic to a manifold M, then M' is a manifold.

Proof: Let $U \subset M$, $U' \subset M'$ and consider the commutative diagram



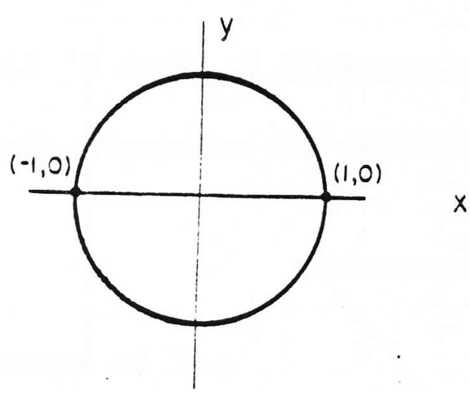

$\phi \circ f$ is homeomorphism

This shows that M' is locally Euclidean. We need to show that M' is Hausdorff and has a countable basis. Let $f: M \longrightarrow M'$ be a homeomorphism onto and let $x, y \varepsilon M$ with $x \neq y$, then there exists neigborhoods $U_x, U_y$, of x,y, respectively with $U_x \cap U_y = \{\phi\}$.

Now $f(U_x)$ and $f(U_y)$ are open and contain $f(x)$ and $f(y)$, respectively. Moreover $f(U_x) \cap f(U_y) = \{\phi\}$ , since if it were not, it's members would necessarily be the image of members of $U_x \cap U_y$ .But this is empty. We leave as an exercise to show that M' also has a countable basis.

Let's now look at some nontrivial examples of manifolds:

<u>Example 2.1</u> <u>circle</u>, $S^1$: The map $f:(0,2\pi) \longrightarrow S^1$ defined by $f(\theta) = (\cos\theta , \sin\theta)$ is a homeomorphism of the open interval $(0,2\pi)$ onto $S^1-(1,0)$. Also the map $g:(-\pi,\pi) \longrightarrow S^1$ is a homeomorphism of the open interval $(-\pi,\pi)$ onto $S^1-(-1,0)$, where $g(\theta) = (\cos\theta , \sin\theta)$. Thus all points of $S^1$ have neighborhoods which are homeomorphic to $\mathbb{R}^1$. $S^1$ is then a manifold, since it inherits a natural metric from $\mathbb{R}^2$. The figure below illustrates the missing points.

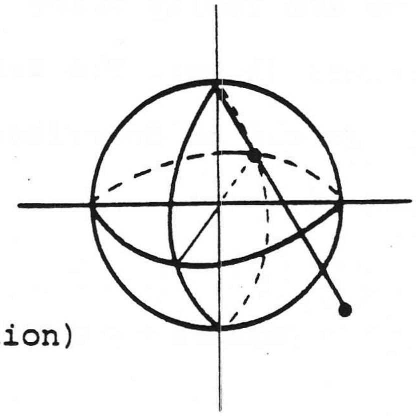



Notice that intervals on the circle $-\varepsilon+\theta_0<\theta<\theta_0+\varepsilon$ are open in the metric topology induced from $R^2$ since the ball $S(p_0,r)$ where $p_0 = (\cos\theta_0 , \sin\theta_0)$, $r = \sin\varepsilon$ is an open set of $R^2$. Such intervals form a basis for the metric topology on $S^1$.

There is another set of homeomorphisms which generalizes easily to spheres, given by the stereographic projections.

<u>Example 2.2</u>: <u>Spheres</u> , $S^n$: We know that

$$S^n = \{\underset{\sim}{x} \varepsilon R^{n+1}: x_1^2 + \ldots + x_{n+1}^2 = 1\}$$



We define the mappings (stereographic projection)

1) $y: S^n \longrightarrow R^n$ by $y^i(\underset{\sim}{x}) = \dfrac{x_i}{1-x_{n+1}}$

2) $y': S^n \longrightarrow R^n$ by $y^i(\underset{\sim}{x}) = \dfrac{x_i}{1+x_{n+1}}$

Notice that the domain of 1) is $S^n - (0,\ldots,1)$ whereas that of 2) is $S^n - (0,\ldots,-1)$. It is not difficult to show that these are indeed homeomorphisms.

Exercise: Show it!

