6.5 The Lie derivative and Derivations

In this section we will extend the operation of Lie derivative to tensor fields and study some of its properties. Consider the set of tensor fields $\underline{T^r_s(M)}$ of type (r,s). It is clear that $\underline{T^r_s(M)}$ is a vector space over $\mathbb{R}$. Moreover, it is a $C^\infty(M)$-module (Show it). Just as in chapter 5 we define

$$(6.26) \quad \mathcal{T}(M) = \sum_{r,s=0}^{\infty} \underline{T^r_s(M)}$$

(Recall that $T^0_0(M) = C^\infty(M)$) and as with tensors there is a natural multiplication $\otimes$ defined on $\mathcal{T}(M)$. That is, if $T \in \underline{T^r_s(M)}$ and $S \in \underline{T^p_q(M)}$, then $T \otimes S \in \underline{T^{r+p}_{s+q}}(M)$ and is clearly C^∞ if T and S are. Thus $\mathcal{T}(M)$ is an algebra — the *algebra of tensor fields* on M. (Here we can restrict our considerations to C^∞ sections of $T^r_s(M)$).

For any algebra the set of *derivations* is usually of considerable interest. A *derivation* of $\mathcal{T}(M)$ is a linear map $D : \mathcal{T}(M) \longrightarrow \mathcal{T}(M)$ which satisfies

$$D(K \otimes K') = DK \otimes K' + K \otimes DK'$$

for any $K, K' \in \mathcal{T}(M)$. We now extend the Lie derivative to $\mathcal{T}(M)$ in the obvious way: Let ϕ_t be a local 1-parameter group of local transformations generated by $X \in \underline{T(M)}$, and as in section 6.2 consider

(6.27a) $$\tilde{\Phi}_t^* = \phi_{-t*} \cdots \otimes \phi_{-t*} \otimes \phi_t^* \otimes \cdots \otimes \phi_t^*$$

and define

(6.27b) $$\left(\underset{X}{\mathcal{L}} K\right)_p = \lim_{t \to 0} \frac{(\tilde{\Phi}_t^* K)_p - K_p}{t}$$

for any $K \in \mathcal{T}(M)$ and for all $p \in \operatorname{dom} \phi_t$. Notice that this reduces to (4.15)–(4.16) for $K \in C^\infty(M)$, $\underline{T^*(M)}$, $\underline{T(M)}$, respectively.

<u>Proposition</u> 6.14: The Lie derivative on $\mathcal{T}(M)$ is a type preserving (i.e. $\underset{X}{\mathcal{L}} K \in \underline{T^r_s(M)}$ if $K \in \underline{T^r_s(M)}$) C^∞-derivation which commutes with

contractions. This also holds on $\underline{S}(M)$ and $\underline{\Lambda}(M)$.

Proof: $\tilde{\Phi}_t^*$ is clearly type preserving and furthermore commutes with contractions (left as an exercise), so $\underset{X}{\mathcal{L}}$ is type preserving and commutes with contractions (i.e. $\underset{X}{\mathcal{L}} C = C \underset{X}{\mathcal{L}}$). For the derivation property we compute

$$\underset{X}{\mathcal{L}}(K \otimes K') = \lim_{t \to 0} \frac{1}{t}\left[\tilde{\Phi}_t^*(K \otimes K') - K \otimes K'\right]$$

$$= \lim_{t \to 0} \frac{1}{t}\left[(\tilde{\Phi}_t^* K) \otimes (\tilde{\Phi}_t^* K') - K \otimes K'\right]$$

$$= \lim_{t \to 0} \frac{1}{t}\left[(\tilde{\Phi}_t^* K) \otimes (\tilde{\Phi}_t^* K') - (\tilde{\Phi}_t^* K) \otimes K'\right]$$

$$+ \lim_{t \to 0} \frac{1}{t}\left[(\tilde{\Phi}_t^* K) \otimes K' - K \otimes K'\right]$$

$$= \lim_{t \to 0} (\tilde{\Phi}_t K) \otimes \frac{1}{t}\left[(\tilde{\Phi}_t^* K') - K'\right]$$

$$+ \lim_{t \to 0} \frac{1}{t}\left[(\tilde{\Phi}_t^* K) - K\right] \otimes K'$$

$$= \left(\underset{X}{\mathcal{L}} K\right) \otimes K' + K \otimes \left(\underset{X}{\mathcal{L}} K'\right).$$

Clearly $\underset{X}{\mathcal{L}}$ is C^∞ since it is C^∞ on $\mathcal{C}^\infty(M)$, $\underline{T}^*(M)$ and $\underline{T}(M)$. The operations used above also commute with $\mathcal{S}$ and $\mathcal{A}$ so this holds on $\underline{S}(M)$ and $\underline{\Lambda}(M)$ as well. Q.E.D.

