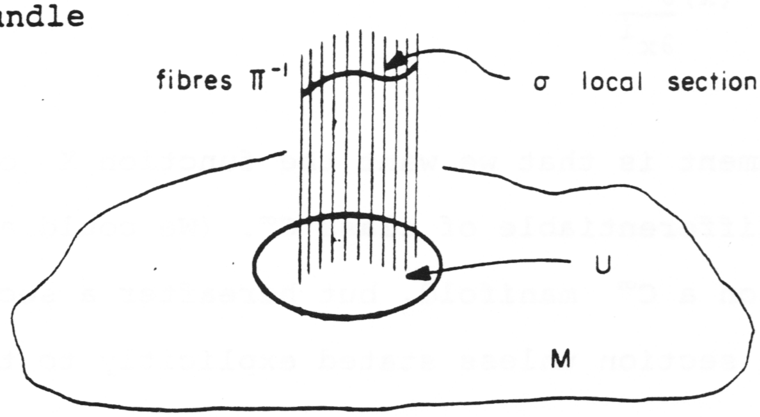
4.5 Sections of Vector Bundles. Vector Fields.

The existence of local C^∞ sections of a C^∞ vector bundle follows immediately from lemma 3.5. A vector bundle also admits global sections, e.g. the zero section which assigns the zero vector to each $p\varepsilon M$. For more general fibre bundles (where the fibre is not a vector space), global sections do not always exist. We have the following intuitive picture of a local section σ of a vector bundle



Thus a section of E is a function σ which assigns a vector of $\pi^{-1}(p)$ to each point $p\varepsilon dom\sigma$. The set of global C^∞ sections of a vector bundle E(M) will be denoted by E(M), whereas the set of local C^∞ sections with domain U will be denoted by E(U). A section of a vector bundle is a generalization of the graph of a product manifold $M \times R^k$.

Exercise: Show that the set of sections E(U) has a natural vector space structure. Moreover, show that one can naturally define multiplication by functions $C^\infty(U)$ on E(U), i.e. E(U) is a $C^\infty(U)$ module.

For example a local section on T(M) is a map $X: U \longrightarrow T(M)$ or in terms of local coordinates

$$X: (x^1,\ldots,x^n) \longrightarrow (x^1,\ldots,x^n, v^1(x),\ldots,v^n(x))$$

Notice $\pi \circ X(x^1,\ldots,x^n) = (x^1,\ldots,x^n)$. When writing such a function we usually suppress the first entrees $(x^1,\ldots,x^n)$ Since at each point $p \in M$ $\left(\frac{\partial}{\partial x^i}\right)_p$ is a basis for $T_p(M)$ we see that the section is just

$$X = \sum_{i=1}^{n} v^i(x)\frac{\partial}{\partial x^i}$$

The only added requirement is that we want the function X or equivalently v^i to be differentiable of class C^∞. (We could also consider a C^r section on a C^∞ manifold, but hereafter a section will always mean a C^∞ section unless stated explicitly to the contrary).

Definition 4.3: A C^∞ section of $T(M)$ is called a vector field on M.

A C^∞ section of $T^*(M)$ is called a differential 1-form on M

The differential 1-forms then take the form

$$\omega = \sum_{i=1}^{n} \omega_i(x)dx^i.$$

In physics and classical mathematics one usually thinks of the components $v^i(x)$ as defining a vector field. But this is coordinate dependent and our definition above is <u>coordinate independent</u>. We know how the coordinates on T(M) transform under a change of chart and we will see that this easily induces a change of coordinates on sections. Indeed if $F:M \longrightarrow M$ is the identity map we have from theorem 4.2

$$\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{n} \left(\frac{\partial y^j}{\partial x^i}\right)_{\phi(p)} \left(\frac{\partial}{\partial y^j}\right)_p$$

for a change of coordinate chart around p. But this holds for any $p \varepsilon M$ where $\phi = (x^1,\ldots,x^n)$ and $\psi = (y^1,\ldots,y^n)$ are local coordinates. Thus for a vector field or $U \cap V$ we have

$$X = \sum_i v^i(x) \sum_{j=1}^{n} \left(\frac{\partial y^j}{\partial x^i}\right) \frac{\partial}{\partial y^j} = \sum_{i=1}^{n} v'^i(y) \frac{\partial}{\partial y^i}$$

where

$$(4.12) \quad v'^i(y) = \sum_{j=1}^{n} \left(\frac{\partial y^i}{\partial x^j}\right) v^j(x)$$

<u>Example 4.3</u>: The Coulomb field. Let $M = R^3 - \{0\}$, and let ϕ be the Cartesian chart on M, $\phi = (x^1, x^2, x^3)$. Then

$$X = \sum_{i=1}^{3} v^i(x) \frac{\partial}{\partial x^i}$$

and

$$v^i(x) = \frac{x^i}{[(x^1)^2 + (x^2)^2 + (x^3)^2]^{3/2}}$$

Now let ψ be the spherical chart on $M-\{x^2 = x^3 = 0,\ x^1 > 0\}$ then $\psi = (r, \theta, \phi) = (y^1, y^2, y^3)$. We have $r = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2}$

$\tan\phi = \dfrac{x^1}{x^2}$, $\tan\theta = \dfrac{[(x^1)^2 + (x^2)^2]^{1/2}}{x^3}$ then

$$\frac{\partial r}{\partial x^i} = \frac{x^i}{r}, \quad \frac{\partial \theta}{\partial x^\alpha} = \frac{x^3\, x^\alpha}{r^2[(x^1)^2 + (x^2)^2]^{1/2}} \quad \alpha = 1,2, \quad \frac{\partial \theta}{\partial x^3} = \frac{-[(x^1)^2 + (x^2)^2]^{1/2}}{r^2}$$

$$\frac{\partial \phi}{\partial x^1} = \frac{x^2}{[(x^1)^2 + (x^2)^2]^{1/2}} \qquad \frac{\partial \phi}{\partial x^2} = \frac{-x^1}{[(x^1)^2 + (x^2)^2]^{1/2}} \qquad \frac{\partial \phi}{\partial x^3} = 0$$

We find

$$v'^1(y) = \frac{1}{r^2} \qquad v'^2(y) = v'^3(y) = 0$$

so in spherical coordinates, we have $X = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}$. The same computation is valid on the open set $M-\{x^2 = x^3 = 0\ ,\ x^1 < 0\}$.

