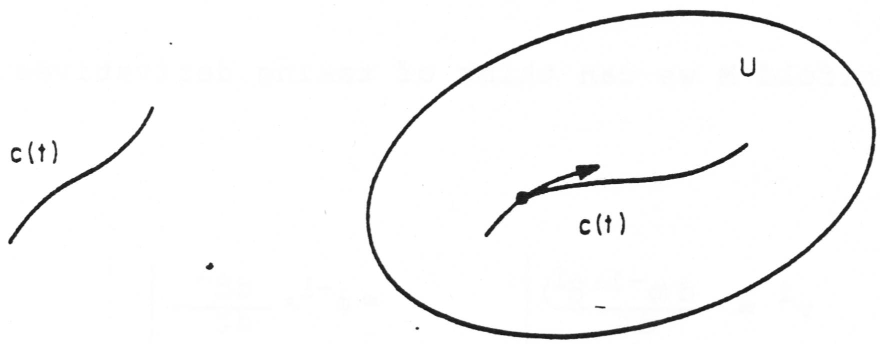
4. The Tangent Bundle

In order to proceed further in our study of differential geometry, we will need the very important concept of tangent vector. There are several definitions of tangent vector in the literature, all more or less equivalent. The definition that I will give is short and very convenient; however, it is less geometric than others. We will, therefore, then tie our definition to the more intuitive idea of a tangent to a curve and see that it really is the same thing.

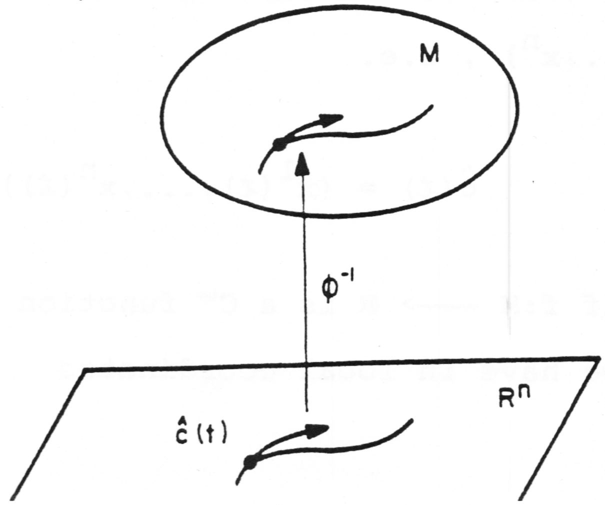
4.1. Tangent vectors and the tangent space.

To motivate our definition we consider a curve in an open set $U \subset M$ parameterized by a real parameter t

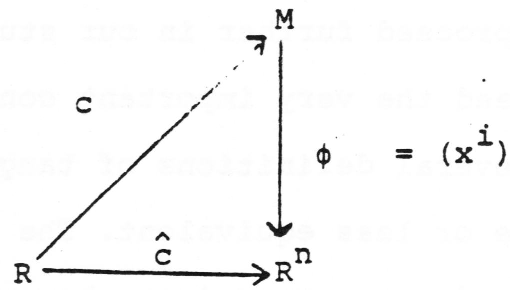


What we really mean by this is a function on U, f, whose coordinate representative depends on a parameter t. That is

$$c(t) = \phi^{-1} \circ \hat{c}(t)$$



Now what we mean by $\hat{c}(t)$ is $\hat{c}:R \longrightarrow R^n$ and if the chart (U,ϕ) assigns the local coordinates x^i to U then we have



commutes and $\hat{c}(t) = (x^1 \circ c(t), \ldots x^n \circ c(t))$.

The tangent to a curve at a point in R^n is given by its derivative at the point, i.e.

$$\hat{v}^i = \left.\frac{d\hat{c}^i}{dt}\right|_{t = t_0}$$

So on the manifold M we can think of taking derivatives with respect to t

$$v^i = \left.\frac{d(\phi^{-1} \circ \hat{c}^i)}{dt}\right|_{t = t_0} = \phi^{-1} \circ \left.\frac{d\hat{c}^i}{dt}\right|_{t = t_0}$$

We can think of the curve $\hat{c}$ in R^n in terms of the coordinates $(x^1, \ldots, x^n)$, i.e.

$$\hat{c}(t) = (x^1(t), \ldots, x^n(t))$$

Thus if $f:M \longrightarrow \mathbb{R}$ is a C^∞ function which depends on the curve $c(t)$ we have in local coordinates

$$vf = \frac{df}{dt} = \frac{d}{dt}(f\circ\phi\circ c(t)) = \sum_i \frac{d\hat{c}^i(t)}{dt}\frac{\partial\hat{f}}{\partial x^i}$$

and this satisfies the classical Leibnitz formula

$$v(fg) = (vf)g + f(vg)$$

Now we denote by $C^\infty(U)$ the set of C^∞ functions on $U \subset M$. Likewise for $p\varepsilon M$ $\tilde{C}^\infty(p)$ is the set of C^∞ functions on M whose domain is a neighborhood of $p\varepsilon M$.

Proposition 4.1: The set $C^\infty(U)$ forms an associative commutative algebra with unit.

Proof: An algebra is a vector space with an additional multiplication operation. We show that $C^\infty(U)$ is a vector space. Let $f,g\varepsilon C^\infty(U)$ then for $\alpha,\beta\varepsilon R$ $\alpha f+\beta g$ is a C^∞ function on U. Thus $\alpha f+\beta g\varepsilon C^\infty(U)$. Clearly, there is a unique function 0_U which is zero on U, and $0_U\varepsilon C^\infty(U)$. Moreover, the function $-f\varepsilon C^\infty(U)$ (take $\alpha=-1$, $\beta=0$ above) and $f+(-f) = 0_U$. So $C^\infty(U)$ is a vector space. It is also an algebra since multiplication of functions fg can be defined and $fg\varepsilon C^\infty(U)$ if f,g are. Clearly this algebra is associative $f(gh) = (fg)h = fgh$ and commutative $fg = gf$. Furthermore, the algebra has a unit, the function 1_U satisfying $1_U = f$. Q.E.D.

