

on tensor fields $\underline{T_s^r(M)} \longrightarrow \underline{T_{s-1}^{r+1}(M)}$ or

$\underline{T_s^r(M)} \longrightarrow \underline{T_{s+1}^{r-1}(M)}$. Classically this is what is called raising and lowering indices. Show that if $T_{j_1 \dots j_s}^{i_1 \dots i_r}$ are the components of a tensor field of type (r, s) then $\sum_n g_{jk} T_{j_1 \dots j_s}^{i_1 \dots i_r k}$ and $\sum_n g^{jk} T_{j_1 \dots j_s}^{i_1 \dots i_r}$ are the components of tensor fields of type $(r+1, s)$ and $(r, s-1)$, respectively.

6.4 Exterior Bundles and Exterior Differential Calculus.

Just as with the case of the symmetric bundles, the antisymmetric tensor product is functorial, so we can construct antisymmetric bundles from $T(M)$ and $T^*(M)$. Of particular importance is the case of $T^*(M)$. We define the vector space

$$\Lambda_p^k(M) = \mathcal{A}(T_p^*(M) \otimes \dots \otimes T_p^*(M)) \quad k=1, \dots$$

$$\Lambda_p^0(M) = \mathbb{R}$$

Then by theorem 6.1

$$\Lambda^k(M) = \bigcup_P \Lambda_P^k(M) \quad k=1, \dots, n$$

is a C^∞ vector bundle. We define

$\Lambda^0(M)$ as the trivial bundle $M \times \mathbb{R}$. Then the set of C^∞ sections of $\Lambda^0(M)$ can be identified with $C^\infty(M)$. The bundle $\Lambda^k(M)$ is called the k^{th} order exterior bundle and its C^∞ sections are k^{th} order differential forms or just k -forms.

We can also take the Whitney sums of the bundles $\Lambda^k(M)$ to get a new bundle.

$$\Lambda(M) = \sum_{k=0}^n \oplus \Lambda^k(M)$$

called the exterior bundle over M . Since $\Lambda_p(M)$ forms an exterior algebra (chapter 5) so does $\Lambda(M)$ (the C^∞ sections of $\Lambda(M)$). This algebra is called the exterior differential algebra over M .

Exercise: Give standard coordinates on $\Lambda^k(M)$ and $\Lambda(M)$.

A member of $\Lambda^k(T_p^*(M) \oplus \dots \oplus T_p^*(M))$ ^{k -times} surely takes the form (from chapter 5)

$$\omega = \sum a_{i_1 \dots i_k} (dx^{i_1})_p \wedge \dots \wedge (dx^{i_k})_p$$

Thus in a local coordinate (U, x) a C^∞ section $\omega \in \underline{\Lambda}^k(M)$ takes the form

$$(6.10) \quad \omega(x) = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where $a_{i_1 \dots i_k} \in C^\infty(M)$.

Proposition 6.5: Let $F: M \rightarrow N$ be a C^∞ map, then $F^*: \underline{\Lambda}^k(N) \rightarrow \underline{\Lambda}^k(M)$ defined ^{(when $k \leq \min(\dim M, \dim N)$)} by extending

$F^*: T^*(N) \rightarrow T^*(M)$ by exterior product, is C^∞ .

Proof: For $\omega \in \underline{\Lambda}^k(N)$, $F^*\omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} F^*(dx^{i_1} \wedge \dots \wedge dx^{i_k})$
 $= \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k} (F^*dx^{i_1}) \wedge \dots \wedge (F^*dx^{i_k})$. But on $T^*(M)$

$(F^*\phi)(\vec{X}) = \phi(F_*\vec{X})$ by definition. Thus,

$$F^*\omega(\vec{X}_1, \dots, \vec{X}_k) = \omega(F_*\vec{X}_1, \dots, F_*\vec{X}_k)$$

But by proposition 4.6 $F_*\vec{X}_i$ is C^∞ and by definition ω is C^∞ ; hence, $F^*\omega$ is C^∞ by (6.9)

Exercise: Show that a similar result holds for $S^k(M)^*$.