Thus $S^n$ is locally homeomorphic to $R^n$. In a way exactly analogous to the circle, $S^n$ with its relative topology from $R^{n+1}$ becomes a separable metric space. Thus $S^n$ is a manifold.

<u>Example 2.3</u>: Torus: $T^2 = S^1 \times S^1$ or more generally $T^n = S^1 \times \ldots \times S^1$. Clearly from the definition of manifold if $M_1$ and $M_2$ are manifolds then $M_1 \times M_2$ is a manifold.

<u>Example 2.4</u>: The Möbius strip: Consider the half open rectangle $S = \{(x,y) \varepsilon \mathbb{R}^2: 0 \leq x \leq 1, -1<y<1\}$. Introduce an equivalence relations on $\mathbb{R}$ by saying that $(x,y)$ and $(x',y')$ are equivalent

if $(x,y) = (x',y')$ or $(x,y) = (x'+1,-y')$.

What we are really doing is identifying the points $(0,y)$ with the points $(1,-y)$. The resulting space $S/\sim = M^2$ is the Möbius strip. It can be described analytically by the map $f:S \longrightarrow \mathbb{R}^3$ defined by

$$f(x,y) = (2\cos 2\pi x + y\cos\pi x\cos 2\pi x,\ 2\sin 2\pi x + y\cos\pi x\sin 2\pi x,\ y\sin\pi x)$$

Notice that $f(0,y) = (2+y,0,0) = f(1,-y)$. The image of f in $\mathbb{R}^3$ is the Möbius strip. The map f on the interior $S^0$ of S given by $S^0 = \{(x,y)\in R^2 : 0<x<1,\ -1<y<1\}$ provides us with a homeomorphism onto $M^2-(2+y,0,0)$. A similar construction with the interval $0<x<1$ replaced by $-\frac{1}{2}<x<\frac{1}{2}$ provides for the homeomorphism onto a neighborhood of $(2+y,0,0)$.

Exercise: Go through this construction.

Almost everybody has made a Möbius strip with a strip of paper by giving it a half twist and glueing the ends together. This is not quite $M^2$ constructed above, but it would be if we would have started with $\bar{S}$ instead of S, i.e. $\bar{S} = \{(x,y)\in R^2 .\ 0\leq x\leq 1,\ -1\leq y\leq 1\}$. The result then is the Möbius strip with boundary. This is not quite a manifold as we have defined, but rather a manifold with boundary. We will give the formal definition shortly. Anyway the $M^2$ constructed above is a manifold. The metric topology being the subspace topology of $R^3$.

Another example somewhat more difficult to visualize is the projective plane. First we discuss some ideas on quotient spaces. Consider a top. space S, an equivalence relation ~ on S, and the quotient space $S/\sim$ with the quotient topology. Now in general the projection $\rho: S \longrightarrow S/\sim$ is not open, but in many cases it is and this leads to

<u>Lemma 2.1.</u>: Let $\rho: S \longrightarrow S/\sim$ be the projection map of a top space $S$ to its quotient $S/\sim$ and let $S/\sim$ have the quotient topology. Furthermore, suppose that $\rho$ is open, then if $S$ has a countable basis $\{U_\alpha\}$ so does $S/\sim$.

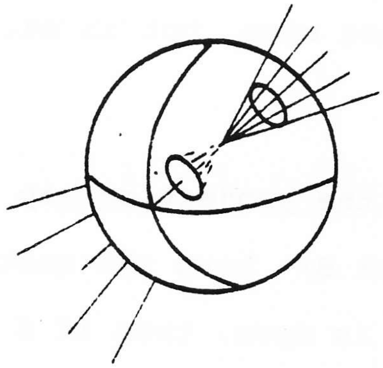
Proof: Let $W \subset S/\sim$, then $\rho^{-1}(W) = \bigcup_i U_{\alpha_i}$ for some subfamily $\{U_{\alpha_i}\}$ of $\{U_\alpha\}$ and $W = \bigcup_i \rho(U_{\alpha_i})$ where each $\rho(U_{\alpha_i})$ is open. Thus $\{\rho(U_{\alpha_i})\}$ is a countable basis for $S/\sim$. (Check this).

