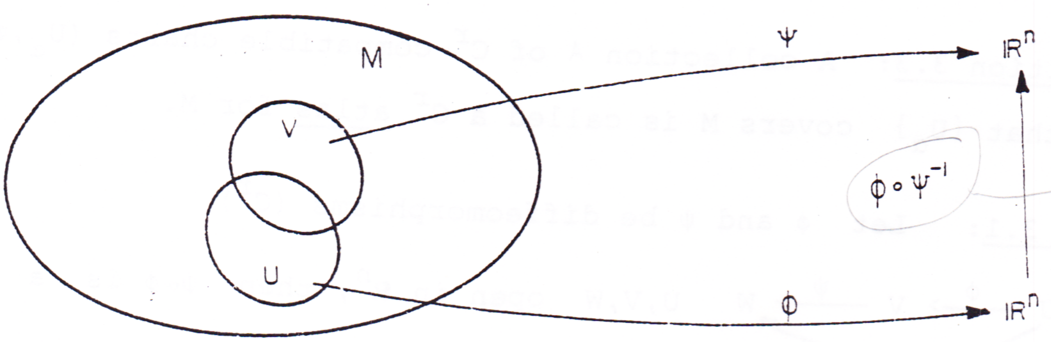
3. Differential Manifolds

3.1 General Definition and Examples.

I will assume that everyone is familiar with the theory of differentiable functions in $\mathbb{R}^n$. Unless otherwise stated we will deal with ordinary manifolds, i.e. not manifolds with boundary. Now consider a top manifold M. Each point $p\varepsilon M$ has associated with it a nbd. U_p and a homeomorphism $\phi : U_p \longrightarrow \mathbb{R}^n$. Now in $\mathbb{R}^n$ we have the natural coordinates x^i, thus we can construct ϕ so that it assigns to each $q\varepsilon U_p$ the coordinates $x^i(q)$, that is the image of q under ϕ is $x^i(q)$. (Remember ϕ has its values in $\mathbb{R}^n$). We refer to x^i as <u>local coordinates</u> for $U_p \subset M$, and the pair (U_p, ϕ) is called a <u>chart</u> at $p\varepsilon M$. In fact, we used this procedure many times in the last section. Now we would like to be able to define the notion of a differentiable function on M. Suppose f is a continuous function on M, $f : M \longrightarrow \mathbb{R}$. We know what this means, namely, that $f \circ \phi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous for a cover of charts for M. Now suppose that $f \circ \phi^{-1}$ is differentiable and let (V, ψ) be another chart for M such that $U \cap V \neq \emptyset$. We do not know that $f \circ \psi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable. In fact it is so if $\phi \circ \psi^{-1}$ is since $f \circ \psi^{-1} = f \circ \phi^{-1} \circ \phi \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$.



transition map

$\phi \circ \psi^{-1}$

$\phi(U \cap V)$

$\downarrow$

$\rightarrow \phi(U \cap V)$

Thus in order define differentiable functions on M we must demand more from our homeomorphisms ϕ and ψ. This leads to

Definition 3.1: A homeomorphism $f:U \longrightarrow V$, where $U,V \subset \mathbb{R}^n$ are open subsets of $\mathbb{R}^n$, is called a diffeomorphism of class C^r $(r=1,\dots,\infty)$ if both f and f^{-1} are differentiable to order r. We can also extend this to the class C^ω of analytic diffeomorphisms, i.e. f and f^{-1} are both required to have power series expansions. We will mainly be concerned with the case $r = \infty$.

Example: Let $f:\mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^3$. f is a homeomorphism and f is a C^∞ function (it is analytic), but f is not a diffeomorphism since $f^{-1}(y) = y^{1/3}$ which is not C^1 at $y = 0$. However; $f:R_+ \longrightarrow R_+$ is a diffeomorphism.

By the way, the following inclusions are clear from the definitions

$$C^\omega \subset C^\infty \subset C^r \subset C^{r-1} \subset \dots \subset C^1 \subset C^0 .$$

Shortly we will give examples to show that $C^\infty \neq C^\omega$.