If (U, x) and (V, y) are local charts for M and N , respectively, with $x = (x^1, \dots, x^m)$ and $y = (y^1, \dots, y^n)$, $n = \dim N$, $m = \dim M$, and if $\omega \in \underline{\Lambda}^k(N)$ with $k \leq \min(n, m)$ then in local coordinates

$$(6.11) \quad (F^*\omega)(x) = \sum_{\substack{i_1 < \dots < i_k \\ j_1 < \dots < j_k}} a_{i_1 \dots i_k}(y) \left(\frac{\partial y^{j_1}}{\partial x^{i_1}} \right) \dots \left(\frac{\partial y^{j_k}}{\partial x^{i_k}} \right) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

by theorem 4.3, so that

$$(6.12) \quad a'_{j_1 \dots j_k}(x) = \sum_{i_1 < \dots < i_k} \left(\frac{\partial y^{j_1}}{\partial x^{i_1}} \right) \dots \left(\frac{\partial y^{j_k}}{\partial x^{i_k}} \right) a_{i_1 \dots i_k}(y)$$

are the components of $F^*\omega$ with respect to $dx^{i_1} \wedge \dots \wedge dx^{i_k}$. If we take F to be the identity map $M \rightarrow M$, then (6.11) gives the expression for a change of coordinates, that is

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1 \dots i_k}(y) dy^{i_1} \wedge \dots \wedge dy^{i_k} = \sum_{i_1 < \dots < i_k} a'_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

with $a'_{i_1 \dots i_k}$ related to $a_{i_1 \dots i_k}$ by (6.12). This shows explicitly that ω is independent of coordinates.

We now come to a fundamental existence theorem:

Theorem 6.8: There is a unique linear map $d: \underline{\Lambda}^k(M) \rightarrow \underline{\Lambda}^{k+1}(M)$ which satisfies

i) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$
for $\omega_1 \in \underline{\Lambda}^p(M)$, $\omega_2 \in \underline{\Lambda}^q(M)$.

ii) For $f \in \underline{\Lambda}^0(M)$, $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$ in local coordinates, that is d is the usual differential on functions.

iii) $d^2 f = 0$ for $f \in \underline{\Lambda}^0(M)$.

Proof: Existence: Let (U, x) be a chart with coordinates $x = (x^1, \dots, x^n)$. Any $\omega \in \underline{\Lambda}^k(M)$ can be written as

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1, \dots, i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

and this is independent of coordinates and thus defined globally on M . Now define d by

$$(6.13) \quad d\omega = \sum_{i_1 < \dots < i_k} da_{i_1, \dots, i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

where $da_{i_1, \dots, i_k}$ is the usual differential on the C^∞ functions $a_{i_1, \dots, i_k}$, i.e. $da_{i_1, \dots, i_k} = \sum_j \frac{\partial a_{i_1, \dots, i_k}}{\partial x^j} dx^j$ on U

It is immediate that d is linear and satisfies ii). Assuming that d is unique then implies that d is defined globally. For if $d\omega|_U$ and $d\omega|_V$ agree on $U \cap V$, then $d\omega$ is defined on $U \cup V$, and is independent of coordinates. ^{at least locally.}
We check iii). If $\omega = \sum_i a_i dx^i$, then (6.13) is

$$d\omega = \sum_{i < j} \left(\frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i} \right) dx^i \wedge dx^j$$

Thus

$$d^2 f = d \left(\sum_j \frac{\partial f}{\partial x^j} dx^j \right) = \sum_{i < j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j = 0$$

We check i). Since d and $\wedge$ are both linear we can take $\omega_1 = f dx^{i_1} \wedge \dots \wedge dx^{i_p}$ and $\omega_2 = g dx^{j_1} \wedge \dots \wedge dx^{j_q}$ for $f, g \in C^\infty(M)$. We have

$$\begin{aligned} d(\omega_1 \wedge \omega_2) &= d(fg) dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q} \\ &= df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge g dx^{j_1} \wedge \dots \wedge dx^{j_q} \\ &\quad + f dg \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q} \\ &= df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge g dx^{j_1} \wedge \dots \wedge dx^{j_q} \\ &\quad + (-1)^p f dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dg \wedge dx^{j_1} \wedge \dots \wedge dx^{j_q} \\ &= d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2 \end{aligned}$$

uniqueness: Again since any $\omega \in \wedge^k(M)$ can be

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written as a linear combination of forms of the type $\omega = f dx^{i_1} \wedge \dots \wedge dx^{i_k}$ we can restrict ourselves to such forms. Let $d': \underline{\Lambda^k(M)} \rightarrow \underline{\Lambda^{k+1}(M)}$ be a linear map satisfying i) - iii), then by i)

$$d'(f dx^{i_1} \wedge \dots \wedge dx^{i_k}) = d'f \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} + f d'(dx^{i_1} \wedge \dots \wedge dx^{i_k})$$

But on functions $d'f = \sum_j \frac{\partial f}{\partial x_j} dx^j = df$ by

ii). Thus $0 = (d')^2 f = d'df$ by iii). So $d'dx^i = 0$ and the last term in the above expression vanishes by i) and induction on k .

