depending on the way we insert a given manifold into another given manifold. Indeed this is the case.

Exercise: Show that the one-sheeted and two-sheeted hyperboloids are C^∞- manifolds. See examples 2.9 and 2.10.

The following exercise will show that two atlases which are not C^∞ - compatible can be diffeomorphic.

Exercise. Consider the real line $\mathbb{R}$ and the usual atlas $(\mathbb{R},\phi)$ where ϕ is the identity map. Now consider $\mathbb{R}$ with the atlas of one chart $(\mathbb{R},\psi)$ where $\psi(t) = t^3$. Show that $(\mathbb{R},\phi)$ and $(\mathbb{R},\psi)$ are not C^∞ - compatible. However construct a diffeomorphism between $(\mathbb{R},\phi)$ and $(\mathbb{R},\psi)$ showing that they are diffeomorphic.

3.2 Partitions of Unity

In this section we will develop some important concepts which will be needed later on for proving certain existence theorems. This section may be skipped until needed. First some topological preliminaries. Let S be a top space. A family of subsets $\{U_\alpha\}$ of S is called <u>locally finite</u> if every $p \varepsilon S$ has a neighborhood which intersects finitely many of the U_α's. Let $\{V_\alpha\}$ and $\{U_\beta\}$ be two covers of S. $\{V_\alpha\}$ is called a <u>refinement</u> of $\{V_\beta\}$ if for every V_α there is a U_β such that $V_\alpha \subset U_\beta$. A hausdorff space S is said to be <u>paracompact</u> if every open cover of S has a locally finite refinement of open sets.

<u>Proposition 3.5:</u> Let S be a locally compact Hausdorff space with a countable basis, then S is paracompact.

Proof: Let $\{U_i\}$ be a countable basis for S. Since S is Hausdorff and locally compact we can take this basis such that $\bar{U}_i$ is compact. We now construct a sequence $\{W_j\}$ of open sets. such that

i) $\bar{W}_j$ is compact

ii) $\bar{W}_{j-1} \subset W_j$

iii) $\bigcup\limits_j W_j = S$

Take $W_1 = U_1$, and suppose that $W_k = U_1 \cup \ldots \cup U_{j_k}$. Let j_{k+1} be the smallest integer greater than j_k such that

$$\bar{W}_k \subset U_1 \cup \ldots \cup U_{j_{k+1}}$$

then define

$$W_{k+1} = U_1 \cup \ldots \cup U_{j_{k+1}}$$

Now let $\{V_\alpha\}$ be any open cover of S. For $j > 2$, the open set $W_{j+2} - \bar{W}_{j-1}$ contains the compact set $\bar{W}_{j+1} - W_j$. The family $\{V_\alpha \cap (W_{j+2} - \bar{W}_{j-1})\}$ is an open cover of $\bar{W}_{j+1} - W_j$, so for each $j > 2$ we can choose a finite subcover $\{\tilde{U}_i\}_j = \{V_{\alpha_i} \cap (W_{j+2} - \bar{W}_{j-1})\}$. For $j = 1$ cover $\bar{W}_2$ with a finite number of V_α's and call this cover $\{\tilde{U}_i\}_1$. The family $\mathcal{B} = \cup_{j=1}^{\infty} \{\tilde{U}_i\}_j$ is an open cover of S since it covers $\bar{W}_2 \cup \{\cup_{j=2}^{\infty} \bar{W}_{j+1} - W_j\} = S'$, it is clearly a refinement of $\{V_\alpha\}$, and for every $p \in S'$, $p \in W_j$ for some j, with $\bar{W}_j$ intersecting only a finite number of the sets of $\mathcal{B}$ since

$$\bar{W}_j = (\bar{W}_j - W_{j-1}) \cup (\bar{W}_{j-1} - W_{j-2}) \cup \ldots \cup \bar{W}_2$$

Hence S is paracompact. QED.

Corollary: A topological manifold M is paracompact.

Proof: Every manifold is a locally compact Hausdorff space with a countable basis.

Paracompactness will allow as to construct partitions of unity on M. Let M be a C^∞ manifold and let $f \varepsilon C^\infty(M)$, the support of f, denoted by supp f, is defined to be the closure of the set of points $p \varepsilon M$ such that $f(p) \neq 0$. A partition of unity on M is a collection $\{\phi_i\}$ of C^∞ functions on M such that

i) the family of supports $\{\text{supp } \phi_i\}$ is locally finite i. e every point has a nbd. which intersects only a finite number of the sets supp ϕ_i.

ii) $\sum_i \phi_i(p) = 1$, $\phi_i(p) \geq 0$ for all $p \varepsilon M$.

We will assume that i runs over a countable index set. A partition of unity $\{\phi_i\}$ (countable) is said to be subordinate to the cover $\{U_\alpha\}$ if for each i there is an α such that supp $\phi_i \subset V_\alpha$.

Proposition 3.6: Let M be C^∞ manifold and $\mathcal{U} = \{U_\alpha\}$ an open cover of M, then there exists on M a partition of unity $\{\phi_i\}$ subordinate to $\mathcal{U}$.

We will first prove some lemmas, the first of which refines proposition 3.5 for differeniable manifolds.

Lemma 3.2: Let M be a C^∞ manifold and $\mathcal{U}$ an open cover of M, then there is an atlas $A = \{(V_i, \phi_i)\}$ of M such that

i) A is a locally finite refinement of $\mathcal{U}$

ii) If (V, ϕ) is a chart of A, then $\psi(V) = S^n(x, 3)$ where $S^n(x, r) = \{x \in \mathbb{R}^n, (x^1)^2 + \dots + (x^n)^2 < r^2\}$ is a spherical nbd of the origin in $\mathbb{R}^n$.

iii) The family of sets $V_i' = \psi_i^{-1}(S^n(x, 1))$ for all ψ_i covers M.

Proof: Let us go back to the proof of proposition 3.5 Since the $\tilde{U}_i$ are open, there is for each $p \in \tilde{U}_i$ a coordinate nbd $V_{p,i} \subset \tilde{U}_i$ and charts $\psi_{p,i}$ such that $S^n(x, 3) = \psi_{p,i}(V_{p,i})$. Put $V_{p,i}' = \psi_{p,i}^{-1}(S^n(x\ 1)) \subset \tilde{U}_i$. A finite number $\{V_{p_1,i_1}', \dots, V_{p_r,i_r}'\}$ covers the compact set $\overline{W}_{j+1} - W_j$. For each j the above family has a finite number so the total set $\{V_i'\}$ is countable and covers M. Moreover, we renumber the countable sets $\{V_{p,i}\}$ and the charts

$\psi_{p,i}$ as (V_i, ψ_i). Both $\{V_i\}$ and $\{\dot{V}_i\}$ are locally finite refinements of $\mathcal{U}$ and have the required properties. Q.E.D.

Lemma 3.3: Let (U, ψ) be a coordinate chart on a C^∞ manifold M. There are an open set V with compact closure $\bar{V} \subset U$ and a C^∞ function h on M such that

$$h = \begin{cases} 1 & \text{on } \bar{V} \\ >0 & \text{on } U - \bar{V} \\ 0 & \text{on } M - U \end{cases}$$

Proof: Consider the C^∞ function $f: \mathbb{R} \longrightarrow \mathbb{R}$ of example 3.1. Define $g: \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$g(x) = g(|x|) = \frac{f(2-|x|)}{f(2-|x|)+f(|x|-1)}$$

where $|x| = [\Sigma_{i=1}^n (x^i)^2]^{1/2}$. g is C^∞ on $\mathbb{R}^n$ which is 0 on $\mathbb{R}^n - S^n(x, 2)$ and 1 on $S^n(x, 1)$. Moreover on $S^n(x, 2) - \overline{S^n(x, 1)}$, $g > 0$. We can always choose ψ so that $\psi(U) = S^n(x, 2)$. Set $V = \psi^{-1}(S^n(x, 1))$ and $h = g \circ \psi$ to give the desired result.

Proof of proposition 3.6: By lemma 3.2 there is a cover $\{V_i\}$

of coordinate neighborhoods which is a locally finite refinement of $\mathcal{U}$. We can always choose $V_i = \psi_i^{-1}(S^n(x,3))$. Moreover, the sets $\{V'_i = \psi_i^{-1}(S^n(x, 1))\}$ cover M. Then by lemma 3.3 there are C^∞ funtions h_i on M which are 1 on $\bar{V}'$ and vanish outside $\psi_i^{-1}(S^n(x, 2))$. Thus $\operatorname{supp} h_i = \psi_i^{-1}(S^n(x,2)) \subset \psi_i^{-1}(S^n(x, 3) = V_i$ which is locally finite. Hence, $\operatorname{supp} h_i$ is locally finite and covers M since $\{V'_i\} \subset \{\operatorname{supp} h_i\}$ does. Thus the sum $h(p) = \Sigma\, h_i(p)$ is finite for each $p \in M$. Define

$$\phi_i(p) = \frac{h_i(p)}{h(p)} .$$

Then $\phi_i(p) \geq 0$ since $h_a(p) \geq 0$ and $\sum_i \phi_i(p) = 1$. And for every i there is an α such that $\operatorname{supp} \phi_i \subset V_i \subset v_\alpha$, so $\{\phi_i\}$ is a partition of unity suboordinate to the cover $\mathcal{U}$. QED.

