Proof: Let $p \varepsilon \text{dom}\ \phi$ and $m = \dim$, $n = \dim N$. By theorem 3.4 there are charts (U,ϕ) and (V,ψ) of p and F(p) respectively with $F(U) \subset V$ and $\phi(p) = (0,\ldots,0)$, $\psi(F(p)) = (0,\ldots,0)$ and

$$\hat{F} = \psi \circ F \circ \phi^{-1}(x^1,\ldots,x^m) = (x^1,\ldots,x^n\ ,\ 0,\ldots,0)$$

(in U $\phi = (x^1,\ldots,x^m)$). Let $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be defined by $f(x^1,\ldots,x^n) = (x^1,\ldots,x^n, 0,\ldots,0)$ which is clearly C^∞. Put $\sigma = \phi^{-1} \circ f \circ \psi$. Then by definition $\sigma:N \longrightarrow M$ is C^∞ . Moreover, $F \circ \sigma = id|_V$.

Proposition 3.2: A submersion F is an open map.

Proof: Let $F: M \to N$ be a submersion and $U \subset M$ be open. Suppose $x \varepsilon F(U)$, then by lemma 3.5 there are a neighborhood V of x and a C^∞ section $\sigma : V \to M$ such that $F \circ \sigma = id_V$. Since σ is continuous $\sigma^{-1}(U)$ is an open set containing x. We show that $\sigma^{-1}(U) \subset F(U)$ which implies F(U) is open since x is arbitrary. Let $y \varepsilon \sigma^{-1}(U)$. Then $\sigma(y) \varepsilon U$, so $y = F \circ \sigma(y) \varepsilon F(U)$. QED.

Let ~ be an equivalence relation on a C^∞ manifold M, and let $\rho:M \longrightarrow M/\sim$ denote the quotient projection. $M/\sim$ is called a quotient manifold (not necessarily Hausdorff) if it is given a differentiable structure which makes ρ a submersion. Notice that since ρ is a submersion it is open by prop.3.2. and thus $M/\sim$ with the quotient topology has a countable basis by lemma 2.1. But a priori we do not know that the topology induced by the differential structure is the same as the quotient topology. In fact they are the same and this follows since ρ is both open and continuous. We emphasize again that a quotient manifold is not necessarily Hausdorff. We will however, be mainly concerned with those which are Hausdorff.

We have seen how a quotient manifold structure makes the projection a submersion. Conversely, if $F:M \longrightarrow N$ is a submersion, then we can define on M an equivalence relation ~ by p~q if $F(p) = F(q)$. This clearly defines an equivalence relation; moreover, F can be considered as the projection map. Thus N has the differential structure of a quotient manifold, and every submersion gives rise naturally to a quotient manifold.

3.7 Discontinuous Tranformation Groups

Here we will introduce some ideas about transformation groups acting on a manifold, in particular discrete transformation groups. The reason for introducing these here is that they provide a rich supply of differentiable manifolds. We will study more general Lie transformation groups in a later chapter.

A group G is said to act on M(or be a transformation group on M if

i) there is a global function $\phi: G\times M \longrightarrow M$ such that the function $\phi_g: M \longrightarrow M$ for fixed $g\varepsilon G$ defined by $\phi_g(p) = \phi(g,p)$, $p\varepsilon M$, is a diffeomorphism.

ii) if $g,h\varepsilon G$, then $\phi_g \circ \phi_h = \phi_{gh}$

The function ϕ is called the action. Notice that if $e\varepsilon G$ is the identity then ϕ_e is the identity diffeomorphism on M since putting $p' = \phi_e^{-1}(p)$, $p \varepsilon M$, we get

$$\phi_e(p) = \phi_e(\phi_e(p')) = \phi_e \circ \phi_e(p') = \phi_{ee}(p') = \phi_e(p') = p.$$