However, the set $\tilde{C}^\infty(p)$ does not fair so well. For example, there is a unique zero function, namely the 0 function on all M. However if $f\varepsilon\tilde{C}^\infty(p)$ with domain U_p then surely $-f\varepsilon\tilde{C}^\infty(p)$ with domain U_p, but $f+(-f) = 0_{U_p}$ that is the zero function on U_p, but this is not necessarily the zero function on M, 0 above.

That is, $0_U \neq 0_M \equiv 0$. (Thus $\tilde{C}^\infty(p)$ is not even a vector space) . To see this explicitly, just consider the functions of example 3.1. Fortunately we can rectify this situation. Indeed, we shall use an algebra something like $\tilde{C}^\infty(p)$ to give the definition of tangent vector.

To circumvent the above problem we introduce the concept of the germ of a function. Let $f,g \varepsilon \tilde{C}^\infty(p)$ and introduce an equivalence relation $f \sim g$ if and only if $f = g$ on some neighborhood containing p. (You should show that ~ indeed defines an equivalence relation on $\tilde{C}^\infty(p)$). The set of all functions in $\tilde{C}^\infty(p)$ equivalent to a given function $f \varepsilon \tilde{C}^\infty(p)$ is called the <u>germ</u> of f. The set of all germs of C^∞ functions at $p \varepsilon M$ is $C^\infty(p) = \tilde{C}^\infty(p)/\sim$.

We will use the same notation, i.e. f to denote a germ, and when we talk about C^∞ functions at $p \varepsilon M$ we will always mean germs unless stated otherwise.

<u>Proposition 4.2</u>: The set $C^\infty(p)$ of germs of C^∞ functions at $p \varepsilon M$ forms an associative commutative algebra with unit.

Proof: <u>Exercise</u>: Now, the zero function on M is equivalent to all functions which vanish on any open set of M.

<u>Definition 4.1</u>: The <u>tangent space</u> $T_p(M)$ to M at $p \varepsilon M$ is the set of all C^∞ mappings $X_p : C^\infty(p) \longrightarrow \mathbb{R}$ which satisfy

i) $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g)$ (linear)

ii) $X_p(fg) = (X_p f)g(p) + f(p)(X_p g)$ (Leibnitz)

for all $\alpha, \beta \varepsilon \mathbb{R}$ and $f,g \varepsilon C^\infty(p)$.

More generally for any algebra A any mapping as defined above is called a <u>derivation</u> of A. We will see later that the set of all derivations of an algebra A forms itself an algebra. Definition 4.1 does not give the correct idea of tangent space in the C^k case for k finite. There is another definition of tangent space for the C^k case which for $k = \infty$ is equivalent to def. 4.1. (see Sternberg). We prefer definition 4.1. here because it is intrinsic and algebraically simple.

We must verify that $T_p(M)$ is in fact a vector space. We define as addition and scalar multiplication

$$(X_p + Y_p)f = X_pf + Y_pf$$

$$(\alpha X_p)f = \alpha(X_pf)$$

for $f \varepsilon C^\infty(p)$, $\alpha \varepsilon \mathbb{R}$, and $X_p, Y_p \varepsilon T_p(M)$. Then, $\alpha X_p + \beta Y_p \varepsilon T_p(M)$ as an easy computation shows. Moreover, $0 \varepsilon T_p(M)$, that is satisfies i) and ii), and $-X_p \varepsilon T_p(M)$ with $X_p + (-X_p) = 0$.

<u>Lemma 4.1</u>: If $X_p \varepsilon T_p(M)$ and f is the constant function (germ) in $C^\infty(p)$. Then $X_pf = 0$.

Proof: Since f is constant it is just a real number thus $(X_pf) = f(X_p1)$ by i). But by ii)

$$X_p(1\cdot 1) = (X_p1)1 + 1(X_p1) = 2X_p(1)$$

$$X_p(1) = 2X_p(1) \Rightarrow X_p(1) = 0 .$$ Q.E.D.

Now if (U,ϕ) is a chart at $p\varepsilon M$ with $\phi = (x^1,\ldots,x^n)$, the function $\left(\frac{\partial}{\partial x^i}\right)_p : C^\infty(p) \longrightarrow \mathbb{R}$ defined by $\left(\frac{\partial}{\partial x^i}\right)_p f = \frac{\partial}{\partial x^i}\left(f\circ\phi^{-1}\right)\Big|_{\phi(p)}$ is a derivation. This is easy to check, and is left as an exercise.

We will need

<u>Lemma 4.2</u>: Let (U,ϕ) be a chart of M such that $\phi = (x^1,\ldots,x^n)$, $\phi(p) = (a^1,\ldots,a^n)$, $p\varepsilon U$. If $f\varepsilon C^\infty(p)$ then there exist functions $h_i \varepsilon C^\infty(p)$ $i = 1,\ldots,n$ and an open set $V\subset U$ containing p such that

$$f = f(p) + \sum_i (x^i-a^i)h_i$$

on V.