Remarks. the Hausdorff requirement is absolutely necessary for partitions of unity. All of the above goes through equally as well in the C^r category for any $r = 0, 1, \ldots, \infty$.

3.3 Differential Maps, Immersions, Submanifolds.

First we recall the definition of the rank of a matrix. Given an nxm matrix A, its rank is defined by the dimension of the subspace spanned by the rows of A or equivalently by the dimension of the subspace spanned by the columns of A. Clearly rank $A \leq \min(m,n)$.

Definition 3.7: Let $U \subset R^n$ and $F: U \longrightarrow R^m$ be a C^r- map ($r \geq 1$). The rank of F at $x \in U$ is the rank of its Jacobian matrix,

$$JF = \begin{pmatrix} \frac{\partial F^1}{\partial x^1} & \cdots & \frac{\partial F^1}{\partial x^n} \\ \vdots & & \\ \frac{\partial F^m}{\partial x^1} & \cdots & \frac{\partial F^m}{\partial x^n} \end{pmatrix},$$

at $x \in U$. The generic rank of F is $\sup_{x \in U} \operatorname{rank}_x F$.

A point $x \in U$ is called generic or regular if the rank of F at x equals the generic rank of F. A point $x \in U$ which is not generic is called critical, and then F(x) is called a critical value of F.

Example 3.9: Consider the map $F(x^1,x^2) = \left((x^1)^2 + (x^2)^2 , 2x^1x^2\right)$

$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$JF = \begin{pmatrix} 2x^1 & 2x^2 \\ 2x^2 & 2x^1 \end{pmatrix} = 2\begin{pmatrix} x^1 & x^2 \\ x^2 & x^1 \end{pmatrix}$$

rank F = Dim (x^1, x^2) , thus

$$\mathrm{rank}_x F = \begin{cases} 0 & x^1 = x^2 = 0 \\ 1 & x^2 = \pm x^1 \\ 2 & \text{otherwise} \end{cases} \quad \text{since rows are} \begin{cases} x^1(1,\pm 1) \\ \pm x^1(1,\pm 1) \end{cases}$$

<u>Exercise</u>: Find $\mathrm{rank}_x F$ for the following:

i) $F:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$: $F(x^1,x^2) = (x^2,x^1)$

ii) $F:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$: $F(x^1,x^2) = (4,(x^2)^3)$

iii) $F:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$: $F(x^1,x^2) = ((x^1)^2,(x^2)^2, 2x^1x^2)$

iv) $F:\mathbb{R}^3 - \{(0,0)\} \longrightarrow \mathbb{R}^3$ $F(x^i) = \left(\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}\right)$

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$$

You can use the following result of linear algebra.

<u>Lemma</u>: The rank of an $n \times m$ matrix A is equal to the maximum order of any nonvanishing minor determinant.

In the study of critical points there is an important theorem which we mention without proof (cf. Sternberg). First, a subset S of $\mathbb{R}^n$ is said to have measure zero if for every $\varepsilon > 0$ there is a covering of S by a countable number of spherical neighborhords $S_i^n(x, \delta_i)$ such that $\sum_i \mathrm{vol}_i < \varepsilon$ where vol_i = volume of $S_i^n(x, \delta_i)$. Then

<u>Theorem (Sard):</u> Let $F: U \longrightarrow \mathbb{R}^m$ be a C^∞ map where U is an open set of $\mathbb{R}^n$. Then the set of critical values of F has measure zero in $\mathbb{R}^m$.

Let us now recall the inverse function theorem of advanced calculus (See Spivak (Cal. on manifolds) or Loomis and Sternberg).

<u>Theorem</u> (Inverse Function): Let W be an open subset of R^n and $F:W \longrightarrow R^n$ a C^r map whose Jacobian $JF(a) \neq 0$ at $a \in W$. Then there are a neighborhood (open) U of $b = F(a)$ and a C^r map $G:U \longrightarrow W$ such that $F \circ G(x) = x$ for all $x \in U$.

Remark: We denote G by F^{-1}. Notice that this says that F is a C^r diffeomorphism on some open set of $a \in W$.

The following theorem is essentially equivalent to the implicit function theorem of advanced calculus.

<u>Exercise</u>: Obtain the implicit function theorem from the following theorem.

<u>Theorem 3.3</u>: (Rank theorem). Let $A_0 \subset R^n$ and $B_0 \subset R^m$ and suppose $F:A_0 \longrightarrow B_0$ is a C^r-map $(r \geq 1)$ whose rank on A_0 is a constant k. If $a \in A_0$ and $F(a) = b \in B_0$, then there exists open sets $A \subset A_0$ and $B \subset B_0$ with $a \in A$, $b \in B$, and C^r diffeomorphisms $G:A \longrightarrow U \subset R^n$, $H:B \longrightarrow V \subset R^m$ such that $H \circ F \circ G^{-1}(U) \subset V$ and

$$H \circ F \circ G^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

Proof: We can assume $a = (0, \ldots, 0) \in R^n$ and $b = (0, \ldots, 0) \in R^m$ since we can always translate to arbitrary points. Since F has constant rank k, we can permute coordinates such that its

Jacobian has a $k\times k$ minor of the form

$$\begin{pmatrix} \frac{\partial f^1}{\partial u^1} & \cdots & \frac{\partial f^1}{\partial u^k} \\ \vdots & & \\ \frac{\partial f^k}{\partial u^1} & \cdots & \frac{\partial f^k}{\partial u^k} \end{pmatrix}$$

with det $\neq 0$. Now we define the C^r-map $G: A_0 \longrightarrow R^n$ by $G(u^1,\dots,u^n) = (f^1,\dots,f^k,u^{k+1},\dots u^n)$. Then its Jacobian has the form

$$\left(\begin{array}{ccc|c} \frac{\partial f^1}{\partial u^1} & \cdots & \frac{\partial f^1}{\partial u^k} & \\ \vdots & & \vdots & * \\ \frac{\partial f^k}{\partial u^1} & \cdots & \frac{\partial f^k}{\partial u^k} & \\ \hline & 0 & & \mathbb{1}_{n-k} \end{array}\right)$$

whose det $\neq 0$ at $u = a$. Thus by the inverse function theorem there is an open set A_1 containing a on which G is a diffeomorphism onto an open set $U_1 = G(A_1)$. Composing our two maps, we have

$$F\circ G^{-1}(x^1,\dots,x^k\ ,\ x^{k+1},\dots x^n) = (x^1,\dots,x^k,\tilde{f}^{k+1},\dots,\tilde{f}^m)$$

where $\tilde{f}^{k+j} = f^{k+j}\circ G^{-1}(x)$, $j = 1,\dots,m-k$, f^{k+j} is the $k + j$ entry of F. The Jacobian matrix of $F\circ G^{-1}$ is

$$J(F\circ G^{-1}) = \left[\begin{array}{c|ccc} \mathbb{1}_k & & 0 & \\ \hline & \frac{\partial \tilde{f}^{k+1}}{\partial x^{k+1}} & \cdots & \frac{\partial \tilde{f}^{k+1}}{\partial x^n} \\ * & \vdots & & \\ & \frac{\partial \tilde{f}^{m}}{\partial x^{k+1}} & \cdots & \frac{\partial \tilde{f}^{m}}{\partial x^n} \end{array}\right]$$

which is nonsingular on U_1. But since JG^{-1} is nonsingular on U_1 and $G^{-1}(U_1) = A_1 \subset A_0$, we have

$$\operatorname{rank} J(F\circ G^{-1}) = \operatorname{rank}(JF\circ JG^{-1}) = k$$

since rank $F = k$, and rank G is k. But this says that the lower-right block must be the zero matrix. That is $\tilde{f}^{k+j}$ depend only on $x^1,\ldots,x^k$. Now we define the map
$H^{-1}: V_1 \longrightarrow B_0$ by

$$H^{-1}(y^1,\ldots,y^k,y^{k+1},\ldots,y^m) = (y^1,\ldots,y^k,\ y^{k+1}+\tilde{f}^{k+1}(y^1,\ldots,y^k),\ldots$$

$$,y^m + \tilde{f}^m(y^1,\ldots,y^k))$$

We choose V_1 such that $y = (y^1,\ldots,y^m)\in V$ and that $\tilde{f}^{k+j}(y^1\ldots,y^k)$ are defined with $H^{-1}(V_1)\subset B_0$. We have $H^{-1}(0) = 0$ and

$$JH^{-1}(y) = \left[\begin{array}{c|c} \mathbb{1}_k & 0 \\ \hline * & \mathbb{1}_{m-k} \end{array}\right]$$