More generally we can ask what happens to vector fields under an arbitrary C^∞ map $F: M \longrightarrow N$. Now by proposition 4.6 and an exercise we know that F is C^∞ if and only if F_* is C^∞ . Now let $X: U \longrightarrow T(M)$ and $Y: V \longrightarrow T(N)$, with $U \subset M$, $V \subset N$, $F(U) \subset V$, be local vector fields on M and N respectively. We say that X and Y are <u>F-related</u> if the diagram

$$\begin{array}{ccc} T(M) & \xrightarrow{\ F_*\ } & T(N) \\ X \uparrow & & \uparrow Y \\ U & \xrightarrow{\ F\ } & V \end{array} \tag{4.13}$$

commutes, i.e. $F_*X = Y \circ F$. We will often write this equation with the domain $F(U)$ of Y understood, that is, $F_*X = Y$. We assume that dom $X = U$, dom $Y = U$, so in the case $U = M$, $V = N$, X and Y are global vector fields. If $M = N$ and X is F - related to itself, i.e. $F_*X = X$, then we say that X is <u>invariant</u> under F. In general we will be interested in when two vector fields are F-related. We have

<u>Theorem 4.6</u>: Let $i:N \longrightarrow M$ be a submanifold. A local vector field X with dom $X = U \subset M$ is <u>tangent</u> to N, i.e. for every $i(p) \varepsilon U \cap i(N)$, $X_{i(p)} \varepsilon\ i_*T_p(N)$ if and only if there is a local vector field Y on N which is i-related to X, i.e. $i_*Y = X$.

Proof: ($\Leftarrow$) Suppose $i_*Y = X \circ i$ and take a $p \varepsilon N$ such that $i(p) \varepsilon U$. Then clearly $X_{i(p)} = i_*Y_p\ \varepsilon\ i_*T_p(N)$.

($\Rightarrow$) Now suppose that X is tangent to N. Since (i,N) is a submanifold, i is injective and $i_*:T_p(N) \longrightarrow T_{i(p)}(M)$ is a monomorphism by theorem 4.2. Thus for every $i(p)\ \varepsilon\ U \cap i(N)$, there is a unique $p \varepsilon N$ and a unique vector $(i_*)^{-1}X_{i(p)}\ \varepsilon\ T_p(N)$, and we put $Y_p = (i_*)^{-1}X_{i(p)}$. Hence, Y is a section of $T(N)$ with domain $i^{-1}(U \cap i(N))$ and we have $Y = (i_*)^{-1} X$. Thus if (V,ψ), $\psi = (y^1,\ldots,y^n)$ and (W,ϕ), $\phi = (x^1,\ldots,x^m)$ are charts of N and M, respectively with $n = \dim N$, $m = \dim M$ and $i^{-1}(W) \subset V$, then using theorem 4.2

$$Y^i(y) = Y(y^i) = \sum_{j=1}^{m} X^j(x) \left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)}$$

Thus $Y^i(y)$ is C^∞ since $X^j(x)$ and $\left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)}$ are. It follows that Y is a vector field i^{-1}-related to X. Q.E.D.

Remarks. Theorem 4.6 is actually a global theorem as well. All we do is take U = M. Notice that in the proof $U \cap i(N)$ is not necessarily open, so (i,N) is not necessarily an embedding.

We can obtain a similar result for quotient manifolds. In particular, we consider the case where the quotient manifold is given by the action of a discontinuous transformation group. More generally suppose $\mu: M \longrightarrow M/\sim$ a quotient projection map and that $M/\sim$ is a quotient manifold. A vector field X on M is called projectable if it is invariant under the equivalence relation ~ , i.e. if $\mu_* X_p = \mu_* X_q$ wherever $p \sim q$, $p,q \in M$.

Proposition 4.7: Let $\mu: M \longrightarrow M/\sim$ be a submersion defined by an equivalence relation ~ on M, so that $M/\sim$ is a quotient manifold. Then $X \in \mathcal{T}(M)$ is projectable if and only if there is an $X' \in \mathcal{T}(M/\sim)$ which is μ-related to X. We say that X is projectable to X'.

Proof: Now $\mu_* X: M \longrightarrow T(M/\sim)$ satisfies $\mu_* X_p = \mu_* X_q$ for all $p \sim q$ and thus only depends on the class $[p] = \mu(p)$. Hence, we can define a section $X': M/\sim \longrightarrow T(M/\sim)$ by $X'_{\mu(p)} = \mu_* X_p$, i.e. $\mu_* X = X' \circ \mu$. We need only show that X' is C^∞. By the rank theorem 3.4, the coordinate representative $\hat{\mu}$ of μ takes the form $\hat{\mu}(x^1,\ldots,x^m) = (y^1,\ldots,y^{m'})$, $\dim M = m \geq m' = \dim M/\sim$ and x^i and y^i are local coordinates on M and $M/\sim$ respectively. We thus have as in the proof of theorem 4.6 (using theorem 4.2)

$$Y^i(y) = \sum_{j=1}^{m} \left(\frac{\partial y^i}{\partial x^j}\right)_{\phi(p)} X^j(x)$$

and this is C^∞. Conversely, if we have a vector field $X':M/\sim \longrightarrow T(M/\sim)$ such that $\mu_* X = X' \circ \mu$, then for all equivalent p and q $\mu_* X_p = X' \circ \mu(p) = X' \circ \mu(q) = \mu_* X_q$. Q.E.D.