So we have

$$\begin{aligned} d'(f dx^{i_1} \wedge \dots \wedge dx^{i_k}) &= d'f \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \\ &= df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} = d(f dx^{i_1} \wedge \dots \wedge dx^{i_k}). \quad \text{Q.E.D.} \end{aligned}$$

We have now an important lemma

Lemma 6.6 : $d^2 = 0$ on all of $\underline{\Lambda(M)}$.

Proof: Again it suffices to check this on elements $\omega = f dx^{i_1} \wedge \dots \wedge dx^{i_k}$ and extend linearly to all of $\underline{\Lambda^k(M)}$.
We have by definition

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$dw = df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_n}$. Then by i)

$$d^2w = (d^2f) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_n} - df \wedge d(dx^{i_1} \wedge \dots \wedge dx^{i_n})$$

The first term vanishes by iii) and second vanishes from what was said above in the uniqueness proof. Since dw is defined globally $d^2w = 0$ globally and is independent of coordinates.

The map d is called the exterior derivative on $\Lambda(M)$. It should be clear that all our considerations hold for w a class C^r function $r \geq 2$. Then dw is of class C^{r-1} .

Lemma 6.2 reflects a very important algebraic property used in algebraic topology. A sequence of graded vector spaces $\underline{\Lambda}^k(M)$ and a map $d: \underline{\Lambda}^k(M) \rightarrow \underline{\Lambda}^{k+1}(M)$ for all k which satisfies $d^2 = 0$ is called a cochain complex. We write

$$(6.14) \quad 0 \rightarrow \underline{\Lambda}^0(M) \xrightarrow{d} \underline{\Lambda}^1(M) \xrightarrow{d} \dots \xrightarrow{d} \underline{\Lambda}^k(M) \xrightarrow{d} \dots$$

In our case the sequence terminates at $\underline{\Lambda}^n(M)$, $n = \dim M$. An element $w \in \underline{\Lambda}^k(M)$

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satisfying $dw = 0$ is called a cocycle, and element $w \in \Lambda^k(M)$ which can be written as $w = d\alpha$ for some $\alpha \in \Lambda^{k-1}(M)$ is called a coboundary. The sequence (6.14) is known as the de Rham sequence. We define

$$(6.15) \quad H^k(M) = \ker d / \operatorname{im} d$$

where $\ker d$ is the kernel of d and $\operatorname{im} d$ is the image of d . Thus an element of $\ker d$ is just a cocycle and an element of $\operatorname{im} d$ is just a coboundary. So $H^k(M)$ determines all cocycles which are not coboundaries. $H^k(M)$ is called the k^{th} de Rham cohomology group of M . A cocycle $w \in \Lambda^k(M)$, i.e. $dw = 0$, is also called a closed k -form, whereas a coboundary $w = d\alpha$ is called an exact k -form on M . Lemma 6.6 says that all exact forms are closed, since $dw = d^2\alpha = 0$. This assures that $\operatorname{im} d_{k-1} \subset \ker d_k$ and so $H^k(M)$ is well defined. It should be clear that here d_k means that d is acting on k -forms. A sequence such as (6.14) is

said to be exact at $\underline{\Lambda}^k(M)$ if $H^k(M) = 0$,
 i.e. all cocycles are coboundaries. The sequence
 is called exact if it is exact at all
 its entries.

Remark: $H^k(M)$ is really a vector space, but
 since a vector space is an Abelian (commutative)
 group, $H^k(M)$ is known as a cohomology group.

Proposition 6.6: Let $\phi: M \rightarrow N$ be a
 C^∞ map, then the diagram

$$\begin{array}{ccccccc} \xrightarrow{d} & \underline{\Lambda}^k(N) & \xrightarrow{d} & \underline{\Lambda}^{k+1}(N) & \xrightarrow{d} & & \\ & \downarrow \phi^* & & \downarrow \phi^* & & & \\ \xrightarrow{d} & \underline{\Lambda}^k(M) & \xrightarrow{d} & \underline{\Lambda}^{k+1}(M) & \xrightarrow{d} & & \end{array}$$

commutes for $k+1 \leq \min(\dim M, \dim N)$.