Exercise: Show that Φ_t^* commutes with contractions.

Lemma 6.9: If two derivations D_1 and D_2 which commute with contraction coincide on $C^\infty(M)$ and $\underline{T(M)}$ or $C^\infty(M)$ and $\underline{T^*(M)}$, they coincide on all $\mathcal{T}(M)$. Furthermore, if D_1 and D_2 coincide on $C^\infty(M)$ and $\underline{\Lambda^1(M)} = \underline{T^*(M)}$, they coincide on $\underline{\Lambda(M)}$.

Proof: We first show that a derivation can be localized, i.e. if $K \in \mathcal{T}(M)$ vanishes on an open set U, so does DK. For let $p \in U$ and $f \in C^\infty(M)$ such that $f(p) = 0$ and $f = 1$ outside U. Then

$K = fK$ and hence, $DK = (Df)K + fDK$.

Evaluating at p gives $DK(p) = 0$. But $p \in U$ was arbitrary, so if $K = 0$ in U so is DK. Thus if two tensor fields K and K' coincide on U, then DK and DK' coincide on U. So if K^U denotes the restriction of K to U, then $DK^U = (DK)|_U$.

Now let $D = D_1 - D_2$ and assume D vanishes on $C^\infty(M)$ and $\underline{T(M)}$. Let $K \in \underline{T^r_s(M)}$ have the local expression

$$K = \sum K^{i_1 \cdots i_r}_{j_1 \cdots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

on U with chart (U, x). To show that $DK^U =$

$(DK)|_U = 0$, it suffices to show that $Ddx^i = 0$.

Let $X \in \underline{T(M)}$ be arbitrary, then

$$0 = D\big(X(dx^i)\big) = DC(X \otimes dx^i) = CD(X \otimes dx^i)$$

$$= C(X \otimes Ddx^i) \quad \text{since } DX = 0$$

But then $0 = C(X \otimes Ddx^i) = (Ddx^i)(X)$.

Assuming $D = 0$ on $C^\infty(M)$ and $\underline{T^1(M)} = \underline{\Lambda^1(M)}$, a similar argument shows that D vanishes on $\underline{T(M)}$ and by linearity on all $\underline{\Lambda(M)}$ as well as $\mathcal{T}(M)$. Q.E.D.

Another useful lemma is

<u>Lemma 6.10:</u> Let D be a derivation of $\mathcal{T}(M)$ which commutes with contractions and let $K \in \underline{T^r_s(M)}$. Then

$$DK(X_1, \dots, X_r) = D\big(K(X_1, \dots, X_r)\big) - \sum_{i=1}^{n} K(X_1, \dots, DX_i, \dots X_r)$$

Proof: Compute

$$D\big(K(X_1, \dots, X_r)\big) = DC(K \otimes X_1, \dots, X_r)$$

$$= CD(K \otimes X_1, \dots, X_r) = C(DK \otimes X_1 \otimes \dots \otimes X_r)$$

$$+ C\Big(K \otimes \sum_{i=1}^{n} X_1 \otimes \dots \otimes DX_i \otimes \dots \otimes X_r\Big)$$

$$= DK(Y_1, \ldots, Y_r) + \sum_{i=1}^{n} K(Y_1, \ldots, DY_i, \ldots, Y_r). \quad \text{Q.E.D.}$$

Since the above operations all commute with antisymmetrization $\mathcal{A}$ we have using $\underset{X}{\mathcal{L}} Y = [X, Y]$

<u>Corollary</u>: For $\omega \in \underline{\Lambda^r(M)}$,

$$\left(\underset{X}{\mathcal{L}} \omega\right)(Y_1, \ldots, Y_r) = X\left(\omega(Y_1, \ldots, Y_r)\right) - \sum_{i=1}^{r} \omega(Y_1, \ldots [X, Y_i], \ldots Y_r)$$