<u>Lemma 2.2</u>: Let $\rho: S \longrightarrow S/\sim$ be the natural projection and suppose it is open. Define $E \subset S\times S$ as $E = \{(x,y) \in S\times S : x \sim y\}$. If $E$ is a closed subspace of $S\times S$, then $S/\sim$ is Hausdorff.

Proof: Let $E$ be closed and $\rho(x)$, $\rho(y)$ be distinct points of $S/\sim$. Then $(x,y) \in S\times S - E$, which is open. There is thus an open set $U\times V \subset S\times S$ with $(U\times V) \cap E = \{\emptyset\}$. But this says that $\rho(U) \cap \rho(V) = \{\emptyset\}$, and $S/\sim$ is Hausdorff since $\rho(U)$ and $\rho(V)$ are open.

<u>Remark</u>: We need the condition that $E$ above be closed to conclude that $S/\sim$ is Hausdorff. There are known counter examples. In fact the converse of lemma 2.2 is true. (Prove it!).

<u>Example 2.5</u>: The projective plane $P^n(R) = (\mathbb{R}^{n+1} - \{0\})/\sim$ where $x \sim y$ if and only if $y = tx$ for some $t \neq 0$. The equivalence classes $[x]$ are just the lines the through the origin $\{0\}$, and we have also $P^n(\mathbb{R}) = S^n / \sim$ where $x, y \in S^n$ are equivalent if $y = \pm x$. (Show this!).



If we show that $\rho: R^{n+1}-\{0\} \longrightarrow P^n(R)$ is open, then $P^n(R)$ will have a countable basis by the first lemma since $\mathbb{R}^{n-1}-\{0\}$ does. Call $X = \mathbb{R}^{n+1}-\{0\}$ , and define $\phi_t: X \longrightarrow X$ by $\phi_t(x) = tx$ for $t \neq 0$. $\phi_t$ is a homeomorphism with $\phi_t^{-1} = \phi_{\frac{1}{t}}$. If $U \subset X$, then $\rho^{-1}([U]) = \bigcup_{t\in \mathbb{R}-\{0\}} \phi_t(U)$ is open since each $\phi_t(V)$ is open. But this says that for each open $U$, $\rho(U) = [U]$ is open. Thus $P^n(\mathbb{R})$ has a countable basis by the first Lemma.

Next let's show that for $P^n(R)$ the set E defined in the second lemma is closed. Consider the function $f: X\times X \longrightarrow R$ defined as

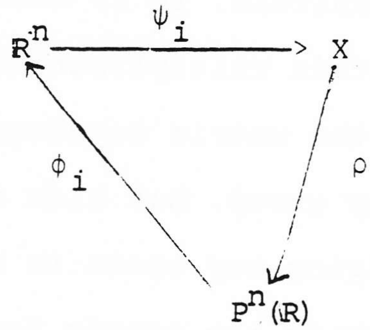
$$f(x_1,\ldots,x_{n+1};y_1,\ldots,y_{n+1}) = \sum_{i\neq j} (x_iy_j-x_jy_i)^2$$

Clearly, f is continuous. Moreover, if $y_i = tx_i$, then $f = 0$. Conversely, assume $f = 0$ then $(x_iy_j-x_jy_i) = 0$, assuming $x_i, y_j \neq 0 \Rightarrow \frac{x_i}{y_i} = \frac{x_j}{y_j} \Rightarrow y_i = tx_i$ , $t \neq 0$. That is, $f = 0$ if and only if $y\sim x$. So $E = \{(x,y): x\sim y\} = f^{-1}(0)$. But $X\times X - E = f^{-1}(\mathbb{R}-0)$ is open, since f is continuous; thus E is closed. Hence, by the second lemma $P^n(R)$ is Hausdorff.