Proof: Let ϕ' be the chart on M defined by the coordinates $z^i = x^i-a^i$. Let $S(0,r)$ denote the spherical neighborhood with center $\{0\}$ in $\mathbb{R}^n$. For any $z \varepsilon S(0,r)$ we have

$$\left[f\circ\phi'^{-1}(z^1,\ldots,z^n) - f\circ\phi'^{-1}(0,\ldots,0)\right]_{S(0,r)}$$

$$= \int_0^1 \frac{d}{dt}\, f\circ\phi'^{-1}(tz^1,\ldots,tz^n)\Big|_{S(o,r)} dt$$

$$= \sum_i \int_0^1 \frac{\partial f\circ\phi'^{-1}}{\partial y^i}(tz^1,\ldots,tz^n)\Big|_{S(0,r)} z_i\, dt \equiv \sum_i H_i(z^1,\ldots,z^n)z^i$$

where $y^i = tz^i$.

Put $h_i = H_i \circ \phi'$ and $V = \phi'^{-1}(S(0,r))$.

Since $f, \phi' \in C^\infty(p)$ so is h_i.

Remark: Notice, however, this lemma is not true in the C^k case since the integrand of H_i is only C^{k-1} making H_i only C^{k-1}.

Theorem 4.1: Let (U,ϕ) $\phi = (x^1,\ldots,x^n)$ be a chart on M with $p \in U$, then the vectors $\left(\frac{\partial}{\partial x^i}\right)_p$ form a basis for $T_p(M)$, i.e. any $X_p \in T_p(M)$ can be written as

$$X_p = \sum_i X_p^i \left(\frac{\partial}{\partial x^i}\right)_p .$$

Moreover, $\text{Dim } T_p(M) = \text{Dim } M = n$.

Proof: Let $\phi(p) = a$ and let $X_p \in T_p \in T_p(M)$ and $f \in C^\infty(p)$. Then by lemma 4.2

$$X_p f = X_p(f(p) + \sum_i (x^i - a^i) h_i)$$

$$= X_p \sum_i (x^i - a^i) h_i \quad \text{by lemma 4.1}$$

$$= \sum_i [X_p(x^i - a^i)]\, h_i(p) + \sum_i (x^i - a^i)_a X_p\, h_i$$

$$= \sum_i (X_p x^i)\, h_i(p) .$$

using again lemma 4.1 and the definition of $T_p(M)$. In particular, if $X_p = \left(\frac{\partial}{\partial x^i}\right)_p$, then

$$\left(\frac{\partial}{\partial x^i}\right)_p f = \sum_j \left(\frac{\partial x^j}{\partial x^i}\right)_{\phi(p)} h_j(p) = h_i(p)$$

Thus

$$X_p f = \sum_i (X_p x^i) \left(\frac{\partial}{\partial x^i}\right)_p f$$

and hence $\left(\frac{\partial}{\partial x^i}\right)_p$ span $T_p(M)$. We must show that they are also linearly independent. Consider

$$\sum_i \alpha^i \left(\frac{\partial}{\partial x^i}\right)_p f \qquad \alpha^i \in R$$

and take $f = x^j$ then

$$\sum_i \alpha^i \left(\frac{\partial}{\partial x^i}\right)_p f^j = \sum_i \alpha^i \left(\frac{\partial x^j}{\partial x^i}\right)_{\phi(p)} = 0$$

implies $\sum \alpha^i \delta^j_i = \alpha^j = 0$. So $\left(\frac{\partial}{\partial x^i}\right)_p$ $i = 1,\ldots,n$ is a basis and so $\text{Dim } T_p(M) = n$.

4.2 Pullbacks and Differentials.

Now suppose $F:M \longrightarrow N$ is a C^∞ map of C^∞ manifolds. What happens to functions in $C^\infty(p)$? Let $f \in C^\infty(q)$ where $q = F(p) \in N$, $p \in M$. Then $f(q) = f \circ F(p)$, so $f \circ F$ is a function on M and it is clearly C^∞ since f and F are, i.e. $f \circ F \in C^\infty(p)$. Thus we have a C^∞ map $F^*:C^\infty(q) \longrightarrow C^\infty(p)$ defined by

$F^*(f) = f \circ F$, F^* is called the pullback map.

Proposition 4.3: Let $F:M \longrightarrow N$ be a C^∞ map of C^∞ manifolds M,N. Then

i) F^* is an algebra homomorphism

ii) If $G:N \longrightarrow L$ is C^∞ where L is a C^∞ manifold and if $H = G \circ F$ then $H^* = F^* \circ G^*$

Proof: i) Let $f,g \in C^\infty(q)$ on N. Then

$$F^*(fg) = (fg)\circ F = (f\circ F)(g\circ F) = F^*(f)\, F^*(g), \text{ and}$$

$$F^*(\alpha f + \beta g) = (\alpha f + \beta g)\circ F = \alpha f\circ F + \beta g\circ F = \alpha F^*(f) + \beta F^*(g), \alpha,\beta \varepsilon R$$

ii) Let $f \in C^\infty(G(q)) = C^\infty(G\circ F(p)) = C^\infty(H(p))$

Then $H^*(f) = f \circ H = f \circ G \circ F = G^*(f) \circ F =$

$= F^*(G^*(f)) = F^* \circ G^*(f)$. Q.E.D.