The following lemma follows immediately from the definition of $\underset{X}{\mathcal{L}}$ and proposition 6.6: (d commutes with pullback)

<u>Lemma 6.11</u>: For any $\omega \in \underline{\Lambda(M)}$ and $X \in \underline{T(M)}$,

$$\underset{X}{\mathcal{L}}\, d\omega = d\left(\underset{X}{\mathcal{L}} \omega\right)$$

Now in chapter 5 we introduced the interior multiplication $\lrcorner$ as the adjoint of $\wedge$. Thus if $X \in \underline{T(M)}$ and $\omega \in \underline{\Lambda^k(M)}$, then we define $X\lrcorner : \underline{\Lambda^k(M)} \longrightarrow \underline{\Lambda^{k-1}(M)}$ by

$$(6.28) \quad (X \lrcorner \omega)(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1})$$

Clearly $X\lrcorner$ is C^∞. Moreover, it is a <u>graded derivation</u> of degree -1, i.e. $X\lrcorner$ takes k-forms to $(k-1)$-forms and satisfies $X\lrcorner(\omega_1 \wedge \omega_2) = (X\lrcorner \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (X \lrcorner \omega_2)$ for $\omega_1 \in \underline{\Lambda^k(M)}$,

$\omega_2 \in \underline{\Lambda}(M)$. Clearly from theorem 6.8, d is a graded derivation of degree $+1$. We will see that there are important relations between $\underset{\mathbb{X}}{\mathcal{L}}$, d, and $\mathbb{X} \lrcorner$. First

Lemma 6.12: Let G_+ and G_- be two graded derivations of $\underline{\Lambda}(M)$ of degree ± 1, i.e. $G_i : \underline{\Lambda}^k(M) \to \underline{\Lambda}^{k\pm 1}(M)$ and

$$G_i(\omega_1 \wedge \omega_2) = (G_i \omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (G_i \omega_2)$$

for $i = \pm$ and $\omega_1 \in \underline{\Lambda}^k(M)$, then $G_+G_- + G_-G_+$ is a derivation of $\underline{\Lambda}(M)$. (degree 0)

Proof: $(G_+G_- + G_-G_+)(\omega_1 \wedge \omega_2) = G_+G_-(\omega_1 \wedge \omega_2) + G_-G_+(\omega_1 \wedge \omega_2)$

$$= G_+\left[G_-\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge G_-\omega_2\right] + G_-\left[G_+\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge G_+\omega_2\right]$$

$$= G_+G_-\omega_1 \wedge \omega_2 + \cancel{(-1)^{k\pm 1} G_-\omega_1 \wedge G_+\omega_2} + \cancel{(-1)^k G_+\omega_1 \wedge G_-\omega_2} + (-1)^{2k}\omega_1 \wedge G_+G_-\omega_2$$

$$+ G_-G_+\omega_1 \wedge \omega_2 + \cancel{(-1)^{k\pm 1} G_+\omega_1 \wedge G_-\omega_2} + \cancel{(-1)^k G_-\omega_1 \wedge G_+\omega_2} + (-1)^{2k}\omega_1 \wedge G_-G_+\omega_2$$

$$= \left[(G_+G_- + G_-G_+)\omega_1\right] \wedge \omega_2 + \omega_1 \wedge \left[(G_+G_- + G_-G_+)\omega_2\right]$$

Theorem 6.11: Let $\mathbb{X}, \mathbb{Y} \in \underline{T}(M)$ and $\omega \in \underline{\Lambda}(M)$, then

i) $$\underset{\mathbb{X}}{\mathcal{L}}\,\omega = \mathbb{X} \lrcorner\, d\omega + d(\mathbb{X} \lrcorner\, \omega)$$

ii) $$\underset{\mathbb{X}}{\mathcal{L}}(\mathbb{Y} \lrcorner\, \omega) = \left(\underset{\mathbb{X}}{\mathcal{L}}\,\mathbb{Y}\right) \lrcorner\, \omega + \mathbb{Y} \lrcorner\, \underset{\mathbb{X}}{\mathcal{L}}\,\omega$$

$$= [\mathbb{X}, \mathbb{Y}] \lrcorner\, \omega + \mathbb{Y} \lrcorner\, \underset{\mathbb{X}}{\mathcal{L}}\,\omega$$