The group G is said to act effectively on M if e is the only element of G which satisfies $\phi_g(p) = p$ for all $p \varepsilon M$. It is said to act freely on M if e is the only element of G which satisfies $\phi_g(p) = p$ for ..., $p \varepsilon M$. Note that if $\phi_g(p) = p$ for all $g \varepsilon G$ then p is a fixed point under the action of G. Thus, a free action means that there are no fixed points under G or any subgroup of G save the identity. The orbit $\mathcal{O}_p$ of $p \varepsilon M$ is the set $\mathcal{O}_p = \{p' = \phi_g(p)$: for all $g \varepsilon G\}$. If for any $p, q \varepsilon M$ there is a $g \varepsilon G$ such that $q = \phi_g(p)$ then G is said to act transitively on M. In this case for any $p \varepsilon M$ $\mathcal{O}_p = M$, there is just one orbit. Notice that the orbits can be used to define equivalence relation on M , i.e $p \sim q$ if there is $g \varepsilon G$ such that $q = \phi_g(p)$. It is easy to check that ~ is equivalence relation (check it!). In fact the equivalence classes are just the orbits. The set of equivalence classes M/G is called the orbit space. The quotient projection $\mu : M \longrightarrow M/G$ sends $p \varepsilon M$ to its orbit $\mathcal{O}_p$. The topology on M/G is the quotient topology. However, M/G may not have a manifold structure, i.e. it may not be a quotient manifold since μ is not necessarily a submersion. Notice that if M is a group, the orbit space is just the space of cosets (left or right).

Example 3.30: $GL(n,\mathbb{R})$ acts on R^n as matrices acting on column vectors.

$$x'^i = \sum_{j=1}^{n} a^i_j \, x^j .$$

Given any $x=(x^1,\ldots,x^n) \neq 0$ we can obtain an arbitrary x' by varying a^i_j . But if $x = 0$ then $x' = 0$ so the origin $(0,\ldots,0)$ is a fixed point of $GL(n,\mathbb{R})$. Thus there are two orbits $\{0\} = (0,\ldots,0)$ and $\mathbb{R}^n - \{0\}$. The later has dimension n while the former has dimension 0. Thus $\mu:\mathbb{R}^n \longrightarrow \mathbb{R}^n/GL(n,\mathbb{R})$ does not have constant rank; it is not a submersion. On $\mathbb{R}^n-\{0\}$ $GL(n,\mathbb{R})$ acts transitively. A more interesting example is found considering the subgroup $O(n) \subset GL(n,\mathbb{R})$. Here the norm $|x| = [\sum_{i=1}^{n} (x^i)^2]^{1/2}$ is left invariant, so the orbits under $O(n)$ are spheres of radius $|x|$. The projection $\mu:\mathbb{R}^n \longrightarrow \mathbb{R}^n/O(n)$ again does not have constant rank since the orbit $\{0\}$ has dimension 0 whereas all other orbits have dimension n-1. However, restricting to $\mathbb{R}^n-\{0\}$, μ is a submersion and the orbit space $\mathbb{R}^n-\{0\}/O(n)$ is diffeomorphic to $\mathbb{R}^1$.

We will now consider certain discrete transformation groups such that the orbit space will always be a C^∞ manifold (Hausdorff). By a <u>discrete group</u> Γ we mean a group with at most a countable number of elements and whose topology is the discrete topology. Thus Γ is a C^∞ manifold since it has a countable basis, and is Hausdorff (discrete topology is Hausdorff) and is locally diffeomorphic to a zero dimensional Euclidean space, i.e. to a point. Thus a discrete group is a zero dimensional Lie group.

Let G be a transformation group which acts freely on M, then G is said to act <u>discontinuously</u> on M if each point $p \in M$ has a neighborhood U such that $U \cap gU = \{\emptyset\}$ for all $g \in G$ $g \neq e$.

(Here and hereafter we write gU instead of $\phi_g(U)$ similarly gp instead of $\phi_g(p)$). This action is said to be <u>properly discontinuous</u> if in addition every pair of nonequivalent points $p, q \in M$ (i.e. p, q lie on different G-orbits) have neighborhoods U_p and U_q, respectively, such that $U_p \cap gU_q = \emptyset$ for all $g \in G$.

<u>Theorem 3.11</u>: Let G be a proper discontinuous transformation group acting freely on M. Then the quotient space M/G is a C^∞-manifold with $n = \dim M = \dim M/G$. Moreover, if μ is the quotient projection map, then rank $\mu = n$, and M/G is a quotient manifold (Hausdorff).