Remarks. All of the above goes through even when $M/\sim$ is nonHausdorff. It is not difficult to show that if $M/\sim$ is nonHausdorff then $T(M/\sim)$ is also nonHausdorff.

Now let's specialize to the case when $M/\sim$ comes from the action of a discontinuous transformation group G. A vector field X on M is invariant under the action $t_a: M \longrightarrow M$ of G if for all $a \in G$, $t_{a*} X = X \circ t_a$.

Theorem 4.7: Let G be a discontinuous transformation group acting freely on M. Then a vector field X on M is projectable to the quotient manifold M/G if and only if it is invariant under G.

Proof: By theorem 3.11 M/G is a C^∞ quotient manifold of dimension n = dim M. Since M/G is precisely the space of orbits, $\mu_* \circ t_{a*} = \mu_*$. So if X is invariant under t_a then upon applying μ_* we get

$$\mu_* X \circ t_a = \mu_* \circ t_{a*} X = \mu_* X$$

for all $a \in G$. Thus $\mu_* X$ is invariant under the equivalence relation defined by $p \sim q$ if there is an $a \in G$ such that $t_a(p) = q$;

hence, X is projectable.

Conversely, suppose that X is projectable. Then for all $a \varepsilon G$ and $p \varepsilon M$ $\mu_* X_{t_a(p)} = \mu_* X_p$, i.e. $\mu_* X \circ t_a = \mu_* X$. But $\mu_* = \mu_* \circ t_{a*}$ so

$$\mu_* X \circ t_a = \mu_* \circ t_{a*} X$$

However, dim M = dim M/G = rank μ. Thus by theorem 4.2 μ_* is an isomorphism on each tangent space. So for all $p \varepsilon M$ and $a \varepsilon G$ $\mu_* X_{t_a(p)} = \mu_* \circ t_{a*} X_p$ implies $X_{t_a(p)} = t_{a*} X_p$, or $X \circ t_a = t_{a*} X$. Q.E.D.

4.6 Integral Curves of Vector Fields.

We will now study the relation between a vector field on M and systems of first order ordinary differential equations on M.

Let $c: \mathbb{R} \longrightarrow M$ be a C^∞ curve on M whose domain is an open interval I of $\mathbb{R}$. Let t be the identity chart on $\mathbb{R}$, then a basis of $T_{t_0}(\mathbb{R})$ is $\left(\frac{\partial}{\partial t}\right)_{t_0}$ and $\frac{\partial}{\partial t}$ is a C^∞ vector field on $\mathbb{R}$.

It follows from proposition 4.6 that

$c_*: T(\mathbb{R}) \longrightarrow T(M)$ is C^∞. Thus $\dot{c} \equiv c_* \circ \frac{\partial}{\partial t}$ is a C^∞ curve in $T(M)$, i.e. the diagram

$$\begin{array}{ccc} T(\mathbb{R}) & \xrightarrow{c_*} & T(M) \\ \uparrow{\scriptstyle \frac{\partial}{\partial t}} & \overset{\dot{c}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ \mathbb{R} & \xrightarrow{c} & M \end{array}$$

commutes since

$$\pi \circ c_* \circ \frac{\partial}{\partial t} = \pi \circ \sum_i \left(\frac{dx^i}{dt}\right)\left(\frac{\partial}{\partial x^i}\right)_{c(t)} = (x^1,\ldots,x^n) = c(t)$$

where we have associated to c(t) the local chart $(x^1(t),\ldots,x^n(t))$, see example 4.1. Now let X be a vector field on M, i.e. a section of T(M) then we can write the first order differential equation on M

(4.14) $\dot{c} = X \circ c$

We consider the initial condition $c(0) = p\varepsilon M$. Then a solution of (4.14) gives an <u>integral curve</u> $c(t)$ starting at $p\varepsilon M$ of the vector field X. We call the curve $\dot{c}$ in T(M) the canonical lift of c into T(M). So every vector field on M should determine an integral curve on M. To prove this we will need the existence theorem for systems of first order ordinary differential equations. First we consider some examples.

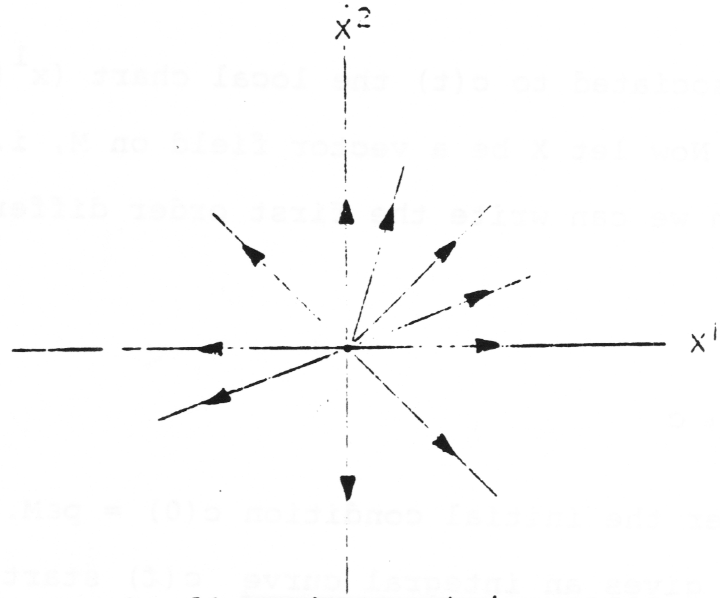
<u>Example 4.4</u>: $X = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$ $\quad M = \mathbb{R}^2$

$c(t) = (x^1(t),\ x^2(t))$. Then (4.14) becomes

$$\frac{dc}{dt} = X \circ c \quad \text{or} \quad \frac{dx^1}{dt} = x^1 \qquad \frac{dx^2}{dt} = x^2$$

Integrating, we have $\frac{dx^1}{x^1} = dt \quad x^1 = \text{cont} \cdot e^t$.