Thus H^{-1} is a C^r-diffeomorphism of a nbd V of $0 \varepsilon V_1$ onto an open set $B \subset R^m$ by the inverse function theorem. Now choose a nbd $U \subset U_1$ of $a \varepsilon R^n$ such that $F \circ G^{-1}(U) \subset B$ and let $A = G^{-1}(U)$. Then $H \circ F \circ G^{-1}(1) \subset V$ and

$$H \circ F \circ G^{-1}(x^1,\ldots,x^n) = (x^1,\ldots,x^k, 0,\ldots 0)$$

since $H(x^1,\ldots,x^k, \tilde{f}^{k+1},\ldots,\tilde{f}^m) = (x^1,\ldots,x^k, 0,\ldots,0)$. Q.E.D.

Now let $F:N \longrightarrow M$ be a C^r mapping of C^r manifolds and let (U,ϕ) and (V,ψ) be charts for $p\varepsilon N$ and $F(\phi)$ M respectively with $F(U) \subset V$, then the coordinate representative of F is

$$\hat{F} = \psi \circ F \circ \phi^{-1} : R^n \longrightarrow R^m.$$

<u>Definition 3.8</u>: The rank of F at $p\varepsilon N$ is defined to be the rank of $\hat{F}$ at $\phi(p)$.

Thus if $\hat{F} = (f^1(x^1,\ldots,x^n), \ldots, f^m(x^1,\ldots,x^n))$, its rank at $\phi(p)$ is the rank of its Jacobian matrix

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1}, \ldots, \frac{\partial f^1}{\partial x^n} \\ \vdots \\ \frac{\partial f^m}{\partial x^1}, \ldots, \frac{\partial f^m}{\partial x^n} \end{pmatrix}_{\phi(p)}$$

For our definition 3.8 to make sense it must be shown to be independent of the coordinate representative. Indeed, let (U',ϕ') and (V',ψ') be charts C^∞-compatible with (U,ϕ) and (V',ψ'), respectively, with $p \in U \cap U'$, $F(U') \subset V'$, coordinate $F(p) \in V' \cap V$. We then have two corresponding coordinate representatives, $\hat{F} = \psi \circ F \circ \phi^{-1}$ and $\hat{\hat{F}} = \psi' \circ F \circ \phi'^{-1}$. Thus in $U \cap U'$ we have $\hat{\hat{F}} = (\psi' \circ \psi^{-1}) \circ \hat{F} \circ (\psi \circ \phi'^{-1})$. But rank $\hat{F}$ = rank $\hat{\hat{F}}$ since $(\psi' \circ \psi^{-1})$ and $\phi \circ \phi'^{-1}$ are diffeomorphisms.

Exercise: Verify in detail this last statement.

Exercise: More, generally show that if $F: N \longrightarrow M$ is nonsingular on N, then rank F = min(n,m) where n=dim N, m = dim M.

We now restate theorem 3.3 for manifolds:

Theorem 3.4: Let $F: N \longrightarrow M$ be a C^∞ map of C^∞ manifolds N,M, with constant rank k. Then there exist charts (U,ϕ) and (V,ψ) of p and F(p) respectively with $F(U) \subset V$ such that $\phi(p) = (0,\ldots,0)$, $\psi(F(p)) = (0,\ldots,0)$ and $\hat{F} = \psi \circ F \circ \phi^{-1}(x^1,\ldots,x^n) = (x^1,\ldots,x^k, 0,\ldots,0)$.

Corollary : If F is a diffeomorphism, then rank F = n = m.

Proof: If F is 1-1, from the above theorem we must have k = n. Applying the theorem to F^{-1} we get k=m.

Now we suppose that $n \leq m$ and $F: N \longrightarrow M$ a C^∞-map, then

Definition 3.9: F is called an immersion at $p \in N$ if rank F = n. F is called an immersion if rank F = n for all $p \in N$. If F is an immersion and injective (1-1) then the pair (N,F) is called an immersed submanifold (or just submanifold) of M. The topology and C^∞-structure on $\tilde{N} = F(N) \subset M$ is that which makes $F: N \longrightarrow \tilde{N}$ a diffeomorphism.

We now consider some examples:

<u>Example 3.9</u> : $F : R \longrightarrow R^2$ given by

$F(t) = (\cos t , \sin t)$ is an immerson since

$$JF = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$ has rank 1 for all

$t \varepsilon R$. It is <u>not</u> an immersed submanifold since F is not 1-1. Clearly $F(R) = S^1$.

<u>Example 3.10</u>: The helix : $F : R \longrightarrow R^3$ given by

$F(t) = (\cos t , \sin t , t)$. F(R) is the helix.

$$JF = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$ has rank 1 for all $t \varepsilon R$.

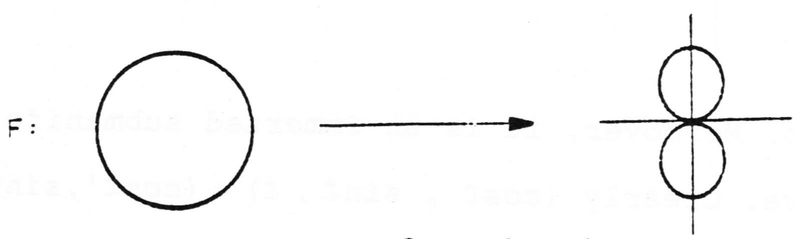
So F is an immersion. Moreover, it is an immersed submanifold since it is injective. Clearly $(\cos t , \sin t , t) = (\cos t', \sin t', t')$ implies $t = t'$.

<u>Example 3.11</u>: The spiral: $F : (1,\infty) \longrightarrow R^2$ given by $F(t) = (\frac{1}{t} \cos t , \frac{1}{t} \sin t)$. JF has rank 1 for all $t \varepsilon (1,\infty)$. Show it. Moreover, $F(1,\infty)$ is 1-1 so it is an immersed submanifold. Show it

<u>Example 3.12</u>: the figure eight: $F:(0,2\pi) \longrightarrow R^2$ given by ϕ^{-1} of example 3.8, i.e. $F(t) = (\sin 2t , \sin t)$. Clearly, rank $F = 1$ for all $t \varepsilon (0,2\pi)$. Moreover, as seen previously F is 1-1 and so it is an immersed submanifold.

Example 3.13: The figure eight again: but now $F : R \longrightarrow R^2$ with the same F as above. But now F is not 1-1 and so is not an immersed submanifold. It is, of course an immersion.

Example 3.14 : The figure eight again: but $F : S^1 \longrightarrow R^2$ given by $F(x,y) = (2xy, y)$. Here since there is no atlas of a single chart which covers S^1 we must check the rank of F for two different charts. First consider the chart (U,ϕ) with $U = S^1 - \{(1,0)\}$, $\phi(\cos t, \sin t) = t$. We have $\phi(U) = (0, 2\pi)$, so $F \circ \phi^{-1} = (\sin 2t, \sin t)$ which is precisely the map of example 3.12. Thus rank F = 1 on U. We need to check that rank F = 1 on $U' = S^1 - \{(-1,0)\}$ with $\phi'(\cos t, \sin t) = t$, $-\pi < t < \pi$. This is left as an exercise. Thus $F(S^1)$ is an immersion. It is, not, however an immersed submanifold since it is not 1-1. $F(1,0) = F(-1,0) = (0,0)$.