Proof: i) Since $\underline{X}\lrcorner$ and d are graded derivations of degree ± 1 respectively, it follows from lemma 6.12 that $\underline{X}\lrcorner\, d + d(\underline{X}\lrcorner\,)$ is a derivation. Thus by lemmas 6.9 and prop. 6.14 it suffices to show that i) holds on $C^\infty(M)$ and $\underline{\Lambda}^1(M)$. On $C^\infty(M)$ $\underline{X}\lrcorner f = 0$ and $\mathcal{L}_{\underline{X}} f = \underline{X}\lrcorner\, df$ so i) holds on $C^\infty(M)$. Again we take $\omega \in \underline{\Lambda}^1(M)$ of the form $\omega = f\,dx^1$ and extend linearly. We have

$$\underline{X}\lrcorner\, d\omega + d(\underline{X}\lrcorner\,\omega) = \underline{X}\lrcorner \sum_{i=2}^{n} \frac{\partial f}{\partial x^i}\, dx^i \wedge dx^1 + d(\underline{X}^1 f)$$

$$= \sum_{i=2}^{n}\left(\frac{\partial f}{\partial x^i}\, \underline{X}^i dx^1 - \frac{\partial f}{\partial x^i}\, \underline{X}^1 dx^i\right) + \underline{X}^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\, dx^i + f\, d\underline{X}^1$$

$$= \underline{X}\lrcorner\, df + f\, d\underline{X}^1 = \sum_i \left(\underline{X}^i \frac{\partial f}{\partial x^i} dx^1 + f \frac{\partial \underline{X}^1}{\partial x^i} dx^i\right)$$

which is precisely $\mathcal{L}_{\underline{X}}\,\omega$ by theorem 4.12. Since ω and $\underline{X}$ are defined globally i) holds globally.

To prove ii) we notice that

$$\mathcal{L}_{\underline{X}}(\underline{Y}\lrcorner\,\omega) = \mathcal{L}_{\underline{X}}\, C(\underline{Y}\otimes\omega) = C\, \mathcal{L}_{\underline{X}}\, \underline{Y}\otimes\omega$$

$$= C\left[(\mathcal{L}_{\underline{X}}\underline{Y})\otimes\omega + \underline{Y}\otimes \mathcal{L}_{\underline{X}}\,\omega\right] = \omega(\mathcal{L}_{\underline{X}}\,\omega) + (\mathcal{L}_{\underline{X}}\,\omega)(\underline{Y})$$

where C is the obvious contraction, i.e. $C(\underline{X}\otimes\omega)(\underline{Y}_1, \dots, \underline{Y}_{k-1}) = \omega(\underline{X}, \underline{Y}_1, \dots, \underline{Y}_{k-1})$ for $\omega \in \underline{\Lambda}^k(M)$.

Q.E.D.

We will now use the Lie derivative to prove a result which is actually independent of it. In fact, the following theorem can be used as a definition of d!

<u>Theorem 6.12</u>: Let $X_0, \ldots, X_r \in \underline{T(M)}$ and $\omega \in \underline{\Lambda^r(M)}$, then

$$d\omega(X_0, \ldots, X_r) = \sum_{i=0}^{r} (-1)^i X_i\left(\omega(X_0, \ldots, \hat{X}_i, \ldots X_r)\right)$$

$$+ \sum_{0 \leq i < j \leq r} (-1)^{i+j} \omega\left([X_i, X_j], X_0, \ldots \hat{X}_i, \ldots \hat{X}_j, \ldots, X_r\right)$$

where $\hat{X}$ means remove that element.

<u>Proof:</u> By induction on r. For $r=0$ we have $\omega = f \in C^\infty(M)$ and the formula reads

$$df(X_0) = X_0 f$$

which is clearly the case. Now we write

$$d\omega(X_0, \ldots, X_r) = C(d\omega \otimes X_0 \otimes \cdots \otimes X_r) = (X_0 \lrcorner d\omega)(X_1, \ldots, X_r)$$

$$= \left(\underset{X}{\mathcal{L}}\, \omega\right)(X_1, \ldots, X_r) - d\left((X_0 \lrcorner \omega)\right)(X_1, \ldots, X_r)$$

by theorem 6.11. But by the corollary to lemma 6.10, this is

$$X_0\left(\omega(X_1, \ldots, X_r)\right) - \sum_{i=1}^{r} \omega(X_1, \ldots, [X_0, X_i], \ldots, X_r)$$

$$-\left(d(X_0 \lrcorner \omega)\right)(X_1, \ldots, X_r)$$

But by the induction hypothesis

$$d(X_0 \lrcorner \omega)(X_1, \dots, X_r) = \sum_{i=1}^{r} (-1)^{i-1} X_i((X_0 \lrcorner \omega)(X_1, \dots \hat{X}_i, \dots X_r))$$

$$+ \sum_{1 \le i < j \le r} (-1)^{i+j} (X_0 \lrcorner \omega)([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X_r)$$

Combining these formulas (keeping track of minus signs) gives the result. Q.E.D.