Definition 3.2: Two charts (U,ϕ) and (V,ψ) are C^r $\overset{r=\infty}{(\text{smoothly})}$ compatible if whenever $U \cap V \neq \emptyset$, $\phi \circ \psi^{-1}$ is a diffeomorphism of class C^r. Notice that if $\phi \circ \psi^{-1}$ is a C^r-diffeomorphism then $(\phi \circ \psi^{-1})^{-1} = \psi \circ \phi^{-1}$ is also. (Note if $U \cap V = \emptyset$ then they are C^r compatible)

Definition 3.3: A collection A of C^r-compatible charts (U_α, ϕ_α) such that $\{U_\alpha\}$ covers M is called a C^r atlas for M.

Lemma 3.1: Let ϕ and ψ be diffeomorphisms (C^r)

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[rr, bend right, "\psi \circ \phi"'] & V \arrow[r, "\psi"] & W \end{tikzcd}$$

U,V,W open in $\mathbb{R}^n$, then $\psi \circ \phi$ is a diffeomorphism.

Proof: By the chain rule $\psi\circ\phi$ is C^r. Both $\psi\circ\phi$ and $(\psi\circ\phi)^{-1} = \phi^{-1}\circ\psi^{-1}$ are homeomorphisms. Moreover, $\phi^{-1}\circ\psi^{-1}$ is class C^r by the chain rule for the C^r maps ϕ^{-1} and ψ^{-1}. Thus $\psi\circ\phi$ is a diffeomorphism.

If an atlas A is not contained in any other atlas, A is called maximal.

Theorem 3.1: If A is any atlas of C^r compatible charts, then A is contained in a unique maximal atlas A'.

Proof: Define A' to be the set of all charts (U',ϕ') which are C^r-compatible with all charts (U,ϕ) of A. Since A' contains A it clearly covers M. The collection A' is clearly unique and maximal. We need to show that all charts of A' are C^r-compatible. Let (U',ϕ') and (V',ψ') be any charts of A'. We can assume $U'\cap V' \neq \{\emptyset\}$. By definition of A', if $U'\cap U \neq \{\emptyset\}$ then $\phi'\circ\phi^{-1}$ is a diffeomorphism for any two charts of A' and A. Thus with $V'\cap U \neq \{\emptyset\}$, $\psi'\circ\phi^{-1}$ is a diffeomorphism. In $U'\cap V'\cap U \neq \{\emptyset\}$ we have $(\psi'\circ\phi^{-1})\circ(\phi\circ\phi'^{-1}) = \psi'\circ\phi'^{-1}$ is a diffeomorphism by lemma 3.1 above. Thus A' is an atlas.

Definition 3.4: A differentiable or C^r-manifold is a pair (M,A) where M is a top manifold and A is a maximal atlas.

Remark. In order to show that a given top. manifold M is a C^r-manifold, theorem 3.1 tells us we need only find any atlas of C^r-compatible charts. We will usually just refer to M as a differentiable manifold and not write the pair (M,A). In the future the word manifold will always mean differentiable manifold of class C^∞ unless stated otherwise.

Definition 3.5: Let M, and M' be C^r-manifolds and $f:M \longrightarrow M'$ a continuous map. We say that f is differentiable of class C^r if for every point p in the domain of f, the coordinate representative $\phi' \circ f \circ \phi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^{n'}$ is C^r, where ϕ and ϕ' are any charts for M and M' at p and f(p), respectively.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\phi"'] & M' \arrow[d, "\phi'"] \\
\mathbb{R}^n \arrow[r, "\phi' \circ f \circ \phi^{-1}"] & \mathbb{R}^{n'}
\end{tikzcd}$$

Remark. Notice that this definition is actually independent of the coordinate representative. This emphasizes the distinction between the modern and classical approaches to differential geometry. In the modern approach almost all definitions are independent of coordinates. After all a sphere, for example, is something intrinsic which does not depend on coordinates in the same way that one studies Euclidean spaces in high school geometry with no notion of coordinates.

