of M. We will see shortly a few generalities concerning bundles. For now it suffices to think of T(M) and T*(M) as topological spaces in the following way: concentrating on T(M) only we notice a projection map $\pi : T(M) \longrightarrow M$ defined by $\pi(v) = p$ where $v \in T_p(M)$, $p \in M$. The topology on T(M) should make π continuous, i.e. if $U \subset M$ is open then $\pi^{-1}(U) \subset T(M)$ is open. Notice that for all $v \in T_p(M)$ $\pi(v) = p$, so $\pi^{-1}(p) = T_p(M)$. $\pi^{-1}(p)$ is called the fibre at p, and here we see that each fibre has the structure of a vector space. We will see that T(M) is an example of a vector bundle. Notice that the topology on T(M) has not been defined uniquely yet.

Definition 4.2: A real C^∞ vector bundle ξ over a C^∞ manifold M is a triple $\xi = (M, E, \pi)$ where:

1) E is a C^∞ manifold called the total space of the bundle.

2) $\pi : E \longrightarrow M$ is a C^∞ surjective submersion with dom π = E, the projection map.

3) For each $p \in M$, $\pi^{-1}(p)$ has the structure of a real vector space. It is called the fibre at p.

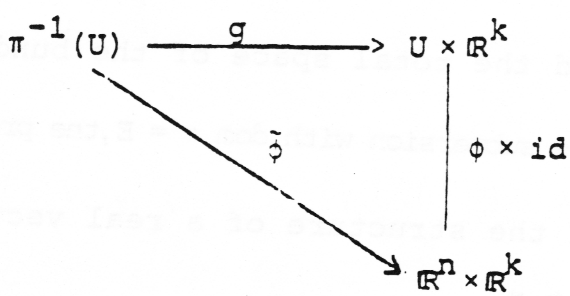
4) For each $p \in M$ there exist a nbd, $U \subset M$, an integer $k \geq 0$, and a diffeomorphism $h : U \times \mathbb{R}^k \longrightarrow \pi^{-1}(U)$ such that for each fixed $p \in M$ the map $h_p : \mathbb{R}^k \longrightarrow \pi^{-1}(p)$ defined by $h_p(v) = h(p,v)$, $v \in \mathbb{R}^k$, is a vector space isomorphism.

Remarks. Of course, this definition can be stated in the C^r category, in particular r = 0. Condition 4) is known as local triviality and if it is possible to choose U = M then the bundle ξ is called trivial. We will use interchangibly the notation ξ for a vector bundle (usually not writing $\xi = (E, M, \pi)$) and the notation E(M). M is called the base space.

Two vector bundles $\xi_1 = (E_1, M_1, \pi_1)$ and $\xi_2 = (E_2, M_2, \pi_2)$ are said to be <u>isomorphic</u> if there is a diffeomorphism $F: E_1 \longrightarrow E_2$ of the total spaces which maps each vector space $\pi_1^{-1}(p)$ isomorphically onto $\pi_2^{-1}(F(p))$. Thus a vector bundle $E(M)$ is trivial if and only if is isomorphic to the product bundle $M \times \mathbb{R}^k$. Thus a vector bundle is a generalization of the direct product of a manifold with $\mathbb{R}^k$.

Now let us see what coordinates look like on a vector bundle $E(M)$. Let (U,ϕ) be a coordinate chart on M, then $\pi^{-1}(U)$ is an open set of E since π is continuous. Moreover, condition 4) says that there is a diffeomorphism $g = h^{-1} : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$. Thus we define the <u>standard coordinate chart</u> $(\pi^{-1}(U), \tilde{\phi})$ of E by the commutative diagram

(4.9)



Clearly, $\tilde{\phi}$ is a homeomorphism and thus defines a coordinate chart of E. We will usually write $(\tilde{U}, \tilde{\phi})$ instead of $(\pi^{-1}(\tilde{U}), \tilde{\phi})$.