Choose $p = (x_0^1, x_0^2)$ then $c(0) = (x_0^1, x_0^2)$ which says that $x^1(t) = x_0^1 e^t$. Similarly, $x^2(t) = x_0^2 e^t$, so $c(t) = (x_0^1, x_0^2)e^t$. Notice that if $c(0) = (0,0)$ then $c(t) = (0,0)$ for all t! This is due to the fact that $X_{(0,0)} = 0$. That is the origin is a zero of the vector field. A zero of a vector field X is called a <u>singularity</u> or better a <u>critical point</u> of X. We draw the integral curves for our X



the arrows indicate the direction as t increases.
Notice that the critical point (0,0) is a fixed point. Also $c(t)$ is defined for all $t \in R$. This is <u>not</u> a general property as the next example shows

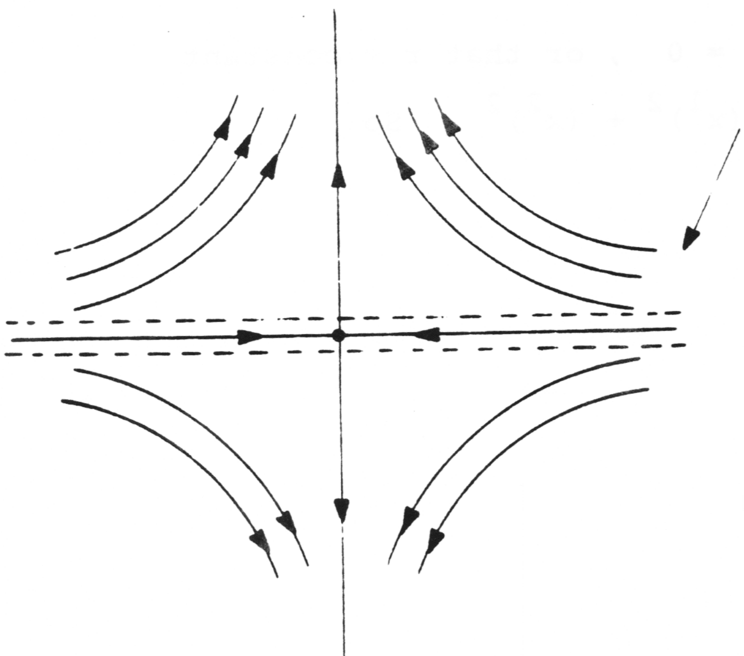
<u>Example 4.5</u>: $M = R^2$ $\quad X = -(x^1)^2 \partial_{x^1} + x^2 \partial_{x^2}$

We have $\quad \frac{dx^1}{dt} = -(x^1)^2 \qquad \frac{dx^2}{dt} = x^2$

If $c(0) = (x_0^1, x_0^2)$, then $x^1 = \dfrac{1}{t + \frac{1}{x_0^1}} = \dfrac{x_1^0}{x_0^1 t + 1}$

$x^2 = x_0^2 e^t$. $C(t) = \left(\dfrac{1}{t + x_0^1}, x_0^2 e^t\right)$, and $c(t) \longrightarrow \infty$

as $t \longrightarrow -\frac{1}{x_0^1}$ so this is not defined for all t, but only for the open interval $I = (-\frac{1}{x_0^1}, \infty)$. An example of this phenomenon occurs in general relativity and its physical interpretation is that there is a singularity in the finite past (big bang). We call a vector field <u>complete</u> if the integral curves are defined for all $t \in \mathbb{R}$. Thus, in this example X is not complete. It should be cautioned that in general the subset of complete vector fields on a manifold does <u>not</u> necessarily form a vector space. We will show that on a compact manifold all vector fields are, however, complete. We also see that $x^1 = x^2 = 0$ is critical point of X (These two phenomenon have nothing to do with each other). We draw the integral curves



reaches $x^1 = +\infty$ in a finite negative time t

$$x^2 = \pm x_0^2 e^{-1} x_0^1$$

The behavior of a vector field near a critical point can be much more complicated than these simple examples show. Example 4.4 is known as a <u>source</u>, and 4.5 as a <u>saddle point</u>. For linear systems, i.e. when $X^i(x)$ are linear functions of the x^j, there is a classification of types. This amounts to considering the system as an eigenvalue problem. Then the classification is given by (in two dimensions) whether the real part of the eigenvalues are all positive (source), all negative (sink), one positive and one negative (saddle point), or zero (center). We now consider an example of a center. (A sink can be obtained from Example 4.4. by taking $X \longrightarrow -X$).

Example 4.6: $X = x^1\partial_{x^2} - x^2\partial_{x^1}$ $M = \mathbb{R}^2$

$$\frac{dx^2}{dt} = x^1 \qquad \frac{dx^1}{dt} = -x^2 \qquad \text{or}$$

$$\frac{dx^2}{x^1} = \frac{dx^1}{-x^2} = dt$$

This implies that $x^1dx^1 + x^2dx^2 = 0$, or that r = constant (independent of t) , where $r^2 = (x^1)^2 + (x^2)^2$. So

$$\frac{dx^2}{\sqrt{r^2-(x^2)^2}} = dt$$

This integrates to

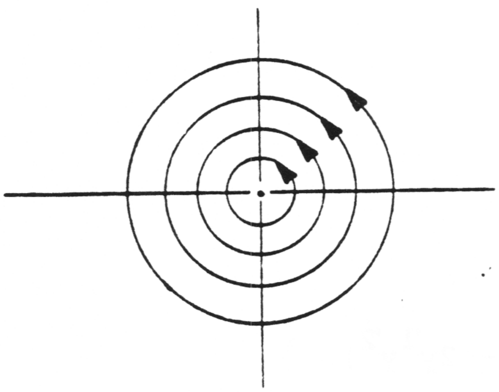
$$x^2 = r \sin (t - t_0)$$

Similarly for x^1 we get

$$x^1 = r \cos (t - t_0)$$

the integral curves are defined for all $t \in \mathbb{R}$ so X is complete.