We write the important case $r=1$ of theorem 6.12 explicitly

$$(6.29)\quad d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

This gives a sort of duality between the Lie bracket and the exterior derivative. In fact let $\{X_a\}$ be a local moving frame on M and $\{\theta^a\}$ its dual moving coframe, then (6.29) becomes

$$d\theta^c(X_a, X_b) = \theta^c([X_a, X_b])$$

So if

$$[X_a, X_b] = \sum_c C^c_{ab} X_c$$

$C^c_{ab} \in C^\infty(U)$, we have also

$$d\theta^a = \sum_{b<c} C^a_{bc} \theta^b \wedge \theta^c$$

An important role of the Lie derivative is that of infinitesimal automorphism. We will now develop this concept. Let K be a tensor field of type (r,s), i.e. $K \in \underline{T^r_s(M)}$. A local diffeomorphism $\phi : M \longrightarrow M$ with dom ϕ and range ϕ some open sets of M is called a <u>local automorphism</u> of K if $\tilde{\Phi}^* K = K$. Similarly a global diffeomorphism $\phi : M \longrightarrow M$ is called a <u>global automorphism</u> or simply <u>automorphism</u> of K if $\tilde{\Phi}^* K = K$. The following proposition is straightforward to prove and is left as an exercise.

<u>Proposition 6.15</u>: The set $\mathrm{Aut}\, K$ of global automorphisms of K is a transformation group on M. The set $\mathrm{Aut}_{loc} K$ of local automorphisms of K is a pseudogroup of transformations on M.

Remark. $\mathrm{Aut}\, K$ is <u>not</u> necessarily a Lie transformation group.

We now wish to introduce the concept of infinitesimal automorphism. A vector field $\underline{X}$ on M is an <u>infinitesimal automorphism</u> of K if

$$\underset{\mathfrak{X}}{\mathcal{L}} K = 0$$

This name is justified by

<u>Proposition 6.16</u>: Let $\mathfrak{X} \in \underline{T(M)}$ and ϕ_t be the local 1-parameter group generated by $\mathfrak{X}$. Then for any tensor field K, $\mathfrak{X}$ is an infinitesimal automorphism of K if and only if ϕ_t is a local automorphism of K.

Proof: The if part is trivial and the only if part follows as a corollary of the next lemma

<u>Lemma 6.13</u>: Let $\mathfrak{X}$, ϕ_t, and K be as in the preceeding proposition. Then

$$\underset{\mathfrak{X}}{\mathcal{L}}\left(\tilde{\phi}_s^* K\right) = \tilde{\phi}_s^*\left(\underset{\mathfrak{X}}{\mathcal{L}} K\right)$$

Proof: We want to show that

$$\underset{\mathfrak{X}}{\mathcal{L}} = \tilde{\phi}_{-s}^* \circ \underset{\mathfrak{X}}{\mathcal{L}} \tilde{\phi}_s^*$$

It is easy to check that the right hand side is a derivation on $\mathcal{T}(M)$ which preserves types and commutes with contractions. Therefore, by

lemma 6.9
n it suffices to prove the formula on $C^\infty(M)$ and $\underline{T(M)}$. On $C^\infty(M)$ we must show that $Xf = \Phi_{-s*} X(f \circ \Phi_s)$ and on $\underline{T(M)}$ that $[X, Y] = [\Phi_{-s*} X, Y]$. But both of these follow from $\Phi_{-s*} X = X$, and we will prove this.
We have

$$(\Phi_{s*} X)_p f = \Phi_{s*} X_{\Phi_{-s}(p)} f = X_{\Phi_{s}(p)} f \circ \Phi_s$$

$$= \lim_{t \to 0} \frac{f \circ \Phi_s \circ \Phi_t \circ \Phi_{-s}(p) - f \circ \Phi_s \circ \Phi_{-s}(p)}{t}$$

$$= \lim_{t \to 0} \frac{f(\Phi_t(p)) - f(p)}{t} = X_p f \qquad \text{Q.E.D.}$$

To finish the proof of proposition 6.16, we notice that

$$\underset{X}{\mathcal{L}} (\tilde{\Phi}_s^* K) = \left(\frac{d}{dt} \tilde{\Phi}_t^* K \right)_{t=s}$$