We need to show that $P^n(R)$ is locally Euclidean. To do this we define the following open sets of $X: \tilde{U}_i = \{x \varepsilon X: x^i \neq 0\}$, that is we remove from $X = \mathbb{R}^{n+1} - \{0\}$, the n-plane $x^i = 0$. Let $U_i = \rho(\tilde{U}_i)$. These are open since we saw that $\rho: X \longrightarrow P^n(\mathbb{R})$ is open. We define the functions $\phi_i : U_i \longrightarrow \mathbb{R}^n$ by

$$\phi_i([x]) = \left( \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right)$$

where x is such that $\rho(x) = [x] \varepsilon U_i$. Clearly, $\phi_i$ is continuous. Moreover, an easy argument as above shows that $\phi_i(x) = \phi_i(y)$ if and only if $y \sim x$. Thus $\phi_i$ is a bijection, that is 1-1 onto. Now $\phi_i^{-1}: \mathbb{R}^n \longrightarrow U_i$ is given by $\phi_i^{-1}(z_1, \ldots, z_n) = \rho(z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_n)$. To see this consider




where $\psi_i(z_1, \ldots, z_n) = (z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_n)$, and we identify $(z_1, \ldots, z_{i-1}, z_i, \ldots, z_n) = \left( \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_{n+1}}{x_i} \right)$. Thus $\phi^{-1}$ is continuous, and thus a homeomorphism. Then $P^n(\mathbb{R})$ is a manifold since $\{U_\alpha\}$ cover $P^n(\mathbb{R})$.

We now return to some simpler examples, before giving a generalization of $P^n(\mathbb{R})$.

<u>Example 2.6:</u> n×m matrices, $\mathcal{M}^{nm}$: obviously $\mathcal{M}^{nm}$ is homeomorphic to $R^{nm}$. A global homeomorphism is $f: \mathcal{M}^{nm} \longrightarrow R^{nm}$ given by

$f(A) = (a_{11},\ldots,a_{1m}\ ,\ a_{21},\ldots,a_{2m}\ ,\ldots,a_{nm})$ where

$$A = \begin{bmatrix} a_{11},\ldots,a_{1m} \\ \vdots \quad\quad \vdots \\ a_{n1},\ldots,a_{nm} \end{bmatrix}$$

. Thus $M^{nm}$ can be considered as a metric space with the metric induced by that of $R^{nm}$. Thus $M^{n}$ is a manifold. The square matrices $M^{nn}$ will be written as $M^{n}$ (homeomorphic to $R^{n^2}$).

Example 2.7: The general linear group. $GL(n,\mathbb{R})$:

$GL(n,\mathbb{R}) = \left\{ A\varepsilon M^{n}: \det A \neq 0 \right\}$. Note that $\det: M^{n} \longrightarrow \mathbb{R}$ is continuous, and $\mathbb{R} - \{0\}$ is open, so $GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ is open in $M^{n}$. It is thus a manifold. It is also a group with group composition given by matrix multiplication. Clearly, this composition is continuous in the metric topology on $GL(n,\mathbb{R})$. Thus $GL(n,R)$ is not only a top group, but also a group manifold, i.e. a top. group whose underlying top. space is a manifold and whose composition is continuous in the metric toplogy.

Example 2.8: Grassman manifolds $G(k,n)$. Consider the manifold $M^{kn}$ of all $k\times n$ real matrices and recall the definition of the rank of $A\varepsilon M^{kn}$. This is the dimension of the subspace spanned by the rows of A (or equivalently the columns). Denote by $F(k,n)$ the subset of $M^{kn}$ of all matrices of rank k. $F(k,n)$ is an open submanifold of $M^{kn}$. Any linear independent set of k-elements of $\mathbb{R}^{n}$ is called a k-frame in $\mathbb{R}^{n}$. Thus a k-frame is just a $k\times n$

matrix $(X_1,\ldots,X_k)$ where $X_i = \begin{pmatrix} X_i^1 \\ \vdots \\ X_i^n \end{pmatrix}$. There is a natural action of $GL(k,\mathbb{R})$ on $F(k,n)$ given by $X_i' = \sum_{j=1}^{k} A_i^j X_j$. Since A is nonsingular $X_i'$ defines another k-frame. The set of all k-planes through the origin of $\mathbb{R}^n$ is denoted by $G(k,n)$ and is called a Grassman manifold. A k-frame thus determines a k-plane. Moreover, two k-frames $X_i'$ and $X_i$ determine the same k-plane if and only if there is an $A \in GL(k,R)$ such that $X_i' = \sum_j A_i^j X_j$. It is easy to check that this defines an equivalence relation ~ on $F(k,n)$ and we have $G(k,n) = F(k,n)/\sim$. Let $\rho: F(k,n) \longrightarrow G(k,n)$ denote the projection. We leave as an exercise to show that $\rho$ is open. We show here that $G(k,n)$ is Hausdorff (assuming $\rho$ is open). Consider the $n \times 2k$ matrix $M \in F(k,n) \times F(k,n)$ $(k<n)$