The maps F and F^* determine a map on tangent spaces. As before $F:M \longrightarrow N$ is C^∞ , $f \in C^\infty(q)$, and let $X_p \in T_p(M)$. Define the map $F_*:T_p(M) \longrightarrow T_q(N)$ by $F_*(X_p)f = X_p(F^*(f))$. F_* is called the differential map. We must show that $F_*(X_p)$ is, in fact, in $T_q(N)$. This will follow from

Proposition 4.4: Let F, G, H, M, N, and L be as in proposition 4.3. Then i) $F_*: T_p(M) \longrightarrow T_{F(p)}(N)$

is a vector space homomorphism

ii) $H_* = G_* \circ F_*$

Proof: linearity: $F_*(X_p)(\alpha f + \beta g)$

$= X_p(F^*(\alpha f + \beta g)) = X_p(\alpha F^*(f) + \beta F^*(g))$ by Prop. 4.3

$= \alpha X_p(F^*(f)) + \beta X_p(F^*(g)) = \alpha F_*(X_p) f + \beta F_*(X_p) g$

Leibnitz: $F_*(X_p)(fg) = X_p(F^*(fg)) = X_p(F^*(f)F^*(g))$

$= X_p(F^*(f))F^*(g)(p) + F^*(f)(p)X_p(F^*(g))$

$= F_*(X_p)(f)\ g(F(p)) + f(F(p))F_*(X_p)(g).$

homomorphism: i) $F_*(\alpha X_p + \beta Y_p)f = (\alpha X_p + \beta Y_p)(F^*(f))$

$= \alpha X_p(F^*(f)) + \beta Y_p(F^*(f)) = \alpha F_*(X_p)(f) + \beta F_*(Y_p)(f)$

$= [\alpha F_*(X_p) + \beta F_*(Y_p)] f$

ii) $H_*(X_p) f = X_p(H^*(f)) = X_p(F^* \circ G^*(f))$

$= F_*(X_p)(G^*(f)) = G_* \circ F_*(X_p) f$. Q.E.D.

Remark: If (U,ϕ) is a chart on M and $\phi = (x^1, \ldots, x^n)$ then $\left(\frac{\partial}{\partial x^i}\right)_p = \phi_*^{-1} \frac{\partial}{\partial x^i}$. Now suppose (V,ψ) is another chart on M with $U \cap V \neq \emptyset$ and $\psi = (y^1, \ldots, y^n)$. The change of

<u>coordinates</u> is given by the diffeomorphism $\psi\circ\phi^{-1}(x^1,\ldots,x^n) = (y^1,\ldots,y^n)$. Then for $p \in U\cap V$ we have

$$\left(\frac{\partial}{\partial y^i}\right)_p = \psi_*^{-1}\frac{\partial}{\partial y^i}\,,\quad \left(\frac{\partial}{\partial x^i}\right)_p = \phi_*^{-1}\frac{\partial}{\partial x^i}$$

and on R^n: $$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

so $$\left(\frac{\partial}{\partial x^i}\right)_p = \phi_*^{-1}\frac{\partial}{\partial x^i} = \phi_*^{-1}\sum_j \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} = \sum_j \frac{\partial y^j}{\partial x^i}\,\phi_*^{-1}\circ\psi_*\left(\frac{\partial}{\partial y^i}\right)_p$$

That is,

$$\left(\frac{\partial}{\partial x^i}\right)_p = \sum_j \frac{\partial y^j}{\partial x^i}\,\phi_*^{-1}\circ\psi_*\left(\frac{\partial}{\partial y^i}\right)_p \tag{4.1}$$

More generally, we have

<u>Theorem 4.2</u>: Given a C^∞ map $F:M \longrightarrow N$ with (U,ϕ) and (V,ψ) as charts for M and N respectively, with $F(U)\subset V$. Suppose that in the local coordinates $\phi = (x^1,\ldots,x^m)$ and $\psi=(y^1,\ldots,y^n)$ the map F has the form ($m = \dim M$, $n = \dim N$)

$$y^i = \hat{F}^i(x^1,\ldots,x^m) = [\psi\circ F\circ\phi^{-1}(x^1,\ldots,x^m)]^i \quad i = 1,\ldots,n$$

Then

$$F_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{m}\left(\frac{\partial y^j}{\partial x^i}\right)_{\phi(p)}\left(\frac{\partial}{\partial y^j}\right)_{F(p)}$$

Furthermore, the rank of F at p is exactly the dimension of the image $F_*(T_p(M))$, and F_* isamonomorphism (1-1) if and only if rank F = m, and F_* is an epimorphism (onto) if and only if rank F = n.

Proof: By theorem 4.1 $\left(\frac{\partial}{\partial x^i}\right)_p$ and $\left(\frac{\partial}{\partial y^i}\right)_{F(p)}$ are bases for $T_p(M)$ and $T_{F(p)}(N)$, resp. Moreover, by Prop. 4.4, $F_*\left(\frac{\partial}{\partial x^i}\right)_p \in T_{F(p)}(N)$ and since $\left(\frac{\partial}{\partial y^i}\right)_{F(p)}$ is a basis for $T_{F(p)}(N)$, we have

$$F_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_j \alpha_i{}^j \left(\frac{\partial}{\partial y^j}\right)_{F(p)}$$

for some $\alpha_i{}^j \in R$. In particular, applying this to the $C^\infty(F(p))$ functions y^i we have

$$\alpha_i{}^j = F_*\left(\frac{\partial}{\partial x^i}\right)_p y^j = \left(F_* \circ \phi_*^{-1} \frac{\partial}{\partial x^i}\right) y^j = \frac{\partial}{\partial x^i} y^j (F \circ \phi^{-1})$$

$$= \frac{\partial}{\partial x^i} \hat{F}^j = \left(\frac{\partial y^j}{\partial x^i}\right)_{\phi(p)} .$$

For the second part, we notice that $\left(\frac{\partial y^j}{\partial x^i}\right)$ is precisely the Jacobian matrix of the map $\psi \circ F \circ \psi^{-1}$ and thus the Jacobian matrix of F. Q.E.D.

<u>Remarks</u>: The conditions on the rank of F that we used so frequently in the previous chapter now appear in terms of the map F_*. In fact, F is an immersion if and only if F_* is a monomorphism for all $p \in M$, and F is a submersion if and, only if F_* is an epimorphism for all $p \in M$. Furthermore, F is a local diffeomorphism if and only if F_* is an isomorphism for all $p \in M$. (Recall local diffeomorphism means that each $p \in M$ has a neighborhood U such that $F|_U$ is a diffeomorphism).