In fact we see that the integral curves are compact and periodic. There is of course a critical point at the origin $r = 0$. $(x^1, x^2) = (0,0)$. In fact it is a general feature of critical points of the type "center" that the integral curves are periodic. The integral curves are easily drawn



concentric circles

We will now consider some examples on manifolds other than $\mathbb{R}^n$. The theory of general differential equation on arbitrary manifolds is much less well developed than on $\mathbb{R}^n$.

Example 4.7: $M = S^2$ and consider the stereographic atlas of example 3.6. Let $U = S^2 - (0,0,1)$ and $V = S^2 - (0,0,-1)$. On U we have

$$y^i = \frac{x^i}{1-x^3} \qquad i = 1,2$$

and a vector field

$$X_U = (y^1 - y^2)\partial_{y^1} + (y^1 + y^2)\partial_{y^2}$$

whereas on V we have

$$y'^i = \frac{x^i}{1+x^3}$$

and a vector field

$$X_V = -(y'^1 + y'^2)\partial_{y'^1} + (y'^1 - y'^2)\partial_{y'^2}$$

Moreover on $U \cap V$ we have

$$X_U \Big|_{U\cap V} = X_V \Big|_{U\cap V}$$

which can be checked from the Jacobian

$$\left[\frac{\partial y'^i}{\partial y'^j}\right] = \frac{1}{[(y'^1)^2 + (y'^2)^2]^2} \begin{pmatrix} (y'^2)^2-(y'^1)^2 & -2y'^1y'^2 \\ -2y'^1y'^2 & -(y'^2)^2-(y'^1)^2 \end{pmatrix}$$

Thus we have a C^∞ vector fiel X on M. Let us find the integral curves. We have on U

$$\frac{dy^1}{dt} = y^1 - y^2 \qquad\qquad \frac{dy^2}{dt} = y^1 + y^2$$

To find the integral curves we write this as a matrix equation

$$\frac{d\underset{\sim}{y}}{dt} = A\underset{\sim}{y} \qquad\qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

If we diagonalize A we can integrate the system. The condition $\det(A - \lambda \mathbb{1}) = 0$ gives $\lambda = 1 \pm i$. The eigenvectors satisfy $(A-\lambda\mathbb{1})\underset{\sim}{y} = 0$. In the new system we have thus

$$\frac{dz^1}{dt} = (1+i)z^1 \qquad \frac{dz^2}{dt} = (1-i)z^2$$

So

$$z^1(t) = z_0^1 e^t e^{it} \qquad z^2(t) = z_0^2 e^t \, e^{-it}$$

The eigenvectors are $(1,-i)$ and $(1,i)$ belonging to $1+i$ and $1-i$, respectively. The transformation matrix is thus

$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

so

$$\begin{pmatrix} y^1(t) \\ y^2(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}.$$

or

$$y^1(t) = z^1 + z^2 = ae^t \cos(t + b) \qquad z_0^1 + z_0^2 = a \cos b$$

$$y^2(t) = \frac{z^1 - z^2}{i} = ae^t \sin(t + b) \qquad z_0^1 - z_0^2 = ia \sin b$$

This holds on U which contains the point $(0,0,-1)$, i.e. $y^1 = y^2 = 0$ which is a critical point since this implies the initial data is given at $z_0^1 = z_0^2 = 0$. A similar analysis can be made on V_1 where $(0,0,1)$ is a critical point. Since our solution is defined for all $t \in \mathbb{R}$, X is complete. The integral curves are spirals and are shown below:

Notice the these curves intersect any great circle from (0,0,1) to (0,0,-1) at a constant angle $\pi/4$. This follows from the fact that stereographic projection is a conformal transformation, i.e. preserves angles as example 3.6. We have

$$\frac{x^1}{x^2} = \frac{y^1}{y^2} \qquad \frac{x^i}{[(x^1)^2 + (x^2)^2]^{1/2}} = \frac{y^i}{[(y^1)^2 + (y^2)^2]^{1/2}} \quad ; i = 1,2$$

where $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$.

<u>Example 4.8</u>: $M = T^2 = R^2/G$, $G = \{t_{n,m}\}$ where $t_{n,m}:(x^1,x^2) \longrightarrow (x^1+n\ ,\ x^2+m)$, $n,m \in \mathbb{R}$. The vector field

$$X = (\sin 2\pi x^1)\frac{\partial}{\partial x^1} + (\sin 2\pi x^2)\frac{\partial}{\partial x^2}$$

on $\mathbb{R}^2$ is invariant under G and thus projectable by theorem 4.7. Hence, by proposition 4.7 it is μ-related to a vector field on T^2 The system of equations on T^2 is

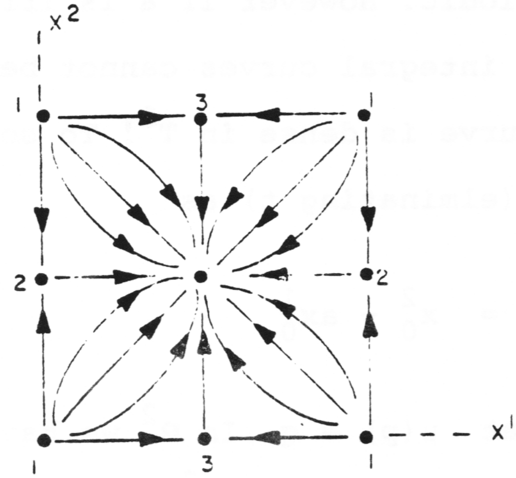
$$\frac{dx^1}{dt} = \sin 2\pi x^1 \qquad \frac{dx^2}{dt} = \sin 2\pi x^2$$

The integral curves are given by

$$x^1(t) = \frac{1}{\pi}\tan^{-1}\left[(\tan \pi x_0^1)e^{2\pi t}\right]$$

$$x^2(t) = \frac{1}{\pi}\tan^{-1}\left[(\tan \pi x_0^2)e^{2\pi t}\right]$$

You should verify the integration. Since X is defined for all t, it is complete. There are critical points $(\frac{1}{2}k, \frac{1}{2}\ell)$, $k,\ell \in \mathbb{Z}$ on $\mathbb{R}^2$. There are 4 critical points on T^2. This is shown in the figure below. Remember on the torus the points marked with the same number are identified.