Example 3.15: $F : R \longrightarrow T^2 = S^1 \times S^1$. It is useful to consider the circle S^1 as a subset of the complex plane. Putting $z = x+iy$ then $S^1 = \{z \in \mathbb{C} : |z|^2 = 1\}$ or equivalent $S^1 = \{z \in \mathbb{C} : z = e^{it}, t \in R\}$. Then $T^2 = (e^{it_1}, e^{it_2})$. Now F above is given by $F(t) = (e^{it}, e^{i\alpha t})$, α = irrational. Clearly, rank F = 1 since $e^{it} \neq 0$. So F is an immersion. Moreover, F is 1-1 since $(e^{it}, e^{i\alpha t}) = e^{it'}, e^{i\alpha t'})$ implies that $t' - t = 2\pi k$, $\alpha(t'-t) = 2\pi k_2$ for two integers k_1, k_2. But this implies that $k_2 = \alpha k_1$ whose only solution is $k_1 = k_2 = 0$ since α is irrational. Thus, F(R) is an

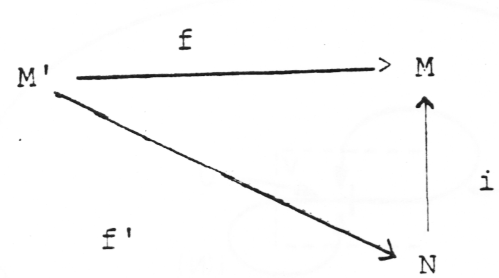
immersed submanifold. The interesting fact about $F(R)$ is that it is dense in T^2! It is also not locally connected at any point! This example illustrates vividly that the topology on $F(R)$ is not the relative topology. With our other examples, the topology on the image of F coincided with relative topology on all but possibly one or two points. Here they coincide at no point. This is also an example of a noncompact submanifold of a compact manifold, but this is not so strange.

Exercise: Show that S^1 is a Lie group, i.e. that it is a top. group such that the composition map $f: S^1 \times S^1 \longrightarrow S^1$, $f(x,y) = x\,y^{-1}$ is C^∞. Thus $T^2 = S^1 \times S^1$ is a Lie group.

The above example gives an example of a noncompact Lie subgroup of a compact Lie group which is dense and nowhere locally connected.

The following proposition will be very useful later on.

Proposition 3.1 : Let $i: N^n \longrightarrow M^m$ be a immersed submanifold and let $f: M' \longrightarrow M$ be C^∞ such that $f(M') \subset i(N)$. Then the there is a map $f': M' \longrightarrow N$ such that the diagram



commutes. Moreover, if f' is continuous it is C^∞.

Proof: The existence of f' is clear since $f(M') \subset i(N)$.

Now i is injective and $f' = i^{-1} f$. Assume f' is continuous, and dim N = k, dim M = n. By theorem 3.4, if $p' \varepsilon M'$ there are coordinate charts (U,ϕ) and (V,ψ) about $f'(p) \varepsilon N$ and $f(p) \varepsilon M$ respectively such that $i(U) \subset V$ and

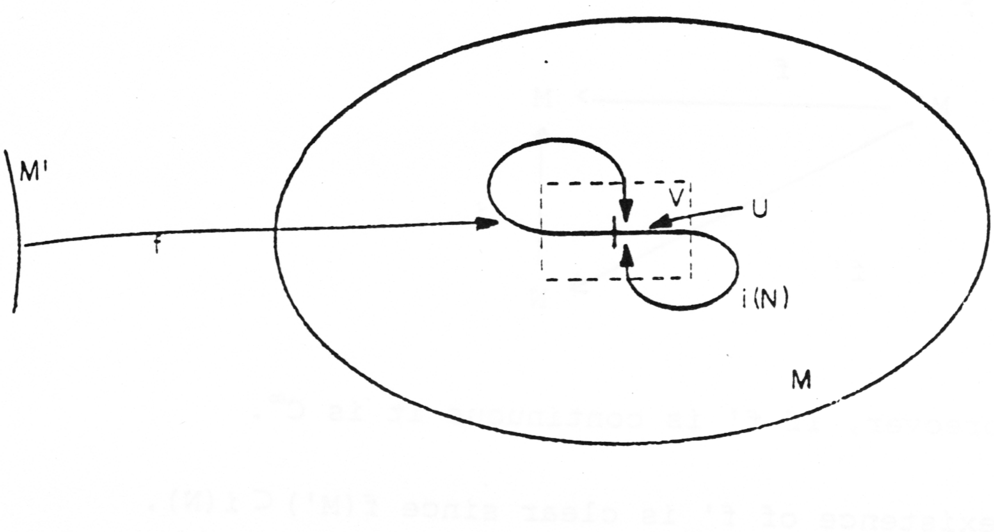
$$\hat{i}(x^1,\ldots,x^k) = \psi \circ i \circ \phi^{-1}(x^1,\ldots,x^k) = (x^1,\ldots,x^k, 0,\ldots,0)$$

Now $f' = i^{-1} \circ f$ is continuous, so $f^{-1} \circ i(U)$ is an open set U' of M' and $f(U') = i(U) \subset V$. Thus on U' f' has coordinate representative

$$\hat{f}' = \phi \circ i^{-1} \circ f(q') = (x^1,\ldots,x^k) \quad , \quad q' \varepsilon U'$$

Thus f' is C^∞ .   Q.E.D.

<u>Remark</u>: It should be mentioned within the above context that for $f = i \circ f'$, f' is not necessarily continuous even when f and i are C^∞ . But the proposition says that if f' is continuous then it is C^∞ . Thus all that can go wrong is that f' be not continuous. This is illustrated by



Exercise: Discuss for examples 3.9-3.14 whether the topology on F(˙) differs from the relative topology.

3.4 Imbeddings

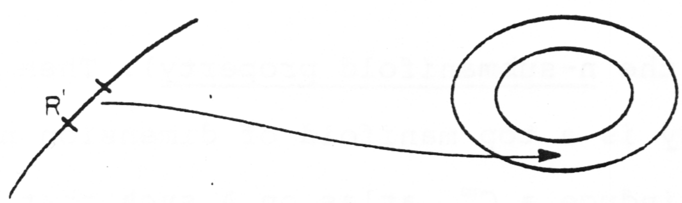
Definition 3.10: Let $F : N \longrightarrow M$ be an immersed submanifold. Then (F,N) is called an imbedding (or an imbedded submanifold) if F is a homeomorphism onto its image $\tilde{N} = F(N)$ with the relative topology.

The distinction between an immersed submanifold and an imbedded submanifold is a global one. Indeed

Proposition 3.2: Let $F : N \longrightarrow M$ be an immersion. Then every $p \in N$ has a nbd U_p such that $F|U_p$ is an imbedding of U_p in M. ($F|U_p$ denotes F restricted to U)

Proof: Let $V \subset M$ and such that $V \cap F(N) \neq \{\phi\}$ suppose $F(p) \in V \cap F(N)$. Since F is continuous $F^{-1}(V)$ is open in N and contains p. Put $U_p = F^{-1}(V)$ and use the rank theorem.

Thus locally (in N) every submanifold is an imbedding. Let's see what this says with pictures for example 3.15 where globally the induced topology is not the relative topology for any nbd in M.



We remark that an imbedded submanifold is also called a regular submanifold. We will now prove a theorem which tells us when we can turn a variety as discussed previously into an imbedded submanifold. First, it will be convenient in our discussion to use instead of the spherical neighborhoods $S(x,r)$ as a basis for the Euclidean topology, the <u>cubical neighborhoods</u> defined by $C^n_\varepsilon(x) = \{y \varepsilon R^n : |x^i - y^i| < \varepsilon, \ i = 1,\dots,n\}$. It is left as an exercise to show that such neighborhoods form a basis for the Euclidean topology on R^n and are completely equivalent to the spherical neighborhoods. We mention that we can always choose our neighborhoods in $\mathbb{R}^n$ and $\mathbb{R}^m$ in theorem 3.4 to be cubical neighborhoods, i.e. $\phi(U) = C^n_\varepsilon(0)$ and $\psi(V) = C^m_\varepsilon(0)$.

<u>Theorem 3.5</u>: Let N and M be C^∞ manifolds of dimensions n and m, respectively, and let $F : N \longrightarrow M$ be a C^∞ map. Suppose that F has constant rank k on N and that $q \varepsilon F(N)$. Then $F^{-1}(q)$ is a closed, imbedded submanifold of N of dimension n-k.

First we prove a lemma.

<u>Lemma 3.1</u>: Let A be a subset of a C^∞ manifold M such that each $p \varepsilon A$ has a coordinate chart (U,ϕ) on M with local coordinates $(x^1,\dots,x^m)$ such that

i) $\phi(p) = (0,\dots,0)$

ii) $\phi(U) = C^m_\varepsilon(0)$

iii) $\phi(U \cap A) = \{x \varepsilon C^m_\varepsilon(0) : x^{n+1} = \dots = x^m = 0\}$

(Such an A is said to have the <u>n-submanifold property</u>). Then A with the relative topology is a top. manifold of dimension n and the above charts (U,ϕ) induce a C^∞ atlas on A such that the inclusion $i : A \longrightarrow M$ is an imbedding.