Proof: Since each $\phi_g \equiv g$ is a diffeomorphism, $[U] = \bigcup_{g \in G} gU$ is open, so μ is open. Thus, M/G has a countable basis by lemma 2.1. The condition that G acts properly is precisely the condition that M/G be Hausdorff. The condition that G acts discontinuously and freely says that there is a neighborhood U of p in M such that $\mu|_U$ is injective. We choose U such that (U, ϕ) is a chart of M. Let $\nu = (\mu|U)^{-1}$, then $\psi \equiv \phi \circ \nu$ is 1-1 (injection) and continuous. Moreover, $\psi^{-1} = \nu^{-1} \circ \phi^{-1} = \mu|U \circ \phi^{-1}$ is continuous. So $(\mu(U), \psi)$ is a chart for M/G. Doing this at every point of M/G gives us an atlas for M/G. We now show that this is a C^∞ atlas. Let $(\mu(U), \psi)$ and $(\mu(U'), \psi')$ be two such charts of G/M. Then $\psi' \circ \psi^{-1} = \phi' \circ \nu' \circ \nu^{-1} \circ \phi^{-1}$ on [illegible] $\cap$ [illegible] $\neq \{\emptyset\}$. Let $p_1 \in \nu(\mu(U) \cap \mu(U'))$ and $p_2 = \nu' \circ \nu^{-1}(p_1) = \nu' \circ \mu|_U(p_1) = \nu'([p_1]) = (\mu|_{U'})^{-1}([p_1])$ which says $\mu(p_2) = [p_1]$, that is that p_1 and p_2 are equivalent. Thus there is a $g \in G$ such that $p_2 = gp_1$. Hence,

$U'' = U \cap g^{-1}U'$ is open in M and contains p_1. But since $\mu(g^{-1}U') = \mu(U')$, $U'' \subset \nu(\cap)$. Now let $p \varepsilon U''$ be arbitrary. Since $\mu(p) \varepsilon \cap (U)$. We have for the unique $p' \varepsilon U'$ such that $\mu(p') = \mu(p)$ that $\nu' \circ \nu^{-1}(p) = \nu' \circ \mu|_U(p) = \nu'([p]) = (\mu|_{U'})^{-1}([p]) = p'$. But then there is a $g \varepsilon G$ such that $p' = gp$. Hence on U'' $\nu' \circ \nu^{-1} = L_g$ which is a diffeomorphism. Thus M/G is a C^∞ manifold. Moreover, the charts $(\mu(U), \psi)$ locally come from (U, ϕ) so dim M/G = dim M. Moreover on $\phi(U)$ $\hat{\mu} = \psi \circ \mu \circ \phi^{-1}$ is a diffeomorphism, in fact it is the identity function so rank μ = dimM = dim M/G , μ is a submersion and M/G is a Hausdroff quotient manifold.

Remark: If we drop the condition that G acts properly, then M/G is a non Hausdorff quotient manifold.

We shall now prove a theorem which will provide us with groups acting properly discontinuously. First, we prove

Lemma 3.6: Let G be a Lie group and Γ subgroup of G having the property that there exists a neighborhood U of e (identity) ε G such that $U \cap \Gamma = \{e\}$. Then Γ is a countable closed subset of G and is discrete as a subspace of G. (Such a Γ is called a discrete subgroup).

Proof: Let U be a neighborhood of $e \varepsilon G$. Since the map $(g_1, g_2) \longrightarrow g_1 g_2^{-1}$ is continuous and sends $(e,e) \longrightarrow e$, there is a neighborhood V of e such that $VV^{-1} \subset U$. Let $\{h_n\} \varepsilon \Gamma$ be a sequence and g its limit point, then there is an integer $N>0$ such that for all $n>N$ $h_n \varepsilon Vg$. Now let $v_n, v_m \varepsilon V$ be chosen by $h_n = v_n g$