Let us define coordinates on $T(M)$. Although we do not yet know that $T(M)$ is a vector bundle we will define the standard coordinate chart on $T(M)$ as in (4.9). Let (U,ϕ) be a coordinate chart of M with $\phi = (x^1, \ldots, x^n)$ and let $p \in U$. Now on $T(M)$ we have a projection map $\pi: T(M) \longrightarrow M$ defined by $\pi(v_p) = p$, $v_p \in T_p(M)$.

Moreover, for every $p \varepsilon U$ we can write by theorem 4.1

$$v_p = \sum_{i=1}^{n} v^i \left(\frac{\partial}{\partial x^i}\right)_p$$

where $v^i \varepsilon \mathbb{R}$. The set $\pi^{-1}(U)$ is clearly just $\bigcup_{p\varepsilon U} T_p(M) \subset T(M)$ and we can define a topology on $T(M)$ as follows: Define a map $g: \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ by $g(v\ ) = (p, v^i)$. It is easy to check that g is bijective. We choose the topology on $T(M)$ to be that which makes g a homeomorphism. Clearly π is continuous in this topology. Moreover, $\tilde{\phi} : \pi^{-1}(U) \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ as defined by (4.9) is a homeomorphism and $(\pi^{-1}(U), \tilde{\phi})$ defines a standard chart for $T(M)$. In local coordinates we have

(4.10) $\tilde{\phi} = (\phi \times id) \circ g = (x^1, \ldots, x^n, v^1, \ldots, v^n)$

Thus $T(M)$ is locally Euclidean and we are ready for

<u>Theorem 4.4</u>: $T(M)$ is a C^∞ vector bundle over M.

Proof: We need to show that $T(M)$ is a C^∞ manifold, π is a C^∞ map and that the condition of local triviality is satisfied.

i) $T(M)$ is Hausdorff: Let $v_1, v_2 \varepsilon T(M)$ be distinct. If $\pi(v_1) \neq \pi(v_2)$, they admit disjoint nbds U_1, U_2 and $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint since π is continuous and the continuous image of a connected space is connected. If $\pi(v_1) = \pi(v_2)$, then $g(v_1) = (\pi(v_1), v_1^i)$ and $g(v_2) = (\pi(v_2), v_2^i)$ and since v_1^i v_2^i are distinct vectors of $\mathbb{R}^n$ they admit disjoint nbds and thus so do v_1 and v_2 since g is continuous.

ii) $T(M)$ has a countable basis: Let $\{U_i\}$ be a countable basis for M then $T(M) = \bigcup_i \pi^{-1}(U_i)$. Moreover, if $v \in T(M)$ with $\pi(v) = p \in U_i \cap U_j$ then there is a $U_k \subset U_i \cap U_j$ with $p \in U_k$. Thus $v \in \pi^{-1}(U_i \cap U_j) = \pi^{-1}(U_i) \cap \pi^{-1}(U_j)$ and $\pi^{-1}(U_k) \subset \pi^{-1}(U_i) \cap \pi^{-1}(U_j)$ with $v \in \pi^{-1}(U_k)$ since $\pi(v) \in U_k$.

iii) $T(M)$ is a C^∞ manifold. Let $(\pi^{-1}(U), \tilde{\phi})$ and $(\pi^{-1}(V), \tilde{\psi})$ be two standard charts on $T(M)$ with $\pi^{-1}(U) \cap \pi^{-1}(V) \neq \emptyset$, we must show that $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is a diffeomorphism. Let $\phi = (x^1, \ldots, x^n)$ and $\psi = (y^1, \ldots, y^n)$ then $\psi \circ \phi^{-1}$ is a diffeomorphism. Moreover, if $v \in \pi^{-1}(U) \cap \pi^{-1}(V)$ then for every $p \in U \cap V$ if we write

$$v = \sum_i v^i \left(\frac{\partial}{\partial x^i}\right)_p = \sum_i v'^i \left(\frac{\partial}{\partial y^i}\right)_p \quad \text{then we saw that}$$

$$v'^j = \sum_i v^i \left(\frac{\partial y^j}{\partial x^i}\right)_{\phi(p)}$$

and this is a C^∞ diffeomorphism. So $\tilde{\psi} \circ \tilde{\phi}^{-1}(x^1, \ldots, x^n, v^1, \ldots, v^n) = (y^1, \ldots, y^n, v'^1, \ldots, v'^n)$ is a C^∞ diffeomorphism.