Proof: Let A be as above and give it the relative topology. Consider coordinate charts (U,ϕ) satisfying i)-iii) above. We define an atlas by the charts $(V,\tilde{\phi})$ where $V = U \cap A$ and $\tilde{\phi} = \pi \circ \phi | V$ where $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is the projection map. By iii) $\tilde{\phi}$ is a homeomorphism. We must show that two such charts are C^∞-compatible. We assume that $V \cap V' = U \cup U' \cap A \neq \{\emptyset\}$. Since we have given A the relative topology $\tilde{\phi}' \circ \tilde{\phi}^{-1}$ is a homeomorphism. Define $\theta : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ such that $\pi \circ \theta$ = identity by $\theta(x^1,\ldots,x^n) = (x^1,\ldots,x^n, 0,\ldots,0)$. Clearly θ is C^∞, so $\theta|_{C^n_\varepsilon(0)}$ is C^∞. Now $\tilde{\phi}' \circ \tilde{\phi}^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a homeomorphism. Furthermore, $\tilde{\phi}' \circ \tilde{\phi}^{-1} = \pi \circ \phi' \circ \phi^{-1} \circ \theta$ which is clearly C^∞ since π, θ, and $\phi' \circ \tilde{\phi}^{-1}$ are. Thus, $\tilde{\phi}' \circ \tilde{\phi}^{-1}$ is C^∞ on its domain $\tilde{\phi}(V \cap V')$. Since $\tilde{\phi}$ and $\tilde{\phi}'$ are arbitrary $\tilde{\phi} \circ \tilde{\phi}'^{-1}$ is also C^∞. Thus $(V,\tilde{\phi})$ and $(V,\tilde{\phi}')$ are C^∞- compatible. But by hypothesis each $p \varepsilon A$ has such charts and their totality forms a C^∞ atlas for A, making A a C^∞-manifold. Now consider the inclusion map $i : A \longrightarrow M$. Its coordinate representative on $V \subset A$ is just

$$\hat{i}(x^1,\ldots,x^n) = (x^1,\ldots,x^n, 0,\ldots,0)$$

So i is clearly an immersion. But since A was given the relative topology i is a homeomorphism onto its image i(A), so i is an imbedding. Q.E.D.

Proof of theorem 3.5: Let $A = F^{-1}(q)$. Then A is a closed subset of N since it is the inverse image of a point under a continuous map. Let $p \varepsilon A$, since F has constant rank, by the rank theorem

(3.4) we can find charts (U,ϕ) and (V,ψ) for p and q respectively such that $\phi(p) = \{0\}\varepsilon R^n$ and $\psi(q) = \{0\}\varepsilon R^m$ with

$$\hat{F}(x^1,\ldots,x^n) = \psi\circ F\circ\phi^{-1}(x) = (x^1,\ldots,x^k, 0,\ldots,0)$$

and we can take $\phi(U) = C^n_\varepsilon(0)$ and $\psi(V) = C^m_\varepsilon(0)$.

Thus we have

$$\hat{F}^{-1}(0) = \phi\circ F^{-1}\circ\psi^{-1}(0) = \{x\varepsilon C^n_\varepsilon(0): x^1 =\ldots= x^k = 0\}$$

But since $A = F^{-1}(q)$ we have, restricting to U, $A\cap U = \phi^{-1}(\phi\circ F^{-1}\circ\psi^{-1})(0)$ $= \phi^{-1}\circ F^{-1}(0)$. So

$$\phi(A\cap U) = \{x \varepsilon C^n_\varepsilon(0): x^1 =,\ldots x^k= 0\}$$

and A satisfies the hypothesis of lemma 3.1 with n replaced by n-k. Hence $A = F^{-1}(q)$ is a closed, imbedded submanifold of dimension n-k. Q.E.D.

<u>Exercise</u>: Construct explicitly the map $\tilde{\phi}'\circ\tilde{\phi}^{-1}$ in the proof of lemma 3.1.

We will now apply theorem 3.5 to many examples.

<u>Example 3.16</u>: The sphere S^n: let $F:\mathbb{R}^{n+1}-\{0\} \longrightarrow \mathbb{R}$ by $F(x^1,\ldots,x^{n+1}) = \sum_{i=1}^{n+1}(x^i)^2$. Clearly F has rank 1 and is C^∞ on all of $R^{n+1}-\{0\}$ which contains $S^n = F^{-1}(1)$. Thus by theorem 3.5 S^n is a closed, imbedded submanifold of $\mathbb{R}^{n+1}$ of dimension $n = n+1-1$.

More generally we have

<u>Example 3.17</u>: The hyperboloids H^n_m with $F:\mathbb{R}^{n+1}-\{0\} \longrightarrow \mathbb{R}$ given by $F(x^1,\ldots,x^n) = \sum_{i=1}^{n+1} \varepsilon_i (x^i)^2$, where $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_m = -1$, $\varepsilon_{m+1} = \ldots = \varepsilon_{n+1} = 1$, are closed imbedded submanifolds since F has rank 1 on $\mathbb{R}^{n+1}-\{0\}$ and $H^n_m = F^{-1}(1)$.

Notice that the cone of example 2.11 fails to satisfy the hypothesis of theorem 3.5 since the rank of F vanishes at the origin.

<u>Example 3.18</u>: The torus T^2 as an imbedded submanifold of R^3. The C^∞ map $F:\mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by $F(x_1,x_2,x_3) = (\rho-a)^2+(x_3)^2$, $\rho=\sqrt{(x_1)^2 + (x_2)^2}$, has rank one on $\mathbb{R}^3-\{\rho = a\ ,\ x_3 = 0\}$. The locus of points $(\rho-a)^2 + x_3^2 = b^2$ is the torus, thus $T^2 = F^{-1}(b^2)$ for $0<b<a$. Thus T^2 is a closed imbedded submanifold of dimension $3-1 = 2$.

<u>Exercise</u>: Show that T^2 defined in example 3.18 is homeomorphic to $S^1 \times S^1$. Moreover, show that the usual C^∞-structure on $S^1 \times S^1$ and the C^∞-structure given by the above imbedding are C^∞-compatible.

<u>Exercise</u>: Show that the ellipsoid $E^n = \left\{x \in \mathbb{R}^{n+1} : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} +\ldots+ \frac{x_{n+1}^2}{a_{n+1}^2} = 1 \ ,\ a_i \in \mathbb{R}\right\}$ is a closed imbedded submanifold of R^{n+1}.

The Mobius strip considered in example 2.4 is an example of an imbedding which is not of the type considered by theorem 3.5. Here is another

Example 3.19: The projective plane $P^2(\mathbb{R})$ as an imbedding in R^4. Consider the C^∞ map $F:S^2 \longrightarrow R^4$ defined by $F(x,y,z) = (yz,xz, xy, x^2+2y^2+3z^2)$. Clearly $F(p) = F(-p)$ so $F:P^2(\mathbb{R}) \longrightarrow R^4$ is well defined. To see that F is injective suppose $F(x,y,z) = F(x',y',z')$. Then $yz = y'z'$, $xz = x'z'$, $xy = x'y'$. Assume $x' \neq 0$. Then we have $y' = \frac{xy}{x'}$, $z' = \frac{xz}{x'}$ giving $y'z' = \left(\frac{x}{x'}\right)^2 yz$. But this implies $\left(\frac{x}{x'}\right)^2 = 1$ if $y'z' \neq 0$ so $x' = \pm x$. Similarly assuming again $y' \neq 0$, $z' \neq 0$ then $y' = \pm y$, $z' = \pm z$. So F is 1-1 restricted to $X = P^2(\mathbb{R}) - \{x=0\} \cup \{y=0\} \cup \{z=0\}$, and we have only used the first three coordinates. Thus we have a 1-1 map $\tilde{F}:X \longrightarrow R^3$ given by $\tilde{F}(x,y,z) = (yz,xz,xy)$. Moreover, it is easy to check that $\tilde{F}$ is C^∞ of constant rank 2 on all of $P^2(\mathbb{R})$. Now suppose that $x' = 0$. Then if $x \neq 0$ then we must have $z = y = 0$ and $y'z' = 0$. But then $(x,y,z) = (\pm1,0,0)$ $(x',y',z') = (0,\pm1,0)$ or $(0,0,\pm1)$ and this contradicts the equation $x^2+2y^2+3z^2 = x'^2 + 2y'^2 + 3z'^2$. Thus we must have $x = 0$. Our equations are then $yz = y'z'$, $2y^2+3z^2 = 2y'^2 + 3z'^2$, $y^2 + z^2 = y'^2 + z'^2 = 1$. From the first and last of these equations we find the roots $y'^2 = y^2$, $(1-y^2)$, $z'^2 = z^2$, $(1-z^2)$. The middle equation eliminates the roots $(1-y^2)$, $(1-z^2)$. By similar computations for the case $y' = 0$ and $z' = 0$ we conclude that $\tilde{F}:P^2(\mathbb{R}) \longrightarrow \mathbb{R}^3$ is an immersion which is not 1-1, but $F:P^2(\mathbb{R}) \longrightarrow \mathbb{R}^4$ is a 1-1 immersion. Furthermore, we can check that sets of the form $F(P^2(\mathbb{R})) \cap$

$S(x,r)$ are open in $F(P^2(\mathbb{R}))$ where $S(x,r)$ is a spherical neighborhood in $\mathbb{R}^4$. (Show it). Thus $F:P^2(\mathbb{R}) \longrightarrow \mathbb{R}^4$ is an imbedding.