<u>Exercise</u>: Find a counterexample to show that this last statement is not true globally.

Notice that our formula (4.1) for a change of coordinates appears as a special case of theorem 4.2 by taking M = N and F the identity map on $U \cap V$. We can also compute from theorem 4.2 how the components of any two vectors with respect to the <u>canonical bases</u> $\left(\frac{\partial}{\partial x^i}\right)_p$ and $\left(\frac{\partial}{\partial y^j}\right)_q$ transform. Indeed, if

$$X_p = \sum_i X^i \left(\frac{\partial}{\partial x^i}\right)_p \qquad Y_p = \sum_y Y^j \left(\frac{\partial}{\partial y^j}\right)_{F(p)}$$

then if

$$Y_p = F_* X_p = \sum_i X^i F_* \left(\frac{\partial}{\partial x^i}\right)_p = \sum_i \sum_j X^i \left(\frac{\partial y^j}{\partial x^i}\right)_{\phi(p)} \left(\frac{\partial}{\partial y^y}\right)_p$$

$$= \sum_j Y^j \left(\frac{\partial}{\partial y^j}\right)_{F(p)}$$

So

$$(4.2) \qquad Y^j = \sum_i X^i \left(\frac{\partial y^j}{\partial x^i}\right)_{\phi(p)}$$

Exercise: Let $F:N \longrightarrow M$ be a C^∞ map, show that for $p \in N$

$$\dim F_* T_p(N) + \dim \ker(F_*)_p = \dim T_p(N)$$

Exercise: Prove the converse to lemma 3.5, i.e..
Let $F:M \longrightarrow N$ be C^∞ and suppose that for every $p \in \operatorname{dom} F$ there are a neighborhood U of F(p) and a C^∞ section σ of F such that $p \in \sigma(U)$. Then F is a submersion.

Now consider the product manifold $M_1 \times M_2$ of two C^∞ manifolds M_1 and M_2. Charts of $M_1 \times M_2$ are of the form $(U_1 \times U_2, \phi_1 \times \phi_2)$ with local coordinates $\phi_1 = (x^1, \ldots, x^{m_1})$ $\phi_2 = (y^1, \ldots, y^{m_2})$ $\phi_1 \times \phi_2 = (x^1, \ldots, x^{m_1}, y^1, \ldots, y^{m_2})$ where $m_i = \dim M_i$, $i = 1,2$. Let $p_i \in M_i$ $(i = 1,2)$. There is a natural identification of $T_{(p_1,p_2)}(M_1 \times M_2)$ with $T_{p_1}(M_1) + T_{p_2}(M_2)$. To see this we notice by theorem 4.1 that every tangent vector $X_p \in T_p(M_1 \times M_2)$, $p = (p_1, p_2)$ can be written as

$$X_p = \sum_{i=1}^{m_1} X_p^i \left(\frac{\partial}{\partial x^i}\right)_p + \sum_{i=1}^{m_2} Y_p^i \left(\frac{\partial}{\partial y^i}\right)_p$$

But if we define the natural projection maps $\pi_i : M_1 \times M_2 \longrightarrow M_i$ by $\pi_i(p) = \pi_i(p_1, p_2) = p_i$, $i = 1,2$, then $\left(\frac{\partial}{\partial x^i}\right)_p = \left(\frac{\partial}{\partial x^i}\right)_{\pi(p)} = \left(\frac{\partial}{\partial x^i}\right)_{p_1}$

and similarly for $\left(\frac{\partial}{\partial y^i}\right)_p$. Thus $\pi_{i*}X_p \in T_{p_i}(M_i)$, and

$$(\pi_1,\pi_2)_* X_p = \sum_{i=1}^{m_1} X_p^i \left(\frac{\partial}{\partial x^i}\right)_{p_1} + \sum_{i=1}^{m_2} Y_p^i\left(\frac{\partial}{\partial y^i}\right)_{p_2} \in T_{p_1}(M_1) + T_{p_2}(M_2).$$

It is easy to check that $(\pi_1,\pi_2)_*$ is an isomorphism. Notice $(\pi_1\ ,\pi_2): M_1 \times M_2 \longrightarrow M_1 \times M_2$ is the identity map. Moreover, it follows from the change of coordinate formulae (4.1) and (4.2) that this identification is natural (i.e. independent of coordinates).

Now let $F: M_1 \times M_2 \longrightarrow N$ be a C^∞ map of C^∞ manifolds $M_1 \times M_2$ and N. Identify $Z_p \in T_p(M_1 \times M_2)$ with $(X_{p_1}, Y_{p_2}) \in T_p(M_1) + T_{p_2}(M_2)$, i.e. $(\pi_1,\pi_2)_* Z_p = (X_{p_1}, Y_{p_2})$ and define $F_i : M_i \longrightarrow N$ by $F_i = F\circ j_i$ where $j_i : M_i \longrightarrow M_1 \times M_2$ is the natural injection.

F_i is clearly C^∞ . Moreover, we have

$$(4.4)\quad F_*(Z_p) = F_{1*}(X_{p_1}) + F_{2*}(Y_{p_2}) .$$

To see this we write the identity on $M_1 \times M_2$ as (j_1, j_2). Then

$$(j_1,j_2)_*(X_{p_1},Y_{p_2}) = (j_1,j_2)_*(\pi_1,\pi_2)_* Z_p$$

$$= ((j_1,j_2)\circ(\pi_1,\pi_2))_* Z_p = Z_p ,$$ and $F_* Z_p =$

$$F_*(j_1,j_2)_*(X_{p_1}, Y_{p_2}) = (F_{1*},F_{2*})(X_{p_1}, Y_{p_2}) =$$

$$= F_{1*}(Y_{p_1}) + F_{2*}(X_{p_2}).$$

Remark: Notice that in a trivial way the maps $\pi_i : M_1 \times M_2 \longrightarrow M_i$ are submersions and the maps $y_i : M_i \longrightarrow M_1 \times M_2$ are global sections of π_i.