Proof: By proposition 6.5 ϕ^* is C^∞ and again
 we can prove this locally and when $\omega \in \underline{\Lambda}^k(M)$
 has the form $\omega = f dx^{i_1} \wedge \dots \wedge dx^{i_k}$. Proof by
 induction (for $k=0$, $\omega = f$ it is just the chain rule)

$$\begin{aligned} d\phi^*\omega &= d\phi^*(f dx^{i_1} \wedge \dots \wedge dx^{i_k}) = d\phi^*((f dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}) \wedge dx^{i_k}) \\ &= d(\phi^*(f dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}})) \wedge \phi^* dx^{i_k} \\ &= \phi^*(df \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}) \wedge \phi^* dx^{i_k} \end{aligned}$$

by induction hypothesis
and theorem 6.6

$$\begin{aligned}
 &= \phi^*(d(f \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k})) = \phi^*d(f dx^{i_1} \wedge \dots \wedge dx^{i_k}) \\
 &= \phi^*d\omega. \quad Q.E.D.
 \end{aligned}$$

Corollary: ϕ induces a map on cohomology

$$\phi^*: H^k(N) \longrightarrow H^k(M)$$

If ϕ is a diffeomorphism, ϕ^* is an isomorphism.

Proof: Since ϕ^* and d commute, closed forms are mapped into closed forms and exact forms are mapped to exact forms, so ϕ^* passes to the quotient. If ϕ is a diffeomorphism, it follows from theorem 4.3 that ϕ^* is isomorphism both on $\mathcal{L}^k(M)$ and $H^k(M)$.

This corollary says that the de Rham cohomology $H^k(M)$ is a differential invariant of differential manifolds. It turns out that $H^k(M)$ is actually a topological invariant, that is, it depends only on the topological nature of M and not on the differential structure. This follows from the ~~well known~~ de Rham theorem, which we will not prove. We will however, make much use of de Rham cohomology.

Example 6.12: If $\omega \in \underline{\Lambda}^1(M)$, then

$$\omega = \sum_i a_i(x) dx^i$$

and we had

$$d\omega = \sum_{i < j} \left(\frac{\partial a_j}{\partial x^i} - \frac{\partial a_i}{\partial x^j} \right) dx^i \wedge dx^j$$

We recognize ^{from the classical language} the components as $\text{Rot } \underline{a}$, where $\underline{a} = (a_i)_i$

$$d\omega = \sum_{i < j} (\text{Rot } \underline{a})_{ij} dx^i \wedge dx^j$$

Thus $d\omega = 0$ is just

$$\text{Rot } \underline{a} = 0$$

~~for a vector~~ $\underline{a}$, and if $a_i = \frac{\partial f}{\partial x^i}$, that is
if $\omega = df$, then $d\omega = d^2 f = 0$ says

$$\text{Rot grad } f = 0$$

We will see shortly that if $d\omega = 0$ for $\omega \in \underline{\Lambda}^1$ then locally there is an $f \in C^1(U)$ such that $\omega = df$. This may not be true globally and the cohomology $H^1(M)$ give the "obstruction" to the existence of such an f globally. On $\mathbb{R}^n$ we will see that $H^1(M) = 0$ and $\omega = df$ is true globally.

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Example 6.13: $H^0(M)$. The first entry in (6.14) is

$$0 \longrightarrow \underline{\Lambda^0(M)} \xrightarrow{d}$$

Clearly, there are no coboundaries so $H^0(M) = \ker d$, i.e. all $f \in \underline{\Lambda^0(M)} = C^0(M)$, such that $df = 0$.

If M is connected then $f = \text{constant}$ on all of M , so $H^0(M) \cong \mathbb{R}$. On the other hand if M has k components then f is a different constant on each component so $H^0(M) \cong \mathbb{R}^k$.

We denote by $B \subset \mathbb{R}^n$ the ^{open} unit ball, i.e. $B = S(n, 1) = \{x \in \mathbb{R}^n : (x^1)^2 + \dots + (x^n)^2 < 1\}$. We will now prove the important Poincaré lemma. First, we prove a useful lemma from algebraic topology.

Lemma 6.7: Let

$$\longrightarrow C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \longrightarrow$$

be any cochain complex with differential map d , i.e. $d^2 = 0$. If there exists a homotopy map $h_k : C^k \longrightarrow C^{k-1}$ such that $h_{k+1} \circ d + d \circ h_k = \text{identity}$

then $H^k(C) = 0$.

Proof: Let $\omega \in C^k$ be a cocycle, i.e. $d\omega = 0$.

Then $(h_{k+1} \circ d + d \circ h_k)\omega = \omega = h_{k+1} \circ d\tilde{\omega} + d \circ h_k \omega$
 $= d(h_k \omega)$. So $h_k \omega \in \text{im } d_{k-1}$, which implies

that $H^k = 0$.