So using the lemma $\underset{X}{\mathcal{L}} K = 0$ implies that $\tilde{\Phi}_t^* K$ is independent of t. Thus $\tilde{\Phi}_t^* K = K$ since $\tilde{\Phi}_0^* = id^*$.

Remarks. In the special case when $K = g$, the metric tensor, the infinitesimal automorphisms are called Killing vectors and the (local) automorphisms are, of course, (local) isometries. In this case it can be shown that every local isometry ϕ can be given in terms of a local 1-parameter group ϕ_t. This is not true in general.

Exercise. Prove the following and its corollary:

Proposition 6.17: The Lie derivative $\underset{\mathbf{X}}{\pounds}$ is a Lie algebra homomorphism in the sense that

$$\underset{[\mathbf{X},\mathbf{Y}]}{\pounds} = [\underset{\mathbf{X}}{\pounds}, \underset{\mathbf{Y}}{\pounds}] \equiv \underset{\mathbf{X}}{\pounds}\underset{\mathbf{Y}}{\pounds} - \underset{\mathbf{Y}}{\pounds}\underset{\mathbf{X}}{\pounds}$$

Corollary: The set $\mathcal{L}_K$ of infinitesimal automorphisms is a Lie subalgebra of $\mathcal{X}(M)$.

Let Ω be a volume element on the orientable manifold M. Consider the Lie algebra $\mathcal{L}_\Omega(M)$ (or just $\mathcal{L}_\Omega$) of volume preserving infinitesimal automorphisms. We denote by $\mathcal{L}_\Omega^{(1)}$ the derived subalgebra, i.e. $\mathcal{L}_\Omega^{(1)} = \{[\mathbf{X},\mathbf{Y}] : \mathbf{X},\mathbf{Y} \in \mathcal{L}_\Omega\}$.

Proposition 6.18: With the above notation assumed, there is a vector space isomorphism ψ between $\mathcal{L}_\Omega(M)$ and the vector space $Z^{n-1}(M)$ of closed $n-1$ forms on M. Moreover, ψ restricted to $\mathcal{L}^{(1)}_\Omega(M)$ is a monomorphism into the space $B^{n-1}(M)$ of exact $(n-1)$-forms on M. Hence, there is a vector space epimorphism $\psi' : \mathcal{L}_\Omega / \mathcal{L}^{(1)}_\Omega \longrightarrow H^{n-1}(M)$ (de Rham cohomology).

Proof: On M the volume element Ω vanishes nowhere, so $\psi(\mathbf{X}) = \mathbf{X} \lrcorner \Omega$ establishes an isomorphism (as $C^\infty(M)$-modules) between $\underline{T(M)}$ and $\underline{\Lambda^{n-1}(M)}$. (check this). But $\mathbf{X} \in \mathcal{L}_\Omega(M)$ if and only if

$$0 = \underset{\mathbf{X}}{\pounds} \Omega = d(\mathbf{X} \lrcorner \Omega)$$

using (ii) of theorem 6.11. This establishes the isomorphism between $\mathcal{L}_\Omega$ and Z^{n-1}. But also by (iii) of theorem 6.11 if $\mathbf{X}, \mathbf{Y} \in \mathcal{L}_\Omega(M)$

$$\underset{\mathbf{X}}{\pounds}(\mathbf{Y} \lrcorner \Omega) = [\mathbf{X}, \mathbf{Y}] \lrcorner \Omega$$
$$= d(\mathbf{X} \lrcorner \mathbf{Y} \lrcorner \Omega)$$

Thus to every element $[\mathbf{X}, \mathbf{Y}] \in \mathcal{L}^{(1)}_\Omega$ there is an exact $(n-1)$-form. To establish the last result we define $\psi'([X]) = [\psi(X)]$ for $X \in \mathcal{L}_\Omega$. Since ψ maps $\mathcal{L}_\Omega^{(1)}$ into $B^{n-1}(M)$ this map is well-defined (i.e depends only on the class $[X]$). Moreover, if $[\alpha] \in H^{n-1}(M)$ for any representative $\alpha \in Z^{n-1}(M)$ there is an $X \in \mathcal{L}_\Omega$ such that $\psi(X) = \alpha$. But then $\psi'([X]) = [\psi(X)] = [\alpha]$ and this only depends on the class of α, so ψ' is surjective. Clearly, ψ and ψ' are linear maps. Q.E.D.