Example 4.1: Curve in M: Let $c : R \longrightarrow M$ be C^∞ in an open set $(a,b) \subset R$. Let t be the coordinate on R, then $\left(\frac{d}{dt}\right)_{t_0}$ is a tangent vector at $t_0 \in (a,b)$. If c is an immersion at $c(t_0) \in M$, then

$$c_*\left(\frac{d}{dt}\right)_{t_0} \quad \text{is tangent to the curve } c(t) \text{ in M. If } (x^i) \text{ are}$$

local coordinates for the chart (U,ϕ) in M, then we have

$$c_*\left(\frac{d}{dt}\right)_{t_0} = \sum_i \alpha^i \left(\frac{\partial}{\partial x^i}\right)_{c(t_0)}$$

We can think of the coordinates on $c(a,b) \subset M$ as functions of t, i.e.

$$\hat{c}(t) = \phi \circ c(t) = (x^1(t), \ldots, x^n(t))$$

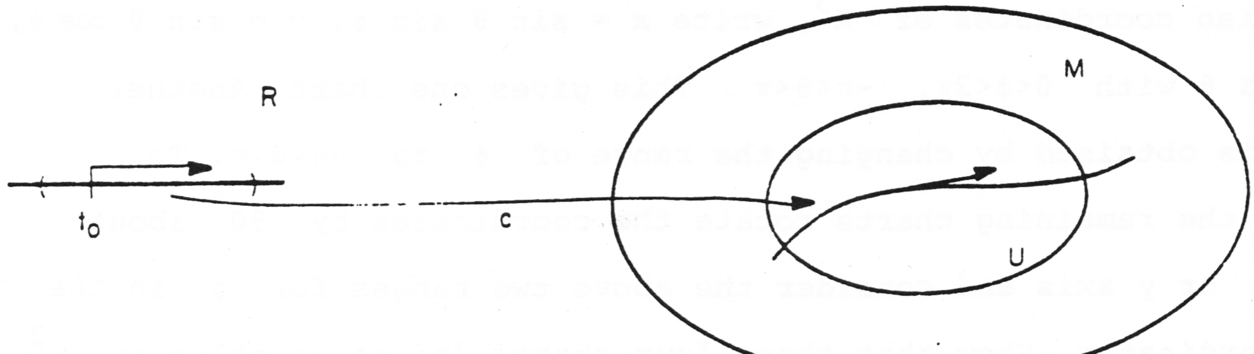
so we have

$$c_*\left(\frac{d}{dt}\right)_{t_0} \phi = \left(\frac{d}{dt}\right)_{t_0} \phi \circ c = \left(\frac{dx^1}{dt}\right)_{t_0}, \ldots, \left(\frac{dx^n}{dt}\right)_{t_0}$$

$$= \left(\frac{d}{dt}\right)_{t_0} (x^1, \ldots, x^n) = \sum_i \alpha^i \left(\frac{\partial}{\partial x^i}\right)_{c(t_0)} (x^1, \ldots, x^n) = (\alpha^1, \ldots, \alpha^n)$$

So we have

$$(4.3) \qquad c_*\left(\frac{d}{dt}\right)_{t_0} = \sum_i \left(\frac{dx^i \circ c}{dt}\right)_{t_0} \left(\frac{\partial}{\partial x^i}\right)_{c(t_0)}$$



Example 4.2: Surface in R^3: Let $F: M \longrightarrow R^3$ be an immersion with dim $M = 2$. If (U,ϕ) is a chart of M with $\phi = (u,v)$. On R^3 we have the identity chart (R^3,ψ) $\psi = (x,y,z)$. So the coordinate representative of F is

$$\hat{F}(u,v) = F \circ \phi^{-1}(u,v) = (f(u,v), g(u,v), h(u,v))$$

with

$$x = f(u,v) \qquad y = g(u,v) \qquad z = h(u,v)$$

Then the tangent vectors to $F(M)$ at a point $F(p)$ are

$$(X_u)_{F(p)} = F_*\left(\frac{\partial}{\partial u}\right)_p = \frac{\partial x}{\partial u}\left(\frac{\partial}{\partial x}\right)_{F(p)} + \frac{\partial y}{\partial u}\left(\frac{\partial}{\partial y}\right)_{F(p)} + \left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial}{\partial z}\right)_{F(p)}$$

$$(X_v)_{F(p)} = F_*\left(\frac{\partial}{\partial v}\right)_p = \left(\frac{\partial x}{\partial v}\right)\left(\frac{\partial}{\partial x}\right)_{F(p)} + \left(\frac{\partial y}{\partial v}\right)\cdot\left(\frac{\partial}{\partial y}\right)_{F(p)} + \left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial}{\partial x}\right)_{F(p)}$$

Since F is an immersion it has rank = 2 at each point of M. Thus the tangent space to the surface $F(M) \subset R^3$ at $F(p)$ is spanned by $(x_u)_{F(p)}$ and $(X_v)_{F(p)}$.