$$M = \begin{pmatrix} X_1^1 & \cdots & X_1^k & X_1'^1 & \cdots & X_1'^k \\ & & \vdots & \vdots & & \\ X_n^1 & \cdots & X_n^k & X_k'^1 & \cdots & X_k'^k \end{pmatrix}$$

and all $(k+1)\times(k+1)$ minor determinants consisting of k unprimed columns and one primed column. A typical example is

$$\begin{vmatrix} X_1^1 & \cdots & X_1^k & X_1'^1 \\ \vdots & & \vdots & \vdots \\ X_{k+1}^1 & \cdots & X_{k+1}^k & X_{k+1}'^1 \end{vmatrix}$$

A well known theorem from linear algebra says that such a determinant vanishes if and only one of the columns (or rows) is a linear combination of the others. But since the $X^1_\alpha,\ldots,X^k_\alpha$ are linearly independent the only possibility is

$$X'^1_\alpha = \sum_{j=1}^{k} A^1_j X^j_\alpha \qquad \alpha = 1,\ldots,k+1$$

Running through all such possibilities we find

$$X'^i_\alpha = \sum_{j=1}^{k} A^i_j X^j_\alpha \qquad i = 1,\ldots,k \quad \alpha = 1,\ldots,n$$

Moreover, since $(X'^1_\alpha,\ldots,X'^k_\alpha)$ are themselves linearly independent $(A^i_j)$ must be nonsingular, i.e. $(A^i_j)\varepsilon GL(k,R)$. It follows that $X, X' \varepsilon F(k,n)$ are equivalent if and only if all such minor determinants vanish. Thus the set $E \subset F(k,n)\times F(k,n)$ defined by $E = \{(X,X')\varepsilon F(k,n)\times F(k,n): X\sim X'\}$ is the zero set of all such $(k+1)\times(k+1)$ minor determinants. Since determinants are continuous, it follows that E is closed, and by lemma 2.2 that G(k,n) is Hausdorff.

<u>Exercise</u>: Show that the quotient projection $\rho:F(k,n) \longrightarrow G(k,n)$ is open and thus that G(k,n) has a countable basis.

<u>Exercise</u>: Show that G(k,n) is a manifold of dimension (n-k)k. Hint. Let $X\varepsilon F(k,n)$ and consider the $k\times k$ submatrix $\tilde{X}$. Let $\tilde{U}$ be the open set of F(k,n) consisting of $k\times n$ matrices X such that $\tilde{X}$ is nonsingular. Put $U = \rho(\tilde{U})$. Show that every $Y\varepsilon\tilde{U}$ is equivalent to a $k\times n$ matrix X such that the $k\times k$ submatrix is the identity. Define a map $\phi:U \longrightarrow M^{k(n-k)} \simeq \mathbf{R}^{k(n-k)}$ by deleting the first k columns of the representative X for Y . Show

that $\phi$ is a homeomorphism.

Remark: $G(1,n) = P^n(R)$

Exercise: Show that $G(k,n) = G(n-k,n)$.

Hint. Consider the map which sends a k-plane into its orthogonal complement.

<u>Example 2.9</u>: <u>One-sheeted hyperboloid</u> $H_1^2$ :

$H_1^2 = \{x \in R^3 : x_1^2 + x_2^2 - x_3^2 = 1\}$. We construct a global homeomorphism $f: H^2 \longrightarrow S^1 \times R^1$ defined by $f(x_1,x_2,x_3) = \left(\frac{x_1}{\sqrt{1+x_3^2}}, \frac{x_2}{\sqrt{1+x_3^2}}, x_3\right)$. Here we consider $S^1 \times R^1 = \{(x_1,x_2,x_3) \in R^3 : x_1^2 + x_2^2 = 1\}$.

It is easy to see that f is 1-1 and continuous. The inverse function $f^{-1}$ is also continuous. (show it). Thus $H_1^2$ is homeomorphic to $S^1 \times \mathbb{R}^1$ and is thus a manifold since $S^1 \times \mathbb{R}^1$ is.