Definition 3.6: A diffeomorphism $f: M \longrightarrow M'$ is an injection (1-1) such that both f and f^{-1} are class C^r. Two manifolds are said to be diffeomorphic if there exists a global diffeomorphism between them, i.e. f is a bijection and dom f = M.

Notice that the same top manifold may be given different differentiable structures. For example, if $M = \mathbb{R}^n$ we can ask whether it can have inequivalent smooth structures (say of class C^∞). The answer is quite surprising. There is a unique differentiable structure on $\mathbb{R}^n$ for all $n \neq 4$. When $n = 4$ it follows from the work of M. Freedman and S. Donaldson in the mid 1980s that there are uncountably many inequivalent smooth structures on $\mathbb{R}^4$! Distinct differentiable structures on manifolds are often called *diffeomorphism types.* A new branch of mathematics called *differential topology* was created when J. Milnor in 1956 showed that there are inequivalent differentiable structure on S^7. Later Kervaire and Milnor treated differential structures on spheres of all dimensions. In particular, they showed that there are precisely 28 oriented differentiable structures on S^7, the standard one and 27 *exotic* structures. Today the study of differential topology is a vibrant one, and determining the distinct diffeomorphism types that occur on a given topological manifold is both interesting and important. Again dimension 4 behaves very differently than other dimensions. See the book: "The Wild World of 4-Manifolds" by A. Scorpan, AMS 2005. Note that there are topological manifolds that do not admit any differentiable structure.

added 1/26/11

Before considering many examples of C^∞-manifolds, I want to convince you that C^∞ and C^ω are indeed different classes.

Example 3.1: The following functions are C^∞ but not analytic:

1)

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$\leq$

$f : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$

That f is C^∞ at $x = 0$ follows from the fact (well-known from calculus) that $x^n e^{-x} \longrightarrow 0$ as $x \longrightarrow \infty$ for any n.(check this). But f is *not* analytic at $x = 0$.

2) $g: \mathbb{R} \longrightarrow \mathbb{R}$

$$g(x) = \begin{cases} e^{-\frac{1}{(x^2-a^2)}} & -a < x < a \\ 0 & x \geqslant a \text{ and } x \leqslant a \end{cases}$$

Again $g(x)$ is C^∞ at $x = \pm a$, in fact $g(x)$ has <u>compact support</u>, i.e. $g(x) \neq 0$ only in a bounded region. Clearly g is not analytic at $x = \pm a$. The only analytic function which vanishes on an open set is the zero function.

<u>Example 3.2</u>: $\mathbb{R}^n$: since $\mathbb{R}^n$ is covered by an atlas which consists of one chart, the identity chart, it is a C^∞ manifold, in fact analytic.

<u>Theorem 3.2</u>: An open subset A of a C^∞-manifold M is a C^∞-manifold.

Proof: Let $\mathcal{A}$ be an atlas on M with charts (U_α, ϕ_α). Then $V_\alpha = A \cap U_\alpha$ are open in A and $\psi_\alpha = \phi_\alpha\big|_A$ are homeomorphism onto open sets of $\mathbb{R}^n$. Check that the required atlas is the collection of charts (V_α, ψ_α).

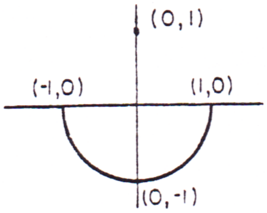
We will now consider examples of C^∞-manifolds. In particular, the top manifolds of the previous section can be given a C^∞-structure.

<u>Example 3.3</u> : M^{nm} is a C^∞- manifold since it is homeomorphic to R^{nm} and we give R^{nm} the identity atlas consisting of one chart, the identity chart.

<u>Example 3.4</u>: GL(n,R) is a C^∞- manifold by Theorem 3.2. Group

composition given by matrix multiplication is thus seen to be a C^∞ - mapping of $GL(n,R) \times GL(n,R) \longrightarrow GL(n,R)$. The inverse is also C^∞. This is our first example of a <u>Lie group</u> which we will define later.