You can check that the arrows are going in the right direction from the integral curves and which integral curves should be identified. The integral curves are of course only given above for one chart, but the expression will hold when we change to another chart (of the same type) as in example 3.5 on S^1.

<u>Example 4.9</u>: $M = T^2 = R^2/G$. The vector field

$$X = \frac{\partial}{\partial x^1} + a\frac{\partial}{\partial x^2}$$

on $\mathbb{R}^2$ is invariant under G and thus projectable to a vector field X on T^2. The integral curves are

$$x^1(t) = x_0^1 + t \qquad\qquad x^2(t) = x_0^2 + at$$

X is complete and has no critical points. If a is rational we can write $a = \frac{p}{q}$ where p and q are relative prime integers. Thus on T^2

$$x^1(q) = x^1(0) \quad \text{and} \quad x^2(q) = x^2(0)$$

so the integral curves are periodic. However if a is irrational, then no such q exists and the integral curves cannot be periodic. In fact, each integral curve is dense in T^2! To see this we write the integral curves (elminating t) as

$$x^2 = b + ax^1 \qquad b = x_0^2 - ax_0^1$$

Let $q \varepsilon T^2$ and let $p \varepsilon \mathbb{R}^2$ such that $\mu(p) = q$. In $\mathbb{R}^2$ we have the spherical nbd. of radius ε about p, $S(p,\varepsilon)$. Let $U \subset T^2$ be open and contain q, we can choose ε such that $\mu^{-1}(U) \subset S(p,\varepsilon)$. Now let $p^1 = (y^1 + n,\ y^2 + m)$ where (y^1, y^2) are coordinates at $p \varepsilon \mathbb{R}^2$. Then $\mu(p^1) = q$ and there is a nbd. $S_{n,m}(p^1,\varepsilon^1)$ such that $S_{n,m}(p^1,\varepsilon^1) \subset \mu^{-1}(U)$. An integral curve on $\mathbb{R}^2$ is given by $x^2 = ax^1 + b$. So the distance of p^1 from an integral curve will be

$$|y^2 + m - a(y^1 + n) - b)|$$

But it follows from the following lemma, whose proof will be relagated to an appendix, that this distance can be made as small as needed by a choice of m and n.

Lemma: If a is irrational and $b \varepsilon R$ then given $\varepsilon > 0$ there are integers n,m such that

$$|an - m - b| < \varepsilon.$$

Thus any integral curve of X is dense in T^2.

Example 4.10: $M = R^2/G \quad G = \{t_{n,m}\}$ where $t_{n,m}: (x^1,x^2) \longrightarrow (x^1 + n, (-1)^n x^2 + m)$, i.e. M is the Klein bottle of example 3.31. Notice that example 4.8 transfers without changing the computations to the Klein bottle. The vector field

$$X = \sin 2\pi x^1 \frac{\partial}{\partial x^1} + \sin 2\pi x^2 \frac{\partial}{\partial x^2}$$

is invariant under G since $\sin 2\pi((-1)^n x^2 + m)\frac{\partial}{\partial(-1)^n x^2} =$ $\sin 2\pi x^2 \frac{\partial}{\partial x^2}$. It thus projects to a vector field X on the Klein bottle. The same analysis for example 4.8 now holds here. The only difference in the drawing of the integral curves is that now the vertical integral curve starting on the x^2-axis just above the origin is identified with the integral curve pointing directly down from the upper right hand corner, etc.

Exercise: Let

$$X = X^1(x^1,x^2)\frac{\partial}{\partial x^1} + X^2(x^1,x^2)\frac{\partial}{\partial x^2}$$

be a vector field on $\mathbb{R}^2$ in the usual Cartesian coordinates. Find necessary conditions on the functions $X^1(x^1,x^2)$, $X^2(x^1,x^2)$ so that X projects to a vector field on

1) the torus

2) the Möbius band

3) the Klein bottle

Exercise: Find the integral curves, critical points, the interval of $\mathbb{R}$ on which the curves are well defined (i.e. are they complete?), and draw the integral curves for the following vector fields

1) $M = R^2 \qquad X = x^1\partial_{x^1} - x^2\partial_{x^2}$

2) $M = R^2 \qquad X = e^{-x^1}\partial_{x^2}$

3) $M = R^2 \qquad X = (x^1)^2\partial_{x^2}$

4) $M = S^2 \qquad X = \partial_\phi$ in the spherical atlas with chart $x^1 = \sin\theta\,\sin\phi$, $x^2 = \sin\theta\,\cos\phi$, $x^3 = \cos\theta$, $0 < \phi < 2\pi$, $0 < \theta < \pi$. Use the atlas of charts obtained by a previous exercise and check that X extends to a global vector field on S^2.

5) $M = S^1 \times R^1 \quad X = \cos x^1\partial_{x^1} + a\partial_{x^2}$ where x^1 is the local coordinate on S^1 and x^2 the coordinate on R^1. Construct atlas which covers all $S^1 \times R^1$ and check that X defines a vector field globally.

6) $M = M^2$, the Möbius band. (See Example 3.30) $X = \partial_{x^1} + x^2\partial_{x^2}$.

7) Does the vector field of 4) project to a vector field on $P^2 = S^2/G$ where $G = \{t\}$ consisting of 2 elements

$t: x^i \longrightarrow \pm x^i$?