$h_m = v_m g$ for $n,m > N$. Then $h_n h_m^{-1} = v_n v_m^{-1} \varepsilon U$. But $U \cap \Gamma = \{e\}$ so $h_n h_m^{-1} = e$ or $h_n = h_m$ for all $n,m>N$. Thus $h_n = h_m = g$ for all $n,m>N$, so $g \varepsilon \Gamma$; hence Γ is closed. Moreover, for $h \varepsilon \Gamma$ hU is a neigborhood of h with $hU \cap \Gamma = \{h\}$. Thus, Γ is discrete. To see that Γ is countable we first notice that the family of sets $\{hV , h\varepsilon\Gamma\}$ is indexed by Γ . Furthermore, if $h_1 V \cap h_2 V \neq \emptyset$ then there are $v_1, v_2 \varepsilon V$ such that $h_1 v_1 = h_2 v_2$ which implies $h_2^{-1} h_1 = v_2 v_1^{-1} \varepsilon VV^{-1} \subset U$ implying $h_1 = h_2$ since $U \cap \Gamma = \{e\}$. So the sets $\{hV\}, h\varepsilon\Gamma$, are disjoint open sets of G. But G has countable basis, so the family $\{hV, h\varepsilon\Gamma\}$ must be countable; hence Γ is countable. Q.E.D.

<u>Theorem 3.12</u>: A discrete subgroup Γ of a Lie group G acts freely and properly discontinuously on G by left (or right) translations.

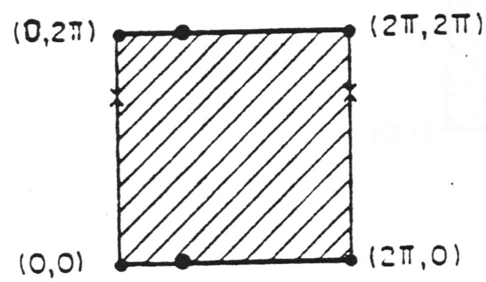
Proof: The action is clearly free since if $gh = g$ for $h\varepsilon\Gamma$, $g\varepsilon G$, then we must have $h = e$. To see that the action is discontinuous we choose neighborhoods U,V of e as above with $VV^{-1} \subset U$ and $U \cap \Gamma = \{e\}$. Suppose $v' \varepsilon V \cap hV$, $e \neq h \varepsilon \Gamma$, then there is a $v \varepsilon V$ such that $v' = hv$. Thus $h = v'v^{-1} \varepsilon VV^{-1} \subset U$ implying $h = e$. Contradiction so $V \cap hV = \{\emptyset\}$. By left translations the argument applies to any point $g\varepsilon G$ with neighborhood Vg. To see that the action is proper we consider two distinct Γ-orbits in in G , Γ_x and Γ_y . Then $x \notin \Gamma_y$ and since Γ is closed and left translation is a homeomorphism Γ_y is closed. But since a manifold is regular by theorem 1.9 there are neighborhoods U of x and $\tilde{U}$ of Γ_y such that $U \cap \tilde{U} = \emptyset$. But every nbd $\tilde{U}$ can be written as $\tilde{U} = \Gamma V$

where V is a neighborhood of y. Thus $U \cap \Gamma V = \emptyset$. Q.E.D.

The following follows immediately from theorems 3.11 and 3.12:

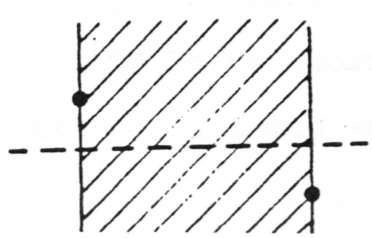
Corollary: If Γ is a discrete subgroup of a Lie group G, then the space of right (or left) cosets G/Γ is a C^∞-manifold and the quotient map $\mu: G \longrightarrow G/\Gamma$ is a C^∞ submersion.