<u>Exercise</u>: Construct a global homeomorphism between $S^1 \times \mathbb{R}^1$ and $\mathbb{R}^2 - \{0\}$.

<u>Example 2.10</u>: The two-sheeted hyperboloids $H_n^n$

Define $H_n^n = \{x \in \mathbb{R}^{n+1} : x_{n+1}^2 = 1 + x_1^2 + \ldots + x_n^2\}$

$H_n^n$ is not connected since there are two ranges for $x_n$, namely $x_n \geq 1$ or $x_n \leq -1$. The component $H_n^{n,+} =$

$\{x \in \mathbb{R}^{n+1} : x_{n+1}^2 = 1 + x_1^2 + \ldots + x_n^2, \; x_n \geq 1\}$ is connected since it is globally homeomorphic to $\mathbb{R}^n$. This follows since projection map $f: H_n^{n,+} \longrightarrow \mathbb{R}^n$ defined by

$$f\left(x^1, \ldots, x^n, + \sqrt{1+x_1^2 + \ldots + x_n^2}\right)$$

is a homeomorphism. (Check this!). Moreover, the map $p:\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ restricted to $H_n^{n,+} \longrightarrow H_n^{n,-}$ defined by $p(x_1,\ldots,x_n) = (x_1,\ldots,-x_n)$ is a homeomorphism so $H_n^{n,-}$ is connected and homeomorphic to $\mathbb{R}^n$. Thus $H_n^n$ has two connected components each homeomorphic to $\mathbb{R}^n$.

We can easily generalize to hyperboloids of the form $H_m^n = \{x\varepsilon\mathbb{R}^{n+1}: x_{n+1}^2+\ldots+ x_{m+1}^2 - x_m^2-\ldots-x_1^2 = 1\}$.

<u>Exercise</u>: Show following example 2.9 that the general one-sheeted hyperboloids $H_1^n$ are homeomorphic to $S^{n-1}\times\mathbb{R}^1$. Why does not an argument similar to that of example 2.10 work?

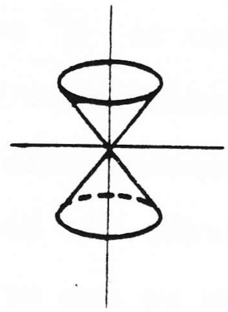
Considering spaces like $S^n$ and $H_m^n$ we make a definition.

Let f be an algebraic function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ the set $f^{-1}(0)$ is called an <u>algebraic variety</u>, $V = f^{-1}(0)$. That is, V is just the set zeros of an algebraic function. (Actually all we need is that f be a polynomial in Cartesian-Coordinates) Clearly $S^n$ and $H^n$ are algebraric varieties. But all algebraic varieties are not manifolds. A familiar example is

<u>Example 2.11</u>: The <u>cone</u> $C^{n,1}$: $n\leq 2$

$$C^{n,1} = \{\underset{\sim}{x}\varepsilon\mathbb{R}^{n+1}: x_1^2 +\ldots+ x_n^2 - x_{n+1}^2 = 0\}$$

Clearly, $C^{n,1}$ is an algebraic variety. It is <u>not</u> a manifold, however. There is no nbd of $0 \varepsilon C^{n,1}$ homeomorphic to $\mathbb{R}^n$. To see this notice that open sets of $C^{n,1}$ containing zero must be of the form $V_0 = C^{n,1} \cap U_0$ where $V_0$ is open in $\mathbb{R}^{n+1}$. So we can reduce our considerations to spherical nbds of $0\varepsilon\mathbb{R}^n$. Now I



claim that $\{0\}$ is a cut point of $C^{n,1}$, that is that $C^{n,1}-\{0\}$ is disconnected. Define $C_{\pm} = \left\{x\varepsilon C^{n,1}: x_{n+1} {>0 \atop <0}\right\}$. Clearly, $C^{n,1}-\{0\} = C_{+} \cup C_{-}$. Moreover $C_{+} \cap C_{-} = \{0\}$. Thus $C^{n,1}-\{0\}$ is disconnected. Let $V_0$ be nbd of $0\varepsilon C^{n,1}$ then $(V_0-\{0\})$ is disconnected. Now suppose that $f: V_0 \longrightarrow \mathbb{R}^n$ is a homeomorphism. Then by a theorem we proved $f(V_0-\{0\})$ must be disconnected. But it is not difficult to show that $\mathbb{R}^n$ — {point} is connected for $n \geq 2$. (See Hocking and Young). Contradiction.