<u>Remark.</u> It is easy to see (show it) that $\mathcal{L}_\Omega^{(1)}$ is an ideal of $\mathcal{L}_\Omega$, so $\mathcal{L}_\Omega / \mathcal{L}_\Omega^{(1)}$ itself is a Lie algebra. A Lie algebra is called <u>simple</u> if it has no proper ideals (i.e ideals other than 0 and itself). Thus if $\mathcal{L}_\Omega(M)$ is simple, $\mathcal{L}_\Omega^{(1)}(M) = \mathcal{L}_\Omega(M)$ (or $\mathcal{L}_\Omega^{(1)}(M) = 0$ which means $\mathcal{L}_\Omega(M)$ is abelian which cannot happen if $\dim M > 1$). Thus we have

<u>Corollary</u>: If $H^{n-1}(M) \neq 0$, $\mathcal{L}_\Omega(M)$ is not simple.

Exercise: Define $\mathcal{L}'_{\Omega}(M) = \{X \in \mathcal{L}_{\Omega}(M) : X \lrcorner \Omega \text{ is exact}\}$. Show that $\mathcal{L}'_{\Omega}$ is an ideal of $\mathcal{L}_{\Omega}$ which contains $\mathcal{L}^{(1)}_{\Omega}$ and that $\mathcal{L}_{\Omega}/\mathcal{L}'_{\Omega}$ is isomorphic to $H^{n-1}(M)$.

Exercise: Let $M = \mathbb{R}^n$ and x be the Cartesian chart. Show that $\mathcal{L}_{\Omega}(\mathbb{R}^n)$ is the subalgebra of $\mathfrak{X}(M)$ such that

$$\operatorname{div} X = \sum \frac{\partial X^i}{\partial x^i} = 0$$

Remark. $\mathcal{L}_{\Omega}(\mathbb{R}^n)$ is known to be simple. In general it would be of interest to know for which M $\mathcal{L}'_{\Omega}(M) = \mathcal{L}^{(1)}_{\Omega}(M)$.

Example 6.21: $M = T^2 = S^1 \times S^1$. Here one can show that $H^1(T^2) = \mathbb{R} \oplus \mathbb{R}$. As with the circle any closed 1-form can be written $\alpha = dg + c_1 d\theta_1 + c_2 d\theta_2$ where θ_1 and θ_2 are the usual coordinates on the torus. We have $\Omega = d\theta_1 \wedge d\theta_2$, so $\partial_{\theta^1} \lrcorner \Omega = d\theta_2$ and $\partial_{\theta^2} \lrcorner \Omega = -d\theta_1$ are not exact, so $\partial_{\theta^1}, \partial_{\theta^2} \in \mathcal{L}_{\Omega}(T^2)/\mathcal{L}'_{\Omega}(T^2) \subset \mathcal{L}_{\Omega}(T^2)/\mathcal{L}^{(1)}(T^2)$.
Clearly, $\mathcal{L}_{\Omega}(T^2)$ is not simple.

<u>Exercise</u>: Generalize proposition 6.16 as follows: ϕ is called a local <u>conformal automorphism</u> if $\hat{\phi}^* H = \sigma H$ for some $\sigma \in C^\infty(M)$. $\underline{X}$ is called an <u>infinitesimal conformal automorphism</u> if $\mathcal{L}_{\underline{X}} H = \lambda H$ for some $C^\infty(M)$-function λ. With ϕ_t, $\underline{X}$, and H the same as proposition 6.16, prove that ϕ_t is a local conformal automorphism if and only if $\underline{X}$ is an infinitesimal conformal automorphism. Show that the set of local (infinitesimal) conformal automorphisms forms a pseudogroup (Lie algebra). When $H = g$, a pseudo-Riemannian metric, $\underline{X}$ is called a <u>conformal Killing vector</u>.