We will now pass to the study of more general properties of vector fields. We will need the existence theorem for systems of ordinary differential equation which is stated without proof.

Theorem 4.8: Let $f^i(x^1,\ldots,x^n,t)$ be C^r functions $(i=1,\ldots,n)$ defined in some neighborhood of the origin in $\mathbb{R}^{n+1}$. Then there are neighborhoods U of the origin in $\mathbb{R}^n$ and I of $0 \in \mathbb{R}$ such that for any $(x_0^1,\ldots,x_0^n) \in U$ and all $t \in I$, there are unique functions $\phi^i(t;x_0^1,\ldots,x_0^n)$ $i=1,\ldots,n$ such that

$$\frac{d\phi^i}{dt} = f^i(\phi^1,\ldots,\phi^n;t)$$

and

$$\phi^i(0, x_0^1,\ldots,x_0^n) = x_0^i$$

Moreover, the functions ϕ^i are of class C^{r+1} in t and of class C^r in x.

Proofs can be found in Spivak (a slightly more general theorem is proved), Arnold, Sternberg, and many books on differential equations.

This important theorem leads to (with $r = \infty$).

Theorem 4.9: Let X be a vector field on a C^∞ manifold M. For every $p \in M$ there exist a neighborhood U of p, an $\varepsilon > 0$, and a unique family of C^∞ maps $\phi_t: U \longrightarrow M$, defined for $|t| < \varepsilon$ such that

1) the map $\phi: I \times U \longrightarrow M$ defined by $\phi(t,p) = \phi_t(p)$ is C^∞ where $I = (-\varepsilon, \varepsilon)$.

2) if t, s and t + s are all in I, and if $q \varepsilon U$ and $\phi_t(q) \varepsilon U$ then

$$\phi_{s+t}(q) = \phi_s \circ \phi_t(q) = \phi_s(\phi_t(q))$$

3) The curve $c: I \longrightarrow M$ defined by $c(t) = \phi(t,p)$ holding p fixed is an integral curve of X. Moreover, X_q is tangent to c(t) at $q \varepsilon U$.

Proof: Let (V, ψ) be a chart at $p \varepsilon M$ with $\psi = (x^1, \ldots, x^n)$. Then in V X has the form

$$X = \sum_i X^i(x) \frac{\partial}{\partial x^i}$$

Consider the system of differential equations

$$*) \quad \frac{d\phi^i}{dt} = X^i(\phi^1, \ldots, \phi^n)$$

By theorem 4.8, there exist $I_1 \subset \mathbb{R}$ and $S(0, \delta_1) \subset \mathbb{R}^n$ such that this system has a unique solution $\phi^i(t^1, x^1, \ldots, x^n)$ with $\phi^i(0, x^1, \ldots, x^n) = x^i$ and $t \varepsilon I_1$, $x \varepsilon S(0, \delta_1)$. We can choose $I \subset I_1$ and a $\delta < \delta_1$ such that $\phi^i(t, x^1, \ldots, x^n) \varepsilon \psi(V)$ for $t \varepsilon I$ and $x \varepsilon S(0, \delta)$. Now clearly the map $\phi_t: U \longrightarrow M$ defined by (in local coordinates)

$$\phi_t^i(x^1, \ldots, x^n) = \phi^i(t, x^1, \ldots, x^n)$$

is C^∞ by the last statement of theorem 4.8., proving part 1).

in $U \cap V$

Moreover, the functions $\tilde{\phi}^i(t;x^1,\ldots,x^n) = \phi^i(s+t;x^1,\ldots,x^n)$ satisfy *) and the initial contidions $\tilde{\phi}^i(0,x^1,\ldots,x^n) = \phi^i(s,x^1,\ldots,x^n)$. Thus by uniqueness, we have

$$\phi^i(s+t;x^1,\ldots,x^n) = \phi^i(t;\phi^i(s;x^1,\ldots,x^n))$$

or

$$\phi^i_{s+t}(x^1,\ldots,x^n) = \phi^i_s \circ \phi^i_t(x^1,\ldots,x^n)$$

proving part 2). Now clearly *) is just the coordinate version of (4.14), so for each fixed p, $\phi(t,p)$ is an integral curve of X, and X_q is tangent to $\phi(t,p)$ for each $q \in U$. Q.E.D.

Actually this theorem introduces a convenient concept. We call any family ϕ_t of C^∞ maps $\phi_t : U \longrightarrow M$ satisfying conditions 1) and 2) of theorem 4.9 a <u>local one-parameter group</u> if in addition we have $\phi_0(p) = p$, i.e. ϕ_0 is the identity transformation. We can always make such a choice by an appropriate choice of initial conditions. Notice that this local group is defined as a group of C^∞ transformations on M. It is an example of a local Lie transformation group. Such a local one-parameter group is also called a <u>flow</u> on M.

We have immediately from theorem 4.9,

<u>Corollary</u>: The integral curves of a C^∞ vector field X on M form a <u>local</u> one-parameter group on M.

If I in theorem 4.9 can be taken as <u>all</u> of $\mathbb{R}$, then ϕ_t form a <u>global one-parameter group</u> , or just a <u>one-parameter group</u>.

It is a Lie group given as a transformation group on M. We have

<u>Corollary</u>: If X is complete, the integral curves ϕ_t form a one-parameter Lie transformation group on M.

The following theorem tells us that if M is compact then every vector field X generates a one-parameter group.

<u>Theorem 4.10</u>: If M is compact, every C^∞ vector field X on M is complete.

Proof: By theorem 4.9, to every $p \in M$ there corresponds an interval I_{ε_p} and a nbd. V_p of p such that ϕ_t is well defined. But on M every open cover has a finite subcover, so we can cover M by a finite number of such V. Moreover, on intersections $V_{p_1} \cap V_{p_2}$ ϕ_t is unique. If t,s and t + s, $\in I = (-\varepsilon,\varepsilon)$, where $\varepsilon = \text{Min}\,\{\varepsilon_p\}$ then the composition law $\phi_{s+t} = \phi_s \circ \phi_t$ holds. If $|t| \geq \varepsilon$ we can define $t = k\varepsilon/2 + r$ and compose $\phi_{\varepsilon/2}$ k times with itself and once with ϕ_r (k integer , $r \in \mathbb{R}$). Do the same for $-\varepsilon$ to complete the proof.