Theorem 3.6: Let N_1 and N_2 be imbedded submanifolds of M_1 and M_2, respectively. Then $N_1 \times N_2$ is an imbedded submanifold of $M_1 \times M_2$.

Theorem 3.7: Let $F:N \longrightarrow M$ and $G:M \longrightarrow P$ be imbeddings. Then $G \circ F:N \longrightarrow P$ is an imbedding.

Exercises: Prove theorems 3.6 and 3.7.

Before ending this subsection, I will mention an important theorem on imbeddings due to H. Whitney.

Theorem (Whitney Imbedding theorem): Let M be a C^∞-manifold of dimension n, then there is a C^∞ map $i:M \longrightarrow \mathbb{R}^{2n+1}$ such that (M,i) is an imbedded submanifold of R^{2n+1}.

The proof of this theorem is beyond the scope of these notes. We refer the reader to (Auslander and Mackenzie) or (Sternberg). For a much weaker imbedding theorem (and much easier proof) see Boothby.

3.5 Lie Groups. Lie groups provide many interesting examples of differentiable Manifolds and we will see them often although the general theory is left until Chapter 8.

Definition 3.11: Let G be a group and a differentiable manifold Then G is called a Lie group if $\phi:G \times G \longrightarrow G$ defined by $\phi(x,y) = xy^{-1}$ is C^∞ for all $x,y \in G$. Here y^{-1} denotes the inverse element of y.

Example 3.20: $\mathbb{R}^n$ with group multiplication given by vector addition is a Lie group since $(x,y) \to x-y$ is C^∞.

Example 3.21: $GL(n,R)$ of example 3.4 is a Lie group since the maps $(A,B) \longrightarrow AB$ and $A \longrightarrow A^{-1}$ are C^∞ maps, or just that $(A,B) \longrightarrow AB^{-1}$ is C^∞.

Theorem 3.8: If G_1 and G_2 are Lie groups, then the direct product $G_1 \times G_2$ with the C^∞ structure of the Cartesian product of manifolds is a Lie group.

Proof: Exercise

In order to prove our next theorem we will need some lemmas. The first is the converse to lemma 3.1.

Lemma 3.2: Let $F:N \longrightarrow M$ be an imbedding of C^∞ manifolds of dimensions n and m respectively. Then N has the n-submanifold property, i.e. $F(N) \subset M$ satisfies the hypothesis of lemma 3.1.

Proof: Let $p \in N$ and $q = F(p) \in F(N)$. By the rank theorem 3.4, there are charts (U,ϕ) and (V,ψ) of p and q such that i) $\phi(p) = (0,\dots,0)$, ii) $\psi(q) = (0,\dots,0)$, iii) $F(U) \subset V$ and iv) $\hat{F}(x) = \psi \circ F \circ \phi^{-1}(x^1,\dots,x^n) = (x^1,\dots,x^n, 0\dots0)$. Moreover, we can always take $\phi(U) = C^n_\varepsilon(0)$. But since F is an imbedding $F(U)$ is open so there is a $W \subset M$ such that $F(U) = W \cap F(N)$ and $q \in W \subset V$. Furthermore we can choose $\psi(W) = C^m_\varepsilon(0)$, so the chart (W,ψ) satisfies conditions i)-iii) of the hypothesis of lemma 3.1 and thus N has the n-submanifold property.

Q.E.D.

The following is a weaker version (i.e. stronger hypothesis) of proportion 3.1.. In fact it says that situation as shown in the illustration following proposition 3.1 do not occur for imbeddings.

Lemma 3.3: Let $F:A \longrightarrow M$ be a C^∞ mapping of C^∞ manifolds and suppose $F(A)\subset N$ where N is an imbedded submanifold of M. Then F is C^∞ as a mapping into N. (dim N=n, dim M=m).

Proof: Since N is an imbedding F is continuous as a mapping into N. Thus F is C^∞ by proposition 3.1.

A Lie subgroup H of a Lie group G is a subgroup H of G which is also a submanifold of G (not necessarily, regular), and a Lie group with respect to its C^∞-structure

We are now ready for

Theorem 3.9: Let G be a Lie group and H a subgroup which is also an imbedded submanifold of G. Then H is a Lie group.

Proof: It follows from theorem 3.6 that the inclusion map $F:H\times H \longrightarrow G\times G$ is an imbedding. Let $F_2:G\times G \longrightarrow G$ be defined by $F_2(g_1,g_2) = g_1g_2^{-1}$. Put $F = F_2\circ F : H\times H \longrightarrow G$. Now the image of F is in H since H is a subgroup and F is C^∞ as a mapping into G. It follows from lemma 3.3 that F is C^∞ as a mapping into H. Thus it is a Lie group.

We will now make use of theorems 3.5 and 3.9 to identify some subgroups of the Lie group $GL(n,R)$. We will show that all of the groups to be considered are Lie groups. Indeed these are Lie groups given as matrix groups. Such groups are known as <u>classical groups</u>. There is an important theorem which we will not prove which says that any Lie group is locally isomorphic to a subgroup of $GL(n,R)$ for some n. (Ado's theorem).

As with matrices given an element g of a Lie group G, we can consider two ways of multiplication: <u>left multiplication</u> L_a or <u>right multiplication</u> R_a for $a \in G$. We have

$$L_a g = ag \quad \text{whereas} \quad R_a g = ga$$

By definition both of these are C^∞ maps and have C^∞ inverses $L_a^{-1} = L_{a^{-1}}$ and $R_a^{-1} = R_{a^{-1}}$.

We will now consider a useful lemma

<u>Lemma 3.4</u>: Let $X \in GL(n,R)$, M a C^∞ manifold and $F: GL(n,R) \longrightarrow M$ a C^∞ mapping. Then rank $F(X)$ = rank $F(\mathbb{1})$.

Proof: Clearly for any $X \in GL(n,R)$, $L_X \mathbb{1} = X \cdot \mathbb{1} = X$. Thus rank $F(X)$ = rank $F \circ L_X(\mathbb{1})$. But since L_X is a diffeomorphism of $GL(n,R)$ onto itself, it follows from the corollary to theorem 3.4 that L_X has rank n^2, Moreover we know that rank $F \leqslant n^2$. Thus rank $F(X)$ = rank $F(\mathbb{1})$. Q.E.D.

<u>Example 3.22</u>: The special linear group, $SL(n,R) = \{A \in GL(n,R): \det A = 1\}$. Then $\det : GL(n,R) \longrightarrow R$ is clearly C^∞ and $SL(n,R) = \det^{-1}(1)$ is closed. We need to show that det has rank 1 on

$GL(n,R)$. But by lemma 3.4 it suffices to show that det has rank 1 at $\mathbb{1} \varepsilon GL(n,R)$. We know that $\det X = \sum_i (-1)^{i+1} X_{1i} M_{1i}$ above M_{ij} is the cofactor of X_{ij} in X. But rank det $(\mathbf{1})$ = rank J det $(\mathbf{1})$ = 1 since the first term $\frac{\partial}{\bar{X}_{11}} \det X (\mathbb{1}) = \frac{\partial X_{11}}{\partial X_{11}} M_{11} \ (\mathbf{1}) = 1$.

Thus by theorem 3.5 $SL(n,R)$ is a closed imbedded submanifold of $GL(n,R)$ of dimension n^2-1. Moreover, for $A,B \ \varepsilon \ SL(n,R)$, $\det AB^{-1} = \det A \det B^{-1} = \det A \det B = 1$, so it is a group and by thoerem 3.9 a Lie group.