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We are now ready for

Theorem 6.9: (Poincaré lemma) $H^k(B) = 0, k \geq 1$, i.e.

every closed k -form on B is exact on B .

Proof: We construct a C^∞ homotopy map, i.e. a C^∞ map $h_k: \underline{\Lambda}^k(B) \rightarrow \underline{\Lambda}^{k-1}(B)$ such that $h_{k+1} \circ d + d \circ h_k = \text{id}$. It suffices to choose $\omega \in \underline{\Lambda}^k(B)$ of the form $\omega = f dx^{i_1} \wedge \dots \wedge dx^{i_k}$ and extend linearly to all of $\underline{\Lambda}^k(B)$. We define

$$h_k \omega = \sum_{j=1}^k (-1)^{j+1} \left(\int_0^1 dt \, t^{k-1} f(t\mathbf{x}) \right) dx^{i_1} \wedge \dots \widehat{dx^{i_j}} \wedge \dots \wedge dx^{i_k}$$

where $\widehat{dx}$ means this element is missing. Then

$$d h_k \omega = \sum_{j=1}^k (-1)^{j+1} \int_0^1 dt \, t^{k-1} f(t\mathbf{x}) (-1)^{j+1} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$+ \sum_{j=1}^k (-1)^{j+1} \int_0^1 dt \, t^{k-1} \sum_{i=1}^n \frac{\partial f(t\mathbf{x})}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \widehat{dx^{i_j}} \wedge \dots \wedge dx^{i_k}$$

$$= + k \int_0^1 dt \, t^{k-1} f(t\mathbf{x}) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$+ \sum_{j=1}^k (-1)^{j+1} \int_0^1 dt \, t^k x^{i_j} \sum_{i=1}^n \frac{\partial f(t\mathbf{x})}{\partial y^i} dx^{i_1} \wedge \dots \widehat{dx^{i_j}} \wedge \dots \wedge dx^{i_k}$$

where $y^i = tx^i$

Notice that d only applies to B and has nothing to do with t . On the other hand

$$h_{k+1} d\omega = \sum_{j=0}^n (-1)^j \int_0^1 dt t^k \sum_{i_0=1}^n x^{i_0} \frac{\partial f(x)}{\partial x^{i_0}} dx^{i_1} \wedge \dots \wedge dx^{i_n} \quad 6.6$$

Thus

$$d\omega = \frac{\partial f(x)}{\partial x^{i_0}} dx^{i_0} \wedge \dots \wedge dx^{i_n}$$

$$h_{k+1} d\omega + dh_k \omega = k \int_0^1 dt t^{k-1} f(tx) dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$+ \int_0^1 dt t^k \sum_{i_0=1}^n x^{i_0} \frac{\partial f}{\partial x^{i_0}} dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$= \int_0^1 dt t^{k-1} \left(k f(tx) + \sum_{i_0=1}^n x^{i_0} \frac{\partial f}{\partial x^{i_0}}(tx) \right) dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$= \int_0^1 dt \frac{d[t^k f(tx)]}{dt} dx^{i_1} \wedge \dots \wedge dx^{i_n} = f dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$= \omega.$$

and the theorem follows by lemma 6.7.

Corollary: Let M be globally diffeomorphic to B , then $H^k(M) = 0$ for all $k > 0$.

Proof: This follows by proposition 6.6 and theorem 6.9.

Example 6.14: Let $M = \mathbb{R}^n$ then since $\mathbb{R}^n$ is connected from example 3 we have $H^0(\mathbb{R}^n) = \mathbb{R}$. Now $\mathbb{R}^n$ is globally diffeomorphic to $B \subset \mathbb{R}^n$. This diffeomorphism is explicitly given by

$$y^i = \frac{x^i}{[1 + |x|^2]^{1/2}}$$

where $|x|^2 = \sum_{i=1}^n (x^i)^2$. Then $|y|^2 = \frac{|x|^2}{1 + |x|^2} < 1$. If

follows from the above corollary that $H^k(\mathbb{R}^n) = 0$ for $k \geq 1$.