<u>Exercise</u>: Construct a global homeomorphism between $\mathbb{R}^n - \{0\}$ and $S^{n-1} \times \mathbb{R}$.
We can use this result to show that $S^n$ is connected, knowing that $\mathbb{R}^{n+1}-\{0\}$ is for $n > 0$.

We had mentioned earlier the need to generalize our definition of manifold to include boundaries. We now do this. Consider the half-space

$$\mathbb{H}^n = \{x\varepsilon\mathbb{R}^n: x_n \geq 0\}$$

(invariance of the domain)

Now $\mathbb{H}^n$ is not homeomorphic to $\mathbb{R}^n$ since considered as a subspace of $\mathbb{R}^n$ it is closed. We define a <u>manifold with boundary</u> of dimension n to be an Hausdorff space with a countable basis satisfying.

i) Every point $x\varepsilon M$ has a neighborhood $V_x$ that is homeomorphic to either $R^n$ or $H^n$.

It can be shown using invariance of the domain that for each $x \in M$ $U_x$ is either homeomorphic to $\mathbb{R}^n$ or $H^n$ but <u>not</u> both. The set of points locally homeomorphic to $\mathbb{R}^n$ is called the interior of M, while the set homeomorphic to $\mathbb{H}^n$ is called boundary of M, denoted $\partial M$. Intuitively, $\partial M$ has dimension n-1. But we do not prove this, here. Now we can write $\mathbb{H}^n = \{x \in R^n : x_n > 0\} \cup \{x \in \mathbb{R}^n : x_n = 0\}$. The first is globally homeomorphic to $\mathbb{R}^n$ while the second is globally homeomorphic to $\mathbb{R}^{n-1}$. Thus up to homeomorphism $\mathbb{H}^n = R^n \cup R^{n-1}$ and $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$.

<u>Example 2.12</u>: The closed ball $\mathcal{B}(0,r) = \{x \in R^n : y_1^2 + \ldots + y_1^2 \leqslant r^2\}$ is a manifold with boundary. Clearly the interior $\mathcal{B}^\circ(0,r)$ is a spherical neighborhood and is thus globally homeomorphic to $\mathbb{R}^n$. The boundary $\partial\mathcal{B}(0,r) = S^{n-1}$ which is locally homeomorphic to $\mathbb{R}^{n-1} = \partial\mathbb{H}^n$. We now show that every point $p \in \partial\mathcal{B}(0,r)$ has a nbd. homeomorphic to $H^n$. First, we can construct a homeomorphism between $S(0,r)$ and the cubical neighborhoods $C^n(0,1) = \{x \in R^n : |x^i| < 1$ for all $i=1,\ldots,n\}$. This is done by composing the homeomorphism of proposition 2.1 with the homeomorphism $x: C^n(0,1) \longrightarrow R^n$ defined by

$$x^i = \frac{z^i}{1-(z^i)^2}$$

Then $y: S^n(0,r) \longrightarrow C^n(0,1)$ is given by

$$y^i = \frac{r\, z^i}{1-(z^i)^2} \cdot \frac{1}{\left[1 + \sum_j \frac{(z^j)^2}{(1-(z^j)^2}\right]^{1/2}}$$

It is straightforward to check that this map extends to a well-defined continuous map on the closures, i.e. $\bar{y} : \overline{S^n(o,r)} = \mathcal{B}(o,r) \longrightarrow \overline{C^n(0,1)}$. Furthermore $\bar{y}$ is a homeomorphism. Clearly, every point on $\partial C^n(0,1)$ with the possible exceptions of the corners has a neighborhood homeomorphic to $H^n$. But the points of $\mathcal{B}(o,r)$ corresponding to the corners of $\overline{C^n(0,1)}$ can be handled by rotating $\mathcal{B}(0,r)$ with respect to $\overline{C^n(0,1)}$.