Example 3.23: The orthogonal group, $O(n) = \{A \varepsilon GL(n,\mathbb{R}) : {}^tAA = 1\}$ where ${}^tA$ denotes the transpose of A. Let $F: GL(n,\mathbb{R}) \longrightarrow GL(n,\mathbb{R})$ be defined by $F(X) = {}^tXX$, $X \varepsilon GL(n,\mathbb{R})$. Notice that $F(X)$ is a symmetric matrix, i.e. ${}^tF = F$ since ${}^tF = {}^t({}^tXX) = {}^tXX = F$. The subset $S(n,\mathbb{R})$ of symmetric matrices has dimension $\frac{n(n+1)}{2}$ and is, moreover, a C^∞ imbedded submanifold of $GL(n,R)$ (check this, it is not difficult). Now $O(n) = F^{-1}(\mathbb{1})$. We check that rank $F(\mathbb{1})$ = dim $S(n,\mathbb{R})$. Clearly, since F is symmetric rank $F \leqslant \dim S(n,\mathbb{R})$. We compute the Jacobian of F

$$(JF)(\mathbf{1})_{ij,rs} = \frac{\partial \sum_n X_{ik}X_{jk}}{\partial X_{rs}} = \sum_k \left(\delta_{ir}\,\delta_{ks}\,X_{jk} + X_{ik}\,\delta_{rj}\,\delta_{sk} \right) \quad (\mathbf{1})$$

$$= \left(\delta_{ir}\,X_{js} + \delta_{rj}X_{is} \right)(\mathbb{1}) = \delta_{ir}\,\delta_{js} + \delta_{rj}\,\delta_{is}\,.$$

There are thus $\frac{n(n+1)}{2}$ independent rows. So $O(n)$ is a closed imbedded submanifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$. Moreover, it is a Lie group since ${}^t(AB^{-1})\,AB^{-1} = {}^tB^{-1}\,{}^tAAB^{-1} = {}^tB^{-1}B^{-1} = 1$ for $A,B\varepsilon O(n)$.

Example 3.24 The special orthogonal group $SO(n)$:
$SO(n) = \{A\varepsilon O(n) : \det A = 1\}$. Notice that for $A\varepsilon O(n)$ we have $\det {}^tAA = \det \mathbb{1} = 1$. Moreover, $\det {}^tA = \det A$, so $(\det A)^2 = 1$. So for $A \varepsilon O(n)$ $\det A = \pm 1$. Since det is a continuous map this implies that $O(n)$ is disconnected with at least two components given by $O^{\pm}(n) = \{A\varepsilon O(n) : \det A = \pm 1\}$. In fact it can be shown that these two components are connected. $O^{+}(n) = SO(n)$ and it is a group (show it). Clearly its dimension is equal to that of $O(n)$. It is also a closed imbedded submanifold of $GL(n,R)$ since $O(n)$ is. $O^{-}(n)$ is not a group since it does not contain the identity.

Example 3.25: The unitary group $U(n)$:

$U(n) = \{A\varepsilon GL(n,\mathbb{C}) : {}^t\bar{A}A = \mathbf{1}\}$, where $\bar{A}$ means complex conjugate. Now our discussion of real C^∞-manifolds has an analogue in the complex case. We just have local homeomorphisms onto $\mathbb{C}^n$ instead of $\mathbb{R}^n$ and we demand that in overlapping neighborhoods the functions $\psi\circ\phi^{-1}$ be complex analytic diffeomorphisms. Then $GL(n,\mathbb{C})$ is a complex open submanifold of $\mathbb{C}^{n^2}$ and we can also always consider $\mathbb{C}^{n^2}$ as a real manifold namely $\mathbb{R}^{2n^2}$. Then following the same reasoning as in example 3.21, $U(n) = F^{-1}(\mathbf{1})$ where $F(X) = {}^t\bar{X}X$ for $X \varepsilon GL(n,\mathbb{C})$, is a closed real submanifold of $GL(n,\mathbb{C})$. Notice that $U(1)$ is just the circle group S^1 considered previously. We define the special unitary group by $SU(n) = \{A\varepsilon U(n) : \det A = 1\}$. Notice that for $A\varepsilon U(n)$ $\det {}^t\bar{A}A = 1$ so $|\det A|^2 = 1$. Now consider the subset of U(n) of matrices of the form $B(\phi) = \begin{pmatrix} e^{i\phi} & 0 \ldots 0 \\ 0 & \mathbb{1} \\ 0 & \end{pmatrix}$

Such matrices from a subgroup of U(n) isomorphic to $U(1) = S^1$. If A is any matrix of U(n) with determinant $e^{i\phi}$ then there is a unique $C \in SU(n)$ such that $A = B(\phi)C$. It follows that the map $f: S^1 \times SU(n) \longrightarrow U(n)$ defined by $f(\phi, C) = B(\phi)C$ is a C^∞ diffeomorphism and by lemma 3.1 SU(n) is a closed imbedded submanifold of U(n) and thus of $GL(n,\mathbb{C})$ by theorem 3.7. Notice that f is <u>not</u> a group homomorphism.

<u>Theorem 3.10</u>: The groups O(n), SO(n), U(n), SU(n) are compact.

Proof: Since these are all closed subgroups of $GL(2n,\mathbb{R})$, we need only check that they are bounded. But we have $\sum_{k=1}^{n} A_{ik} A_{jk} = \mathbb{1}_{ij}$ in the real case and $\sum_{k=1}^{n} \bar{A}_{ik} A_{jk} = \mathbb{1}_{ij}$ in the complex case. We consider only the latter since the former is easily obtained by restricting A to be real. Then we have $\sum_{k=1}^{n} \bar{A}_{ik} A_{ik} = 1 = \sum_{k=1}^{n} |A_{ik}|^2$ which is clearly bounded.

<u>Example 3.26</u>: Of special importance in physics are the groups SO(3) and SU(2), and we will see that they are locally isomorphic. We study now some properties of SU(2). The group SU(2) is the group of matrices of the form

$$A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C}$$

with $\det A = |\alpha|^2 + |\beta|^2 = 1$. To see this check that $A^{-1} = {}^t\bar{A}$.

Thus SU(2) is homeomorphic to $\{\alpha, \beta \in \mathbb{C} : |\alpha|^2 + |\beta|^2 = 1\}$, that is to the sphere S^3. To see the connection with SO(3), we consider the Hermitian matrix H(i.e. ${}^t\bar{H} = H$) defined by

$$H_x = \begin{pmatrix} x^3 & x^1 + ix^2 \\ x^1 - ix^2 & -x^3 \end{pmatrix}$$

$x^1, x^2, x^3 \in \mathbb{R}$ and are to be identified with the components of a vector $x = (x^1, x^2, x^3) \in R^3$. Now let $u \in SU(2)$ and consider the transformation T_u defined by

$$T_u H_x = u H {}^t\bar{u}$$

$T_u H$ is Hermitian since ${}^t\overline{(uH{}^t\bar{u})} = u{}^t\bar{H}{}^t\bar{u} = uH{}^t\bar{u}$. But since ${}^t\bar{u} = u^{-1}$, $T_u H = uHu^{-1}$ and $T_u H$ has 0 trace since H does. Thus

Thus $T_u H$ can be written as

$$H'_x = \begin{pmatrix} x'^3 & x'^1 + ix'^2 \\ x'^1 - ix'^2 & -x'^3 \end{pmatrix} = T_u H_x = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} x^3 & x^1 + ix^2 \\ x^1 - ix^2 & -x'^3 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}$$

It follows that x'^i is a linear combination of the x^i, that is associated with u there is a 3 × 3 matrix $A^i_j(u)$. Now since $T_u H = uHu^{-1}$ we have $\det T_u H = \det H = -\left[(x')^2 + (x^2)^2 + (x^3)^2\right]$.

Hence T_u preserves determinants and $A^i_j(u)$ preserves the quadratic form $\left[(x^1)^2 + (x^2)^2 + (x^3)^2\right]$. But this means that A(u) must be orthogonal, i.e. $A(u) \in O(3)$. However, since T_u is a continuous map containing the identity transformation on the x^i s , $A(u)$ must be connected to the identity. Thus $A(u) \in SO(3)$. We have thus established a map $A: SU(2) \longrightarrow SO(3)$, and it is easy to check that this is a group homomorphism (i.e. $T_{u_1 u_2} = T_{u_1} T_{u_2}$ thus $A(u_1 u_2) = A(u_1)A(u_2)$). We now show that the map A is surjective. Since SO(3) acts as rotations on the sphere $(x^1)^2 + (x^2)^2 + (x^3)^2 = \frac{1}{4}$, we consider this action and the stereographic projection of example 2.2. Define

$$\zeta = y^1 + iy^2 = \frac{x^1 + ix^2}{\frac{1}{2} - x^3} \qquad \text{(here radius of sphere is } \tfrac{1}{2} \text{ instead of 1)}$$

and similarly

$$w = y^2 + iy^3 = \frac{x^2 + ix^3}{\frac{1}{2} - x^1}$$

Now every rotation $g \in SO(3)$ can be written as $g_{\phi'} g_\theta g_\phi$ where $g_{\phi'}$ and g_ϕ are rotations about the z-axis, and g_θ is a rotation about the x-axis. Thus it suffices to show that g_θ and g_ϕ have an inverse image in SU(2) under A. The rotation g_ϕ is given by

$$x'^1 = x^1 \cos\phi - x^2 \sin\phi \qquad x'^2 = x^1 \sin\phi + x^2 \cos\phi \qquad x'^3 = x^3$$