Example 6.15: The circle S^1 . Since S^1 is connected $H^0(S^1) = \mathbb{R}$. Since $\dim S^1 = 1$, any 1-form on S^1 is closed. Now consider $S^1 = U \cup V$ as in example 3. On S^1

$$\omega = \begin{cases} d\theta & \text{on } U \\ d\theta' & \text{on } V \end{cases} \quad \begin{matrix} 0 < \theta < 2\pi \\ -\pi < \theta' < \pi \end{matrix}$$

is a closed 1-form. It is not exact since θ and θ' are not continuous functions on S^1 . To compute $H^1(S^1)$ consider an arbitrary form $\alpha = f\omega$ where $f \in C^\infty(S^1)$. Define the real number

$$c = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta = \frac{1}{2\pi} \int \alpha$$

Put

$$g(\theta) = \int_0^\theta (f(t) - c) dt$$

for $\theta \in \mathbb{R}$. Now $g(\theta + 2\pi) = g(\theta)$, so $g \in C^\infty(S^1)$.

But

$$dg = (f(\theta) - c) d\theta = \alpha - c d\theta$$

Thus the arbitrary 1-form α differs from an exact form dg by a real multiple c of $d\theta$, i.e. $\alpha = dg + c d\theta$; hence $H^1(S^1) = \mathbb{R}$.

In order to compute the de Rham cohomology groups for more complicated manifolds we will need more machinery. We develop some of this here but leave most for a later chapter.

Let us now introduce another important concept. Suppose $f, g: M \rightarrow N$ are C^∞ maps of C^∞ manifolds, and let $I = [0, 1]$ the closed unit interval. f and g are said to be C^∞ homotopic if there is a C^∞ map $H: M \times I \rightarrow N$ which satisfies

$$H(p, 0) = f(p)$$

$$H(p, 1) = g(p) \quad \text{for all } p \in M$$

H is called a C^∞ homotopy between f and g . Consider now a special case. Let $N = M$ and fix a point $p_0 \in M$. Take f to be the constant map and g the identity map, i.e.

$$f(p) = p_0$$

$$g(p) = p$$

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Then M is said to be C^∞ -contractible to a point p_0 if the identity is C^∞ -homotopic to the constant map, i.e. if there exists a C^∞ map $H: M \times I \rightarrow M$ such that

$$H(p, 0) = p_0 \quad \text{and} \quad H(p, 1) = p.$$

Proposition 6.7: Let $p_0 \in \mathbb{R}^n$ and consider a subset $U \subset \mathbb{R}^n$ containing p_0 which satisfies $[p_0 + t(p - p_0)] \in U$ for all $p \in U$ and all $t \in I$. (U is called star-shaped with respect to p_0). Then U is contractible to p_0 .

Proof: The map $H: U \times I \rightarrow U$ defined by

$$H(p, t) = p_0 + t(p - p_0)$$

is C^∞ and

it satisfies $H(p, 0) = p_0$ and $H(p, 1) = p$.

Exercise: Show that $\mathbb{R}^n$ is contractible to a point (say $(0, \dots, 0)$).

We will now use the above ideas to prove a theorem from which a more general version of the Poincaré lemma follows.

Theorem 6.10: Let $f, g: M \rightarrow N$ be two

C^∞ homotopic maps. Then on $H^k(M)$, $f^* = g^*$,
i.e. for any $\omega \in \underline{\Lambda}^k(M)$ with $d\omega = 0$, $f^*\omega - g^*\omega$
is exact.

Proof: We define the natural injection $i_x: M \rightarrow M \times \mathbb{I}$
by $i_x(p) = (p, x)$, which is clearly
 C^∞ . We will define a homotopy operator h which
satisfies

$$*) \quad d \circ h \omega + h \circ d \omega = i_1^* \omega - i_0^* \omega$$

First notice that in $\underline{\Lambda}^k(M \times \mathbb{I})$ there are two types
of decomposable forms, namely in an arbitrary local
coordinate chart,

$$\omega_1 = a(x, t) dt \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}$$

$$\omega_2 = a(x, t) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

It suffices to prove $*)$ on these two types and
then extend by linearity. We define h by

$$h\omega_1 = \left(\int_0^1 a(x, t) dt \right) dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}$$

$$h\omega_2 = 0$$

For ω_1 we have $i_x^* \omega_1 = 0$ since $i_x^* dt = 0$.