<u>Example 3.5</u>: The circle S^1. The homeomorphisms of example 1 of section 2 are $f:(0,2\pi) \longrightarrow S^1-(1,0)$ and $f':(-\pi,\pi) \longrightarrow S^1-(-1,0)$ with $f(\theta) = (\cos\theta, \sin\theta) = f'(\theta)$. We thus have an atlas consisting of two charts (v,ϕ) and (U',ϕ') where $U = S^1-(1,0)$, $v' = S^1-(-1,0)$, $\phi = f^{-1}$ and $\phi' = f'^{-1}$. Clearly $S^1 = U \cup U'$ and $U \cap U' = S^1 -(1,0) \cup (-1,0)$. We must show that the two charts are C^∞- compatible, that is that $\phi' \circ \phi^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^∞ diffeomorphism. But $\phi' \circ \phi^{-1} = f'^{-1} \circ f$ and for $0<\theta<\pi$ we have $f'^{-1} \circ f\ (\theta) = \theta$, that is $\phi' \circ \phi = f'^{-1} \circ f$ is identity on $0<\theta<\pi$. This is clearly a diffeomorphism. Now on $\pi<\theta<2\pi$, we have that (x,y) starts at $(-1,0)$ and ends at $(1,0)$ through negative values of y, that is, traverses the lower semicircle N



$$N = \left\{(x,y) \varepsilon S^1 : \begin{matrix} -1<x<1 \\ 0<y\leqslant 1 \end{matrix}\right\}$$

But $\phi'(N) = f'^{-1}(N) = \{\theta \varepsilon (-\pi,\pi) : -\pi<\theta<0\}$. So $\phi' \circ \phi^{-1}(\theta) = \theta - 2\pi$. This is clearly a C^∞ diffeomorphism. Thus with this atlas S^1 is a C^∞ - manifold, in fact analytic.

<u>Example 3.6</u>: <u>The sphere S^n</u>: The homeomorphisms from example 2.2 are

$$y:\overbrace{S^n - (0,\dots,1)}^{U} \longrightarrow R^n \qquad y_i(x) = \frac{x_i}{1-x_{n+1}}$$

$$y':\underbrace{S^n - (0,\dots,-1)}_{U'} \longrightarrow R^n \qquad y_i'(x) = \frac{x_i}{1+x_{n+1}}$$

We must show that $y' \circ y^{-1}$ is a C^∞ diffeomorphism. The atlas consists of the two charts (U,y) and (U',y'). This is called the stereographic atlas. $U \cap U' = S^n - (0,\dots,1) \cup (0,\dots,-1)$

We have

$$y_i' = y_i \frac{1-x_{n+1}}{1+x_{n+1}} \quad \text{and} \quad \sum_{i=1}^{n} y_i^2 = \frac{\sum x_i^2}{(1-x_{n+1})^2} = \frac{(1-x_{n+1}^2)}{(1-x_{n+1})^2}$$

$$= \frac{1+x_{n+1}}{1-x_{n+1}}$$

So

$$y_i' = \frac{y_i}{\sum_{i=1}^{n} y_i^2}$$

Notice that $y(U \cap U') = \mathbb{R}^n - \{0\}$, so on $U \cap U'$ $y \neq (0,\dots,0)$. Thus $y_i':\mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\}$ is a continuous bijection. It is also C^∞; moreover, the inverse map is

$$y_i = \frac{y'_i}{\sum\limits_i {y'_i}^2}$$

which is also C^∞. Thus the charts are C^∞ compatible, and S^n is a C^∞ - manifold .

Exercise: Show that for $n = 1$ the stereographic atlas is C^∞ - compatible with the atlas of Example 3.5.

Example 3.7: The projective plane $P^n(R)$: We work with the charts given in example 2.5. We have an atlas which consists of n+1 charts (U_i, ϕ_i) where $U_i = \rho(\tilde{U}_i)$, $\tilde{U}_i = \{x \varepsilon R^{n+1} - \{0\} : x^i \neq 0\}$ and

$$\phi_i([x]) = (\frac{x_1}{x_i}, \ldots \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_{n+1}}{x_i}) = (z_1, \ldots, z_n)$$

We consider two such maps ϕ_i , ϕ_j on $U_i \cap U_j$.