Therefore,

$$\zeta' = \frac{x'^1 + ix'^2}{\frac{1}{2} - x'^3} = e^{i\phi} \frac{(x^1 + ix^2)}{\frac{1}{2} - x^3} = e^{i\phi}\zeta$$

Similarly for g_θ we have

$$\omega' = e^{i\theta}\omega$$

But straightforward algebra shows that

$$\omega = i\frac{\zeta + 1}{\zeta - 1} \qquad \omega' = i\frac{\zeta' + 1}{\zeta' - 1}$$

Thus

$$\zeta' = \frac{\zeta \cos\frac{\theta}{2} + i \sin\frac{\theta}{2}}{i\zeta \sin\frac{\theta}{2} + \cos\frac{\theta}{2}}$$

Thus, every rotation $g \in SO(3)$ corresponds to a transformation of the form

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \zeta}$$

on the complex plane. The transformation corresponding to g_θ is a matrix

$$u_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & i \sin\frac{\theta}{2} \\ i \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

and we see that $u_\theta \in SU(2)$. Moreover, corresponding to g_θ we have the matrix

$$u_\phi = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix} \qquad \begin{aligned} \alpha &= e^{i\phi/2} \\ \delta &= e^{-i\phi/2} \end{aligned}$$

and $u_\phi \varepsilon\, SU(2)$. So A is surjective. Notice, however, that the choice of u_θ, u_ϕ is not unique. So let us compute ker A. Going back to the equation $H_x = T_u H_x$, we put x' = x and doing the matrix multiplication we find $\alpha\bar{\beta} = \beta\bar{\alpha} = \alpha\beta = \beta^2 = 0$ $|\alpha|^2 = \alpha^2 = 1$. Thus ker A = Z_2, $Z_2 = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$.

To every rotation g there corresponds two unitary matrices (with det = 1) u and -u. Moreover, the matrix **-1** sends u to -u, thus $\alpha, \beta \to -\alpha, -\beta$. If we write $\alpha = z^1 + iz^2$ and $\beta = z^3 + iz^4$, then $|\alpha|^2 + |\beta|^2 = 1$ corresponds to $(z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = 1$, i.e. the sphere S^3. Furthermore, under Z_2 $z^i \to -z^i$. So

$$SO(3) \approx SU(2)/Z_2 = P^3(\mathbb{R})$$

that is, the group SO(3) is diffeomorphic with the projective plane $P^3(\mathbb{R})$. In quantum mechanics representations (unitary) of SU(2) which are not representations of SO(3) (or are "double-valued") are associated with particles of half-odd integer spin e.g. the electron.

<u>Exercise:</u> Show that $g = g_{\phi'} g_\theta g_\phi$ can be written in terms of

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

as

$$g = \begin{pmatrix} \frac{1}{2}\left[\alpha^2+\bar{\alpha}^2-\beta^2-\bar{\beta}^2\right] & \frac{i}{2}\left[\alpha^2-\bar{\alpha}^2+\beta^2-\bar{\beta}^2\right] & -\alpha\beta-\bar{\alpha}\bar{\beta} \\ \frac{i}{2}\left[-\alpha^2+\bar{\alpha}^2+\beta^2-\bar{\beta}^2\right] & \frac{1}{2}\left[\alpha^2+\bar{\alpha}^2+\beta^2+\bar{\beta}^2\right] & i(\alpha\beta-\bar{\alpha}\bar{\beta}) \\ \alpha\bar{\beta}+\bar{\alpha}\beta & i(\alpha\bar{\beta}+\bar{\alpha}\beta) & \alpha\bar{\alpha}-\beta\bar{\beta} \end{pmatrix}$$

Example 3.27: Pseudo-orthogonal group, $O(p,q)$: Let G denote the diagonal $n \times n$ matrix whose first p diagonal entrees are 1 and whose remaining q diagonal entrees are -1, $p+q = n$. We define the pseudo-orthogonal group as

$$O(p,q) = \left\{ A \in GL(n,\mathbb{R}) : {}^tAGA = G \right\}$$

and the group property is easily checked. Now $G^{-1} = {}^tG = G$, so as in example 3.21, we define the C^∞ map $F: GL(n,\mathbb{R}) \longrightarrow S(n,\mathbb{R})$ by $F(A) = {}^tAGA$, and we see that $O(p,q) = F^{-1}(G)$ and rank $F = \frac{n}{2}(n+1)$. It follows that $O(p,q)$ is a C^∞ manifold of dimension $\frac{n(n-1)}{2}$. It is thus a Lie group. As with $O(n)$, $O(p,q)$ is not connected; however, in this case neither is $SO(p,q) = \left\{ A \in O(p,q) : \det A = 1 \right\}$. Of special interest in physics is the Lorentz group $O(3,1)$. It has four connected components and we will show this in a later chapter. Moreover, $SO(3,1)$ has two connected components. The component connected to the identity is denoted by $SO_0(3,1)$. We mention without proof that a relation similar to the relation between $SO(2)$ and $SO(3)$ exists between $SL(2,\mathbb{C})$ and

$SO_0(3,1)$. Namely, the map $A: SL(2,\mathbb{C}) \longrightarrow SO_0(3,1)$ is a Lie group epimorphism (homomorphism onto) with kernel Z_2.

Exercise: Show that the groups defined by

$$U(p,q) = \left\{A \in GL(n,\mathbb{C}) \ : \ {}^t\bar{A}GA = G\right\}$$

are Lie groups. (G defined as in the previous example).

Exercise: Consider the $2n\times 2n$ matrix

$$J = \left[\begin{array}{c|c} 0 & \mathbb{1} \\ \hline -\mathbb{1} & 0 \end{array}\right]$$

where $\mathbb{1}$ is the $n\times n$ identity matrix. Show that the real <u>symplectic group</u> $Sp(n,\mathbb{R})$ defined by

$$Sp(n,\mathbb{R}) = \left\{A \in GL(2n,\mathbb{R}) \ : \ {}^tAJA = J\right\}$$

is a Lie group.

Exercise: Define the unitary <u>symplectic group</u> $USp(n) = Sp(n,\mathbb{R}) \cap U(2n)$. Prove that $USp(n)$ is a compact Lie group. ($USp(n)$ is sometimes denoted by $Sp(n)$).

Exercise: Show that a nonsingular matrix $A \in O(p,q)$ if and only if it leaves the quadratic form $x_1^2+\dots+x_p^2 - x_{p+1}^2-\dots-x_n^2$ invariant, $x_i \in \mathbb{R}$, $q=n-p$.

3.6 Submersions and Quotient Manifolds.

A C^∞-map $F:M \longrightarrow N$ is a called a <u>submersion</u> if rank $F = \dim N$ for every $p \in \text{Dom } F$.

<u>Example 3.28</u>: The C^∞ map $F:\mathbb{R}^{n+1}-\{0\} \longrightarrow S^n$ defined by

$$F(x^1,\ldots,x^{n+1}) = \left(\frac{x^1}{|x|},\ldots,\frac{x^{n+1}}{|x|}\right), \quad |x|^2 = \sum_{i=1}^{n+1} (x^i)^2 ,$$

is onto S^n and is easily seen to have rank n. Thus ϕ is a submersion.

<u>Example 3.29</u>: The projective plane of example 3.7 (and 2.5) is defined by the projection $\rho:\mathbb{R}^{n+1}-\{0\} \longrightarrow P^n(\mathbb{R})$ which assigns to each point of $\mathbb{R}^{n+1}-\{0\}$ its line through $\{0\}$. For the chart (U_i,ϕ_i) on $P^n(\mathbb{R})$ the projection

$$\phi_i \circ \rho(x^1,\ldots,x^{n+1}) = \left\{\frac{x^1}{x^i},\ldots,\frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i},\ldots,\frac{x^{n+1}}{x^i}\right\}$$

It is easy to check that this has rank n on U^i. Thus ρ is a submersion.

Let $F:M \longrightarrow N$ be a continuous map. A (local) <u>section</u> of F is a continuous map

$$\sigma:N \longrightarrow M \quad \text{such that} \quad F\circ\sigma = id\big|_{\text{Dom } \sigma}.$$

<u>Lemma 3.5</u> : Let $F:M \longrightarrow N$ be a submersion. Then for every $p \in \text{dom } F \subset M$ there are a neighborhood V of $F(p)$ and a C^∞ section σ such that $p \in \sigma(V)$.