Example 3.31: According to example 3.18, $\mathbb{R}^n$ is a Lie group. Let Γ be the subgroup of R^n which acts on $\mathbb{R}^n$ as $x^i \longrightarrow x^i + 2\pi k_i$ where k_i are integers and (x^i) is the Cartesian chart. Consider the neighborhood U of $e = (0,\ldots,0) \in \mathbb{R}^n$ defined by $U = S(0,\varepsilon)$, where $\varepsilon<1$. Now $\Gamma \approx \mathbb{Z}^n$ as a set ($\mathbb{Z}^n = Z \times \ldots \times Z$, n times, Z = integers). So Γ is the set of all points of the form $2\pi(k_1,\ldots,k_n)$ $k_i \in Z$. Clearly $U \cap \Gamma = \{e\}$, so Γ is discrete. It follows that R^n/Γ is a C^∞ manifold. Let us see what this is. If n= 1 then $R/Z = S^1$. But $R^n = R^1 \times \ldots \times R^1$ and is commutative as a group (Abelian), i.e. $x+y = y+x$. So $\mathbb{R}^n/\mathbb{Z}^n = R^1/\mathbb{Z} \times \ldots \times R^1/\mathbb{Z} = S^1 \times \ldots \times S^1$ i.e. $\mathbb{R}^n/\Gamma = T^n$ (n dimensional Torus) as sets. It remains to check that this holds as C^∞ manifolds. Consider the case $R^1/\mathbb{Z} = S^1$. On R^1 take neighborhoods $U_k = (2\pi k, 2\pi(k+1))$, $k = 0,\pm 1,\ldots$. The projection $\mu: R \longrightarrow \mathbb{R}/\mathbb{Z}$ restricted to U_k is a bijection for each k. Moreover, $[U_k] = (U_k) = \mu(U_\ell)$ for all $k,\ell \in Z$, and $[U_k]$ is open since u is open and a representative is $(0,2\pi)$. Then the map $f:(0,2\pi) \longrightarrow S^1 - (1,0)$ of example 3.5 provides a chart on R/Z. Now consider neighborhoods U'_k on R^1 of the form $(-\pi+2\pi k, \pi+2\pi k)$. Again on U'_k μ is a bijection with $\mu(U'_k) = \mu(U'_\ell)$. A representative is $(-\pi,\pi)$. The map f': $(-\pi,\pi) \longrightarrow S^1-(-1,0)$ provides another chart for S^1. These are precisely the two charts of example 3.2. Moreover, $\mu(U_k \cup U'_k) = R/Z$. Doing the same thing for each dimension provides an identification $R^n/Z^n = T^n$ as C^∞ manifolds. Notice that what we have done above is just consider a certain subset F of $\mathbb{R}^n$ which contains just one point from each equivalence class. The closure $\bar{F}$ of such a subset is called a <u>fundamental domain</u>. Here we draw $\bar{F}$ in the case of R^2/Z^2.



We identify all points denoted with •, or ×.

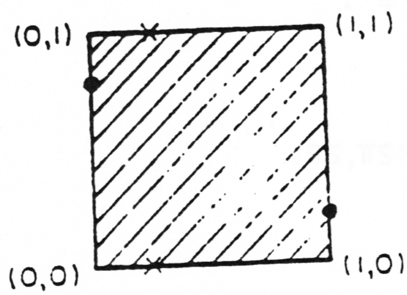
<u>Example 3.32:</u> Consider the discrete group $\Gamma = \mathbb{Z}$ acting on $\mathbb{R}^2$ by sending $(x,y) \longmapsto (x+k, (-1)^k y)$, $k \in \mathbb{Z}$. One can check that this action is free and properly discontinuous (left as an exercise). Thus $\mathbb{R}^2/\Gamma$ is a C^∞ manifold. In fact, since in $\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\}$ the only non-trivial relations are $(0, y) \sim (1, -y)$, $\mathbb{R}^2/\Gamma$ is the infinite Mobuis strip. The fundamental domain is given by



For the next example we need to recall the "twisted" or semidirect product $G \rtimes H$ of two groups G and H. Suppose there is a <u>representation</u> of the group G in the group of automorphisms of H, i.e. a group homomorphism $\rho: G \longmapsto \text{Aut } H$. Then we can define a group structure on the set $G \times H$ by

$$(g',h') \cdot (g,h) = (g'g, (\rho(g')h) \cdot h').$$

<u>Example 3.33:</u> The discrete group $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$ acts on $\mathbb{R}^2$ by sending $(x,y) \longmapsto (x+k, (-1)^k y + \ell)$. Again it is left as an exercise to check that this action is C^∞, free, and properly discontinuous. Thus $\mathbb{R}^2/\Gamma$ is a C^∞ manifold. The fundamental domain is



$\times$: $t_{0,1}$ $(x,y) \sim (x,y+1)$
$(0,y) \sim (0,1+y)$

$\bullet$: $t_{1,1}$ $(x,y) \sim (x+1,-y+1)$
$(0,y) \sim (1,1-y)$

Here $\mathbb{R}^2/\Gamma$ is known as the <u>Klein bottle</u>.

<u>Exercise:</u> Show that the Γ's of examples 3.32 and 3.33 act freely and properly discontinuously.