We have

$$z_1 = \frac{x_1}{x_i}, \ldots, z_{i-1} = \frac{x_{i-1}}{x_i}, z_i = \frac{x_{i+1}}{x_i}, \ldots, z_n = \frac{x_{n+1}}{x_i}$$

and

$$z'_1 = \frac{x_1}{x_j}, \ldots, z'_{j-1} = \frac{x_{j-1}}{x_j}, z'_j = \frac{x_{j+1}}{x_j}, \ldots, z'_n = \frac{x_{n+1}}{x_j}$$

$j > i : \frac{x_j}{x_i} = z_{j-1}$

so that for $j > i$ $z'_1 = \frac{x_1}{x_j} = \frac{x_i}{x_j}\frac{x_1}{x_i} = \frac{z_1}{z_{j-1}}$

$$z'_1 = \frac{z_1}{z_{j-1}} \quad , \ldots , \quad z'_{i-1} = \frac{z_{i-1}}{z_{j-1}} \quad , \quad z'_i = \frac{1}{z_{j-1}} \quad , \quad z'_{i+1} = \frac{z_i}{z_{j-1}}$$

$$\ldots \quad z'_{j-1} = \frac{z_{j-2}}{z_{j-1}} \quad , \quad z'_j = \frac{z_j}{z_{j-1}} \quad , \ldots , \quad z'_n = \frac{z_n}{z_{j-1}}$$

This is a continuous bijection from $\mathbb{R}^n - \{z^{j-1} = 0\}$ onto $\mathbb{R}^n - \{z^i = 0\}$. It is also clearly C^∞ there. The inverse mapping has a similar form and is C^∞ . Thus $P^n(\mathbb{R})$ is a differentiable manifold.

<u>Exercise</u>: Show that the atlas of two charts for the Möbius strip of example 2.4 is C^∞ compatible and thus the Möbius strip is a C^∞ manifold.

<u>Exercise</u>: Show that the Grassman manifold $G(k,n)$ is a C^∞ manifold. (Use the same chart as the exercise of the previous chapter).

<u>Proposition 3.1</u>: Let the top space S be compact and be given an equivalence relation ~. Then S/~ is compact.

Proof: Since $\rho : S \longrightarrow S/\sim$ is a continuous surjection, this follow immediately from theorem 1.7.

Corollary: $P^n(\mathbb{R})$ is compact

It is clear that any compact manifold cannot have an atlas of one chart only, since then it would have to be globally homeomorphic to $\mathbb{R}^n$.

<u>Proposition 3.2</u>: Any C^∞ manifold M has a countable atlas.

Proof: This follows immediately from the Lindelof property that for any top space with a countable basis for its topology, every open cover has a countable subcover. (See a topology book).

<u>Proposition 3.3</u>: Any C^∞-compact manifold has a finite atlas.
Proof: clear from the definitions.

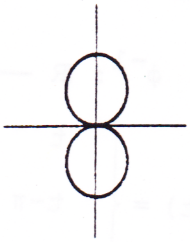
<u>Proposition 3.4</u>: $P^n(\mathbb{R})$ is connected
Proof: The sphere S^n is connected and the map $\rho: S^n \longrightarrow P^n(\mathbb{R})$ is continuous. But we saw in section 1 that the continuous image of a connected space is connected.

<u>Exercise</u>: Prove that the Cartesian product of two C^∞- manifolds is a C^∞ manifold.

<u>Exercise</u>: Construct an atlas of two charts for $\mathbb{R}^2$ using polar coordinates. The two charts should cover $\mathbb{R}^2 - \{(x>0,0)\}$ and $R^2 - \{(x<0,0)\}$ respectively. Show that this atlas is C^∞-compatible with the standard Cartesian atlas for R^2.