Moreover,

$$\begin{aligned} d\omega_1 &= d \int_0^1 a(x,t) dt \, dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}} \\ &= \sum_{i=1}^n \left(\int_0^1 \frac{\partial a}{\partial x^i} dt \right) dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{k-1}} \end{aligned}$$

But

$$\begin{aligned} h \cdot d\omega_1 &= h \sum_i \frac{\partial a}{\partial x^i} dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}} \\ &= - \sum_{i=1}^n \left(\int_0^1 \frac{\partial a}{\partial x^i} dt \right) dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{k-1}} \end{aligned}$$

Hence, $\ast$) is satisfied for this case. For ω_2 we have $i_1^* \omega_2 - i_0^* \omega_2 = [a(x,1) - a(x,0)] dx^{i_1} \wedge \dots \wedge dx^{i_k}$ and $d\omega_2 = 0$. So

$$h \cdot d\omega_2 = \left(\int_0^1 \frac{\partial a}{\partial x} dt \right) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

and $\ast$) is satisfied by the fundamental theorem of integral calculus. To complete the proof, we let $H: M \times I \rightarrow N$ be the homotopy map connecting f and g . Then $f(p) = H(p,0) = H \circ i_0(p)$ and $g(p) = H(p,1) = H \circ i_1(p)$. Now assuming that ω is closed we have using $\ast$)

$$g^* \omega - f^* \omega = (i_1^* - i_0^*) H^* \omega = d(h H^* \omega).$$

So $(g^* - f^*) \omega$ is exact, i.e. $g^* = f^*$ on $H^k(M)$. Q.E.D.

Remark. We will use this theorem later coupled with Stokes theorem to prove that on S^n for n even there is no vector field which vanishes nowhere.

Exercise: Using the theorem prove the following version of the Poincaré lemma: If M is contractible to a point then $H^k(M) = 0$ $k \geq 1$.

Exercise: Consider the direct sum of vector spaces

$$H^*(M) = \sum_{k=0}^{\infty} H^k(M).$$

Define a product $\cup$ on

on $H^*(M)$ by $[\omega] \cup [\eta] = [\omega \wedge \eta]$. Show that this product is well-defined and that the vector space $H^*(M)$ with $\cup$ is an algebra.

Let us now turn to another important topological property of a manifold, namely orientability. Consider the tangent space $T_p(M)$ at $p \in M$. Choose a frame at p , $\{\bar{x}_i(p)\}_{i=1, \dots, n}$. Now any other frame at p is related to $\{\bar{x}_i(p)\}$ by a member of $GL(n, \mathbb{R})$. Introduce an equivalence relation on the set of frames $F_p(M)$ at p by saying that two frames $\{\bar{x}_i(p)\}$ and $\{\bar{y}_i(p)\}$ are equivalent if they are related by a member of the subgroup $GL(n, \mathbb{R})^+ = \{A \in GL(n, \mathbb{R}) \mid \det A > 0\}$ of $GL(n, \mathbb{R})$. (It is easy to check that $GL(n, \mathbb{R})^+$ is a Lie subgroup of $GL(n, \mathbb{R})$, and that this establishes an equivalence relation on the set of frames). This partitions $T_p(M)$ into two equivalence classes and each class is called an orientation for $T_p(M)$. Thus $F_p(M) / \sim = \mathbb{Z}_2 = \{\text{set with two elements}\}$ (which we can choose as ± 1 by pure convention). Now let U be a coordinate neighborhood of M with chart (U, x) . Then $\{\partial x_i\}$ is a moving frame on U and thus provides a parallelization

for U . This clearly determines an orientation for $T_p(M)$ for each $p \in M$. We assign by convention the orientation $+1$ to the frame $\{\partial x^1\}$ and -1 to the frame $\{-(\partial x^1)_p, (\partial x^2)_p, \dots, (\partial x^n)_p\}$. Thus an arbitrary frame will be related to one of these two frames by a member of $GL(n, \mathbb{R})^+$. We denote by $[\mathcal{X}_1(p)]$ the orientation of the frame $\{\mathcal{X}_1(p)\}$. On U the function $\theta: U \rightarrow \mathbb{Z}_2$ defined by

$$\theta(p) = [(\partial x^1)_p] = +1$$

is clearly C^∞ . This provides an orientation for U . Notice that the choice of orientation of $T_p(M)$ is completely arbitrary since we could have equally as well chosen a coordinate system $\tilde{x} = (\tilde{x}^1, \dots, \tilde{x}^n) = (-x^1, x^2, \dots, x^n)$ in which case the orientation is reversed. We say that M is orientable if we can choose $\theta = +1$ or $\theta = -1$ on all of M , in which case θ provides an orientation for M .

Proposition 6.8: A C^∞ manifold M is orientable if and only if there is an atlas of charts (U_α, x_α) which is compatible with the pseudogroup $\mathcal{P}_{GL(n, \mathbb{R})^+}$ of transformations of $\mathbb{R}^n$

into itself with Jacobian matrix in $GL(n, \mathbb{R})$ at each point $p \in M$.