<u>Exercise:</u> Consider the rotation group SO(3) and a regular m-polygon in the x-y plane with center at the origin in $\mathbb{R}^3$. Show that the set of rotations Z_m of angle $2\pi/m$ about the z-axis is a discrete subgroup of SO(3). It is called the <u>cyclic group</u> of order m. Now draw a line L from one of the vertices of

the m-polygon through the origin until it meets an edge or vertex of the polygon. A rotation R about L of angle π is a symmetry of the polygon. Show that the set $\{Z_m, R\}$ is a discrete subgroup of SO(3). It is called the dihedral group of order 2m.

Exercise: Using methods of this section show that $P^n(\mathbb{R})$ is a C^∞ manifold.

Exercise: Show that the infinite Möbius strip of example 3.32 and the Möbius strip of example 2.4 are diffeomorphic.

Theorem 3.13: Let Γ be a discrete group acting freely and properly discontinuously on a C^∞ manifold M. Then M/Γ is compact if and only if there is a compact subset $K \subset M$ such that $\Gamma K = M$.

Proof: Suppose M/Γ is compact and $\rho : M \to M/G$ is the quotient map. For each $\bar{x} \in M/\Gamma$ choose a point $x \in \rho^{-1}(\bar{x})$ M, and let K denote the set of such points. Then $\Gamma K = M$. Let $\{U_\alpha\}$ be an open cover of M. Then $\{\rho(U_\alpha)\}$ is an open cover for M/Γ. Let $\{\rho(U_{\alpha_i})\}$ be a finite subcover. Then $\{U_{\alpha_i}\}$ is a finite subcover for K.

Conversely, let K be a compact subset of M such that $\Gamma K = M$. Since ρ is continuous $\rho(K)$ is a compact subset of M/Γ. But if $\bar{x} \in M/\Gamma$ and $x \in M$, there are $\gamma \in \Gamma$ and $k \in K$ such that $x = \gamma k$. So $\bar{x} = \rho(x) = \rho(\gamma k) = \rho(k) \in \rho(K)$, i.e. $\rho(K) = M/\Gamma$. So M/Γ is compact.

It follows easily from this theorem that the torus and Klein bottle are compact.

Theorem 3.14: Any finite group Γ acting freely on a manifold M acts properly discontinuously.

Proof: Let $\Gamma = \{\gamma_0 = e, \gamma_1, \ldots, \gamma_n\}$. Since M is Hausdorff and Γ acts freely, if we choose any $x \in M$, there are neighborhoods U_j, $j = 0, \ldots n$ of $\gamma_j x$ such that $U_0 \cap U_j = \emptyset$ for all $j = 1, \ldots, n$. Now define $U = \bigcap_{j=0}^{n} \gamma_j^{-1} U_j$, then $x \in U$ and $U \cap \gamma U = \emptyset$ for all $\gamma \neq e$. Moreover if x and y lie in distinct Γ-orbits there is a neighborhood U_y of y such that $U_y \cap U_j = \emptyset$ for all $j = 0, \ldots, n$. Thus $U_y \cap \gamma U = \emptyset$ since $\gamma_j U \subset U_j$.

Example 3.34: Lens spaces: Consider $S^3 = \{(z^1, z^2) \in \mathbb{C}^2 \mid |z^1|^2 + |z^2|^2 = 1\}$. Fix a prime p. The cyclic group $\mathbb{Z}_p$ of order p acts on S^3 as follows: Fix any q relatively prime to p and consider the transformation

$$(z^1, z^2) \longmapsto (e^{\frac{2\pi i}{p}} z^1, e^{\frac{2\pi i q}{p}} z^2)$$

Call this transformation h. The cyclic group $\mathbb{Z}_p = \{e, h, h^2, \ldots, h^{p-1}\}$ acts freely on S^3 since if $(h^r z^1, h^{qr} z^2) = (z^1, z^2)$ for $r = 0, \ldots, p-1$, then either $h^r = 1$ or $h^{qr} = 1$ implying $r = 0$. Thus by theorem 3.14 $\mathbb{Z}_p$ acts properly discontinuously on S^3 and by theorem 3.11 the quotient space $L(q,p)$ (called a lens space) is a C^∞ manifold of dimension 3. Since S^3 is compact, so is $L(q,p)$. Note that $L(1,2) = \mathbb{P}^3(\mathbb{R})$. This construction can be generalized to any odd dimensional sphere.