<u>Example 3.8</u>: The <u>figure 8</u>: This is defined by

$$E = \{(x,y) \varepsilon \mathbb{R}^2 : (x,y) = (\sin 2t, \sin t)\}$$



The injection $\phi: E \longrightarrow \mathbb{R}$ defined by

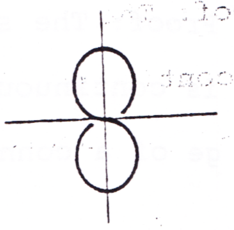
$\phi(\sin 2t, \sin t) = t$ for $0<t<2\pi$ notice $t \neq 0$ or 2π

is a chart whose domain is all of E and whose range is an open interval of $\mathbb{R}$. However, ϕ is <u>not</u> continuous if we give E the relative topology. For clearly in the relative topology (0,0) and $(\sin 2\varepsilon \sin\varepsilon)$ are close for ε small, that is we have

$\sin \varepsilon < \frac{\delta}{2} < \delta$ and $\sin 2\varepsilon < (2\cos\varepsilon)\frac{\delta}{2} < \delta$ but

$$|\phi(\sin 2\varepsilon, \sin\varepsilon) - \phi(0,0)| = |\varepsilon - \pi|$$

We can, however, define E to be the inverse image $\phi^{-1}(t)$ for $0<t<2\pi$ and the topology on E is that which makes ϕ and ϕ^{-1} continuous. In this topology $(\sin 2\varepsilon, \sin\varepsilon)$ and $(0,0)$ are far away.

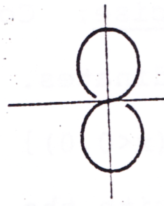


The picture for this is:

In fact ϕ and ϕ^{-1} are now C^∞ and this defines an atlas of 1 chart for E and E is globally homeomorphic to $\mathbb{R}$. Thus E is noncompact.

Similarly, we can give E the topology induced from the map.

$$\phi'(\sin 2t, \sin t) = t \quad \text{for} \quad -\pi<t<\pi$$



This corresponds to the picture:

This map also converts E into a C^∞-manifold. (E,ϕ') is homeomorphic to R. Thus (E,ϕ) and (E,ϕ') are homeomorphic.

They are <u>not</u>, however, C^∞ - compatible since

$$\phi' \circ \phi^{-1} : \mathbb{R} \longrightarrow \mathbb{R} \quad \text{is}$$

$$\phi'\circ\phi^{-1}(t) = \begin{cases} t & 0<t<\pi \\ t-\pi & t = \pi \\ t-2\pi & \pi<t<2\pi \end{cases}$$

and this is not even continuous.

The previous discussion leads us to believe that the important concept of "submanifold" may have sutle distinctions depending on the way we insert a given manifold into another given manifold. Indeed this is the case.

Exercise: Show that the one-sheeted and two-sheeted hyperboloids are C^∞- manifolds. See examples 2.9 and 2.10.

The following exercise will show that two atlases which are not C^∞ - compatible can be diffeomorphic.

Exercise. Consider the real line $\mathbb{R}$ and the usual atlas $(\mathbb{R},\phi)$ where ϕ is the identity map. Now consider $\mathbb{R}$ with the atlas of one chart $(\mathbb{R},\psi)$ where $\psi(t) = t^3$. Show that $(\mathbb{R},\phi)$ and $(\mathbb{R},\psi)$ are not C^∞ - compatible. However construct a diffeomorphism between $(\mathbb{R},\phi)$ and $(\mathbb{R},\psi)$ showing that they are diffeomorphic.

$(\mathbb{R}, \phi)$ $\phi(t) = t$

$(\mathbb{R}', \psi)$ $\psi(t) = t^3$

$\psi \circ \phi^{-1}(t) = t^3$ $\phi \circ \psi^{-1}(t) = t^{1/3}$ not smooth

Construct differ

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f"] \arrow[d, "\phi"'] & \mathbb{R}' \arrow[d, "\psi"] \\
\mathbb{R} \arrow[r, "\psi \circ f \circ \phi^{-1}"] & \mathbb{R}
\end{tikzcd}$$

take $f(t) = t^{1/3}$, then

$\psi \circ f \circ \phi^{-1}(t) = \psi \circ f(t) = \psi(t^{1/3}) = t$

this is a differ.