Proof: Suppose the atlas $\{(U_\alpha, x_\alpha)\}$ is compatible with $GL(n, \mathbb{R})^+$, then in any $U_\alpha \cap U_\beta \neq \emptyset$ we have

$$*) \det \left(\frac{\partial x_\alpha^i}{\partial x_\beta^j} \right) > 0$$

so the orientation is the same on all U_α 's and thus the same on M . Conversely if M is orientable then $\Theta = +1$ or -1 on all of M , which we may choose as $+1$. In particular, in any $U_\alpha \cap U_\beta \neq \emptyset$ $\Theta = +1$, so there are charts (U_α, x_α) such that $*)$ holds. Thus since this must be true for the entire atlas $\{(U_\alpha, x_\alpha)\}$, it must be compatible with $GL(n, \mathbb{R})^+$. Q.E.D.

Remark: It is trivial that any parallelizable manifold is orientable.

Corollary: Suppose M has an atlas of two charts (U, x) and $(\tilde{U}, \tilde{x})$ and suppose $U \cap \tilde{U}$ is connected, then M is orientable.

Proof: Since $\det \left(\frac{\partial x_\alpha^i}{\partial x_\beta^j} \right)$ never vanishes on $U \cap \tilde{U}$, it retains its sign there. If $\det \left(\frac{\partial x_\alpha^i}{\partial x_\beta^j} \right) < 0$, define an atlas with charts $(\tilde{U}, \tilde{x}')$ where $\tilde{x}'^1 = -\tilde{x}^1$, $\tilde{x}'^i = \tilde{x}^i$ $i=2, \dots, n$ and (U, x) . This has positive determinant on $U \cap \tilde{U}$ and by the proposition M

orientable.

Example 6.16: $\mathbb{R}^n$ and the torus T^n are parallelizable and thus orientable. S^n has an atlas (stereographic) of charts with connected intersection, and thus orientable. The Möbius band has an atlas of two charts but the intersection of the two neighborhoods is not connected, so the corollary does not apply. In fact it is left as an exercise to show that the Möbius band is not orientable.

Lemma 6.8: Let M be oriented and let (U, x) and $(\hat{U}, \hat{x})$ any two charts on M with U and $\hat{U}$ connected. Then $\det \left(\frac{\partial x^i}{\partial \hat{x}^j} \right)$ has constant sign on $U \cap \hat{U}$.

Proof: (Notice that if $U \cap \hat{U}$ is connected, we are through). Let Θ^M be an orientation on M , and Θ the orientation on U ^{determined by x} . Define $S = \{p \in U : \Theta_p^M = \Theta_p\}$. We show that S is open. Let V and $\hat{V}$ be any two coordinate neighborhoods of p . Then Θ^M and Θ are determined by parallelisms on V and $\hat{V}$. Let $\{X_i\}$ and $\{\hat{X}_i\}$ $i=1, \dots, n$ denote the moving frames which determine the parallelism on V and $\hat{V}$, respectively. Then on $V \cap \hat{V}$, there is a nonsingular matrix a_{ij}^p of C^∞ functions with $\det a_{ij}^p > 0$. Thus

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every $p \in S$ has a neighborhood, namely $V \cap \dot{U}$, contained in S , so S is open. A similar argument shows that $U - S$ is open. But $U = S \cup (U - S)$ and U is connected. Thus either $S = U$ or $S = \emptyset$, and either $\theta = \theta^M|_U$ or $\theta = -\theta^M|_U$. Similarly on $\dot{U}$ we have an orientation θ' such that $\theta' = \theta^M|_{\dot{U}}$ or $\theta' = -\theta^M|_{\dot{U}}$. Since both θ and θ' are determined by coordinate parallelisms, we must have on all of $U \cap \dot{U}$ either $\det\left(\frac{\partial x^i}{\partial x^j}\right) > 0$ or $\det\left(\frac{\partial x^i}{\partial x^j}\right) < 0$. Q.E.D.

Exercise: Use the above lemma to show that the Möbius band is not orientable.

Exercise: Show that $M \times \dot{M}$ is orientable if and only if both M and $\dot{M}$ are.

Let M be a C^∞ manifold of dimension n . A C^∞ section $^\omega$ of $\Lambda^n(M)$ which vanishes nowhere on M is called a volume element (or form) on M . Since such a form ω vanishes nowhere we may assume that $\omega > 0$ on M . On a local coordinate chart (U, x) ω can be written as

$$\omega = a dx^1 \wedge \dots \wedge dx^n$$

$a \in C^\infty(M)$, $a > 0$. Such a form does not exist on every manifold M . In fact, we have