iv) π is C^∞. Its coordinate representative is $\hat{\pi}(x^1, \ldots, x^n, v^1, \ldots, v^n) = (x^1, \ldots, x^n)$.

v) local triviality. g is a diffeomorphism since its coordinate representative $\hat{g}$ is the identity map. It remains to show that g is a vector space isomorphism. This is easy. Let $v, u \in T_p(M)$ then for fixed p, $g|_{\pi^{-1}(p)} : T_p(M) \longrightarrow \mathbb{R}^n$ is bijective. Moreover, $g(\alpha v + \beta u) = (p, \alpha v^i + \beta u^i) = (p, \alpha v^i) + (p, \beta u^i)$. Q.E.D.

A similar contruction can be made for $T^*(M)$. As with $T(M)$ we have a C^∞ map $\pi: T^*(M) \longrightarrow M$. An arbitrary (covector) 1-form at $q \varepsilon M$ has the form

$$\phi_q = \sum_{i=1}^{n} p_i \, dx^i$$

where $p_i \varepsilon \mathbb{R}$. Again we can consider a vector $(p_1, \ldots, p_n) \varepsilon \mathbb{R}^n$ and this holds for any $\phi \varepsilon T_p(M) \subset \pi^{-1}(U)$. So we have a C^∞ map $g: \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ defined by

$$g(\phi_q) = (q, p_i)$$

The standard chart on $T^*(M)$ then has the form

$$\tilde{\phi}(q, p_i) = (x^1, \ldots, x^n, p_1, \ldots, p_n) \tag{4.11}$$

where $\tilde{\phi}$ is related to the chart $\phi = (x^1, \ldots, x^n)$ on $U \subset M$. We use here p_i as coordinates on $T^*(M)$ since $T^*(M)$ is the generalization of phase space to arbitrary manifolds.

<u>Theorem 4.5</u>: $T^*(M)$ is a C^∞ vector bundle over M.

Proof: <u>Exercise</u> (similar to theorem 4.4)

In section 4.2 we saw that a C^∞ map $F: M \longrightarrow N$ of manifolds induces a map $F_*: T_p(M) \longrightarrow T_{F(p)}(N)$ of tangent spaces. Now by varying $p \varepsilon M$ we can consider F_* as a map on tangent bundles, i.e. $F_*: T(M) \longrightarrow T(N)$, such that for each $p \varepsilon M$ F_* maps the fibre at p into the fibre at $F(p) \varepsilon N$, and is a homomorphism of vector spaces. This leads to the following definition: Let

$\xi_1 = (E_1, M_1, \pi_1)$ and $\xi_2 = (E_2, M_2, \pi_2)$ be C^∞ vector bundles. A <u>bundle map</u> $(\tilde{f}, f)$ is a pair of C^∞ maps $\tilde{f}: E_1 \longrightarrow E_2$ and $f: M_1 \longrightarrow M_2$ such that

1) the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\ \tilde{f}\ } & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\ f\ } & M_2 \end{array}$$

commutes, and

2) for each fixed $p \in M_1$ $(\tilde{f})_p : \pi_1^{-1}(p) \longrightarrow \pi_2^{-1}(f(p))$ is a vector space homomorphism.

<u>Proposition 4.6</u>: Suppose $F: M \longrightarrow N$ is a C^∞ map of C^∞ manifolds. Then $F_*: T(M) \longrightarrow T(N)$ is C^∞ and (F_*, F) is a bundle map. If F is a global diffeomorphism so is F_*; hence $T(M)$ and $T(N)$ are isomorphic as vector bundles.