<u>Exercise:</u> Consider the sphere $F: S^2 \to R^3$. Construct an atlas of spherical coordinates using four charts. Hint, with (x,y,z) being Cartesian coordinates of R^3, write $x = \sin\theta \sin\phi$, $y = \sin\theta\cos\phi$, $z = \cos\theta$ with $0<\phi<2\pi$, $-\pi<\theta<\pi$. This gives one chart. Another chart is obtained by changing the range of ϕ to $-\pi<\phi<\pi$. To obtain the remaining charts rotate the coordinates by 90° about the x or y axis and consider the above two ranges for ϕ in the new coordinates. Show that these four charts define an atlas on S^2 and compute explicitly $(X_\theta)_{F(p)}$ and $(X_\phi)_{F(p)}$. Show that this atlas is C^∞ compatible with the stereographic atlas of example 3.6.

<u>Exercise:</u> Consider the cross <u>product</u> $\times: T_p(\mathbb{R}^3) \times T_p(\mathbb{R}^3) \longrightarrow T_p(\mathbb{R}^3)$ defined on the basis vectors $(\partial_x, \partial_y, \partial_z)$ by

$\partial_x \times \partial_y = \partial_z$, $\partial_y \times \partial_z = \partial_x$, $\partial_z \times \partial_x = \partial_y$. Show that the

C^∞-map $F: M \longrightarrow R^3$ (dim M = 2) is an immersion at $p \in M$ if and only

if $(X_u)_{F(p)} \times (X_v)_{F(p)}$ does not vanish.

<u>Exercise:</u> (Enneper's surface, E). Show that $F: E \longrightarrow R^3$ is an immersion where $F(u,v) = (u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + u^2 v,\ u^2 - v^2)$. Is

(E,F) a submanifold? an imbedding?

4.3 The Cotangent Space

We will now consider another important vector space related to $T_p(M)$. In general, for any vector space V over $\mathbb{R}$ (or $\mathbb{C}$) we can consider its dual space $V^* \equiv \left\{ \phi : V \longrightarrow \mathbb{R} \text{ , } \phi \text{ a linear map} \right\}$.

Exercise: Prove the following theorem of linear algebra: If V is a vector space of dimension n, then V* is also a vector space of dimension n.

In the proof of this result one chooses a basis E_i for V and proves that the elements $E^{*i} \in V^*$ defined by $E^{*i}(E_j) = \delta^i_j$ (Kronecker delta) form a basis for V*. This basis is called the dual basis to E_i. Let us now apply these ideas to the tangent space $T_p(M)$.

We define the cotangent space $T^*_p(M)$ at $p \in M$ to be the vector space dual to $T_p(M)$, that is, the space of all linear functions from $T_p(M)$ to $\mathbb{R}$. We define dx^i_p to be the dual basis for $T^*_p(M)$ of $\left(\frac{\partial}{\partial x^i}\right)_p$, i.e.

$$(4.5) \qquad dx^i_p\left(\frac{\partial}{\partial x^j}\right)_p = \delta^i_j$$

This definition is given, of course, relative to the coordinate chart $\phi = (x^1, \ldots, x^n)$ at $p \in M$, the definition of $T^*_p(M)$, however, is independent of coordinates since the definition of $T_p(M)$ was. Since dx^1_p is a basis for $T^*_p(M)$ any $\phi_p \in T^*_p(M)$ can be written as

(4.6) $\phi_p = \sum_i \phi_i dx^i_p$

where $\phi_i = \phi\left(\frac{\partial}{\partial x^i}\right)_p$. ϕ_p is called a <u>differential 1-form at $p\varepsilon M$</u> (or Pfaffian form at p). In our subsequent study of differential manifolds we will find both $T_p(M)$ and $T^*_p(M)$ extremely useful.

Now suppose $F:M \longrightarrow N$ is a C^∞ map then $F_*:T_p(M) \longrightarrow T_{F(p)}(N)$ is a vector space homomorphism. What happens to $T^*_p(M)$? We define $F^*: T^*_{F(p)}(N) \longrightarrow T^*_p(M)$ to be

(4.7) $(F^*\phi)(v) = \phi(F_*v)$

for $v\varepsilon T_p(M)$ and $\phi\varepsilon T^*_{F(p)}(N)$. F^* is the <u>pullback</u> map on 1-forms and is quite similar to the pullback map on functions introduced previously. In fact we can think of functions as 0-forms on M, thus the same notation F^* is used. We just have to keep in mind whether it acts on functions or 1-forms.

It is clear that the definition of F^* given by (4.7) depends only on the vector space property of $T_p(M)$. That is it could have been defined for any map $F_*:V \longrightarrow W$ of vector spaces. Here is another result which only depends on vector space morphisms and their duals once we have proposition 4.4.

<u>Proposition 4.5</u>: Using the notation of proposition 4.4 and F^*, G^*, H^* defined by (4.7), we have

i) $F^*: T^*_{F(p)}(N) \longrightarrow T^*_p(M)$ is a vector space homomorphism.

ii) If F_* is a monomorphism (epimorphism). Then F^* is an epimorphism (monomorphism).

iii) $H^* = F^* \circ G^*$

Proof: We prove only ii) and leave the proof of i) and iii) for an exercise. Suppose F_* is 1-1, then given $v \varepsilon T_p(M)$, F_*v corresponds to a unique such v. So defining $\psi(F_*v) = \phi(u)$ we see that ψ is well-defined. Thus $\phi = F^*\psi$ and so F^* is onto. Now suppose F_* is onto, then every $w \varepsilon T_{F(p)}(N)$ is the image of a $v \varepsilon T_p(M)$, i.e. $w = F_*v$. Let $\phi, \psi \; \varepsilon \; T^*_{F(p)}(N)$ with $F^*\phi(v) = F^*\psi(v)$. Then $\phi(F_*v) = \psi(F_*v)$ implying $\phi(w) = \psi(w)$ for all $w \varepsilon T_{F(p)}(N)$. Thus $\phi = \psi$ and F^* is 1-1.

Exercise: Prove parts i) and iii) of Proposition 4.5.

Similarly to theorem 4.2 we have

Theorem 4.3: Under the hypothesis of theorem 4.2, the map $F^*: T^*_{F(p)}(N) \longrightarrow T_p(M)$ satisfies

$$F^* dy^i_{F(p)} = \sum_{j=1}^{n} \left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)} dx^j_p$$

Proof: A simple computation using theorem 4.2 is

$$F^*dy^i_p\left(\frac{\partial}{\partial x^j}\right)_p = dy^i_p\left(F_*\left(\frac{\partial}{\partial x^j}\right)_p\right) = dy^i_p\left(\sum_{k=1}^{m}\left(\frac{\partial y^k}{\partial x^j}\right)_{\phi(p)}\left(\frac{\partial}{\partial y^k}\right)_p\right)$$

$$= \sum_{k=1}^{m}\left(\frac{\partial y^k}{\partial x^j}\right)_{\phi(p)} dy^i_p\left(\frac{\partial}{\partial y^k}\right)_p = \sum_{k=1}^{m}\left(\frac{\partial y^k}{\partial x^j}\right)_{\phi(p)} \delta^i_k = \left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)}$$

and the result follows since $\left(\frac{\partial}{\partial x^j}\right)_p$ is a basis of $T_p(M)$.

4.4 Vector Bundles. The tangent and cotangent Bundles.

Many books study $T_p(M)$ in much detail before introducing $T^*_p(M)$, but I believe it is useful to study them more on equal footings, since that is the way they appear in practice. It is frequently useful to make one computation in terms of $T_p(M)$ while another in terms of $T^*_p(M)$. In fact, the use of differential forms, as we shall see later, simplifies many calculations, whereas $T_p(M)$ is usually more geometrically intuitive to visualize

In actual practice the spaces $T_p(M)$ and $T^*_p(M)$ are too restrictive since they are attached to a fixed point $p\varepsilon M$. We thus define

(4.8a) $\quad T(M) = \bigcup_{p\varepsilon M} T_p(M)$

(4.8b) $\quad T^*(M) = \bigcup_{p\varepsilon M} T_p{}^*(M)$

$T(M)$ is called the <u>tangent bundle</u> and $T^*(M)$ the <u>cotangent bundle</u>

of M. We will see shortly a few generalities concerning bundles. For now it suffices to think of $T(M)$ and $T^*(M)$ as topological spaces in the following way: concentrating on $T(M)$ only we notice a projection map $\pi : T(M) \longrightarrow M$ defined by $\pi(v) = p$ where $v \in T_p(M)$, $p \in M$. The topology on $T(M)$ should make π continuous, i.e. if $U \subset M$ is open then $\pi^{-1}(U) \subset T(M)$ is open. Notice that for all $v \in T_p(M)$ $\pi(v) = p$, so $\pi^{-1}(p) = T_p(M)$. $\pi^{-1}(p)$ is called the <u>fibre</u> at p, and here we see that each fibre has the structure of a vector space. We will see that $T(M)$ is an example of a <u>vector bundle</u>. Notice that the topology on $T(M)$ has not been defined uniquely yet.

<u>Definition 4.2</u>: A <u>real C^∞ vector bundle</u> ξ over a C^∞ manifold M is a triple $\xi = (M, E, \pi)$ where:

1) E is a C^∞ manifold called the total space of the bundle.

2) $\pi : E \longrightarrow M$ is a C^∞ surjective submersion with dom $\pi = E$, the projection map.

3) For each $p \in M$, $\pi^{-1}(p)$ has the structure of a real vector space. It is called the fibre at p.

4) For each $p \in M$ there exist a nbd, $U \subset M$, an integer $k \geq 0$, and a diffeomorphism $h : U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U)$ such that for each fixed $p \in M$ the map $h_p : \mathbb{R}^k \longrightarrow \pi^{-1}(p)$ defined by $h_p(v) = h(p,v)$, $v \in \mathbb{R}^k$, is a vector space isomorphism.

Remarks. Of course, this definition can be stated in the C^r category, in particular $r = 0$. Condition 4) is known as <u>local triviality</u> and if it is possible to choose $U = M$ then the bundle ξ is called <u>trivial</u>. We will use interchangibly the notation ξ for a vector bundle (usually not writing $\xi = (E, M, \pi)$) and the notation $E(M)$. M is called the base space.

commutes, and

2) for each fixed $p \in M_1$ $(\tilde{f})_p : \pi_1^{-1}(p) \longrightarrow \pi_2^{-1}(f(p))$ is a vector space homomorphism.

<u>Proposition 4.6</u>: Suppose $F: M \longrightarrow N$ is a C^∞ map of C^∞ manifolds. Then $F_*: T(M) \longrightarrow T(N)$ is C^∞ and (F_*, F) is a bundle map. If F is a global diffeomorphism so is F_*; hence $T(M)$ and $T(N)$ are isomorphic as vector bundles.

Proof: Let $(U, \tilde{\phi})$ and $(V, \tilde{\psi})$ be standard charts of $T(M)$ and $T(N)$ respectively with $F(U) \subset V$ and $\tilde{\phi} = (x^1, \ldots, x^m, v^1, \ldots, v^m)$ and $\tilde{\psi} = (y^1, \ldots, y^n, u^1, \ldots, u^n)$ $(m = \dim M,\ n = \dim N)$. Then by Eq. (4.2)

$$u^i = \sum_j v^j \left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)}$$

where $F_* v = u$. But $\left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)}$ is just the Jacobian matrix of