Proof: Let $(U, \tilde{\phi})$ and $(V, \tilde{\psi})$ be standard charts of $T(M)$ and $T(N)$ respectively with $F(U) \subset V$ and $\tilde{\phi} = (x^1, \ldots, x^m, v^1, \ldots, v^m)$ and $\tilde{\psi} = (y^1, \ldots, y^n, u^1, \ldots, u^n)$ $(m = \dim M,\ n = \dim N)$. Then by Eq. (4.2)

$$u^i = \sum_j v^j \left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)}$$

where $F_* v = u$. But $\left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)}$ is just the Jacobian matrix of

$y^i = [\phi \circ F \circ \phi^{-1}]^i \ (x^1,\ldots,x^n)$ and this is C^∞ since F is. So F_* is C^∞ . Also by proposition 4.4 (F_*, F) is a bundle map. Now suppose F is a global diffeomorphism, then by theorem 4.2 each fibre $T_p(M)$ is mapped isomorphically onto $T_{F(p)}(N)$ and the correspondence $T_p(M) \longleftrightarrow T_{F(p)}(N)$ is a bijection. Moreover, the Jacobian matrix $JF^i_j = \left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)}$ is an invertible square matrix whose inverse $(JF)^{-1} = \frac{(-1)^{i+j} \ \mathrm{cof}(JF)^i{}_j}{\det JF}$ is C^∞. It follows that F_* is a global diffeomorphism. (cof = cofactor).

Remark: Notice from the proof above that the coordinate representative of F_* is just

(4.12) $\hat{F}_* = (\hat{F} \ , J\hat{F})$

The converse of proposition 4.6 also holds. In fact this can be done more generally as the following exercise shows.

Exercise: Suppose the pair $(\tilde{f}, f)$ is a bundle map of vector bundles $\xi_1 = (E_1, M_1, \pi_1)$ and $\xi_2 = (E_2, M_2, \pi_2)$ but only suppose that $\tilde{f}$ is C^∞. Show that f is in fact C^∞. Now suppose that $\tilde{f}$ is a global diffeomorphism, show that f is also a global diffeomorphism.

For general vector bundles proposition 4.6 is not true, but we do have

Lemma 4.3: Let $E_1(M)$ and $E_2(M)$ be vector bundles over the same base space M and consider the bundle map $(\tilde{f}, id)$ where id is the identity map on M. Then if $(\tilde{f})_p$ is an isomorphism of fibres onto fibres, then $\tilde{f}$ is a diffeomorphism.

Proof: Let $(U,\tilde{\phi})$ and $(V,\tilde{\psi})$ be standard charts for E_1 and E_2 respectively with $U \cap V \neq \emptyset$. We must show that $\tilde{\psi} \circ \tilde{f} \circ \tilde{\phi}^{-1}$ is a diffeomorphism. Let $p \in U \cap V$ then by hypothesis $(\tilde{f})_p : \pi^{-1}(p) \longrightarrow \pi_2^{-1}(p)$ is an isomorphism of vector spaces and thus takes the form

$$u^i = \tilde{f}_p(v^i) = \sum_{j=1}^{n} f^i{}_j(p) v^j$$

for some invertible matrix $f^i_j(p)$ where u^i, v^i are the components of the vectors $u \in \pi_2^{-1}(p)$ and $v \in \pi_1^{-1}(p)$, respectively. But since $\tilde{f}$ is a bundle map it is C^∞ as functions of p as well as v. But since $f^i_j(p)$ is invertible we see easily that the inverse matrix

$$(f^{-1})^i_j = \frac{(-1)^{i+j}(\text{cof } f)^i_j}{\det f}$$

exists and is C^∞. Q.E.D.

Exercise: State and prove the analog of proposition 4.6 for co-tangent bundles. Find the coordinate representative $\hat{F}^*$.

Exercise: Consider the product manifold $M_1 \times M_2$ with the natural projections $p_i: M_1 \times M_2 \longrightarrow M_i$, $i = 1,2$. Show that $(p_{i*}\ p_{2*})$: $T(M_1 \times M_2 \longrightarrow T(M_1) \times T(M_2)$ defines a vector bundle isomorphism.