The following proposition guarantees among other things that on quotient manifolds one obtains integral curves by projection.

<u>Proposition 4.8</u>: Let $F: M \longrightarrow M'$ be a C^∞ global map (i.e. dom F = M) and suppose that the vector fields X and X' on M and M', respectively, are F-related (i.e. $X' = F_* X$). Then if c is an integral curve of X, $F \circ c$ is an integral curve of X'.

Proof: The map $F \circ c : \mathbb{R} \longrightarrow M'$ is a curve on M'. Moreover, its lift to $T(M')$ is just $F_* \circ \dot{c}$. But then by (4.14)

$$F_* \circ \dot{c} = F_* X \circ c = X' \circ F \circ c$$

and this is just a system of differential equations of the form (4.14). Solutions exist by theorem 4.9 and are just the integral curves $F \circ c$.

Another useful and important result is the following local theorem originally due to S. Lie.

Theorem 4.11: Let X be a C^∞ vector field on M with $X(p) \neq 0$ (p is a regular point), then there is a coordinate chart (x,U) about p with coordinate representative $(x^1, \ldots, x^n)$ such that $X = \frac{\partial}{\partial x^1}$.

Proof: We choose a chart ψ of U with coordinates $(y^1, \ldots, y^n)$ and $\psi(p) = 0$. Moreover, we can choose $X_p = \left(\frac{\partial}{\partial y^1}\right)_p$ by a linear transformation in $T_p(M)$.

By theorem 4.9, we have a local 1-parameter group ϕ generated by X. We define a function $\chi : \mathbb{R}^n \longrightarrow M$ by

$$\chi(a^1, \ldots, a^n) = \phi_{a^1}(\psi^{-1}(0, a^2, \ldots, a^n))$$

We compute for $a = (a^1, \ldots, a^n) \in \mathbb{R}^n$, $f \in C^\infty(M)$

$$\chi_*\left(\frac{\partial}{\partial y^1}\right)_a (f) = \frac{\partial}{\partial y^1}\bigg|_a f \circ \chi$$

$$= \lim_{h\to 0} \frac{1}{h} [f(\chi(a^1+h, a^2, \ldots, a^n) - f(\chi(a))]$$

$$= \lim_{h\to 0} \frac{1}{h} [f(\phi_{a^1+h} (\psi^{-1}(0, a^2, \ldots, a^n)) - f(\chi(a)]$$

$$= \lim_{h\to 0} \frac{1}{h} [f(\phi_h(\chi(a)) - f(\chi(a))] \qquad \text{by} \quad 2) \text{ of thm. } 4.9$$

$$= (Xf)(\chi(a)) \qquad \text{by} \quad 3) \text{ of thm. } 4.9$$

So we have

$$\chi_*\left(\frac{\partial}{\partial y^1}\right)_a = X \circ \chi(a)$$

Moreover, for $i > 1$, we have

$$\chi_*\left(\frac{\partial}{\partial y^i}\right)_0 f = \frac{\partial}{\partial y^i}\bigg|_0 f \circ \chi$$

$$= \lim_{h\to 0} \frac{1}{h} [f(\chi(0, 0, \ldots, \overset{i^{th}}{h}, 0)) - f(\chi(0))]$$

$$= \lim_{h\to 0} \frac{1}{h} [f \circ \psi^{-1}(0, \ldots h, \ldots 0) - f(p)] \quad \text{since } \phi_0(\ldots) \text{ is the identity transf.}$$

$$= \frac{\partial f \circ \psi^{-1}}{\partial y^i}$$

or

$$\chi_* \left(\frac{\partial}{\partial y^i}\right)_0 = \left(\frac{\partial}{\partial y^i}\right)_p \qquad i = 2,\ldots,n$$

Since $\left(\frac{\partial}{\partial y^i}\right)_p$, $i = 1,\ldots,n$, is a basis for $T_p(M)$, it follows that $\chi_*: T_0(\mathbb{R}^n) \longrightarrow T_p(M)$ is an isomorphism onto. Thus by the inverse function theorem, there is a nbd. V of $0 \in \mathbb{R}^n$ such that $\chi\big|_V$ is a diffeomorphism. So we can take $x = (\chi\big|_V)^{-1}$ as a coordinate chart. Now through any point $q \in U = \chi(V)$ there passes an integral curve of X whose x^1 component is given by

$$t \longrightarrow \chi(a^1 + t,\ a^2,\ldots,a^n) = \phi_t(\psi^{-1}(a))$$

with $x(q) = a$. We have

$$(a^1 + t,\ a^2,\ldots,a^n) = x \circ \phi_t \circ \psi^{-1}(a)$$

and by 3) of thm. 4.9 and the above we have that at q $X = \frac{\partial}{\partial x^1}$, proving the theorem.

4.7 The Lie derivative

We will now develop some further consequences of the fact that X_q is tangent to $\phi_t(q)$ at $t = 0$, i.e. 3) of theorem 4.9. For functions we have

$$\lim_{t\to 0} \frac{f(\phi_t(p)) - f(p)}{t} = (Xf)(p)$$

We define the right hand side to be the <u>Lie derivative</u> of $f \varepsilon C^\infty(M)$ with respect to X. It is denoted as

$$(4.15) \qquad (\underset{X}{\pounds} f)(p) \equiv \lim_{t\to 0} \frac{f(\phi_t(p)) - f(p)}{t} = Xf(p)$$

or

$$\underset{X}{\pounds} f = Xf$$

The point is that this is readily generalized to differential forms and vector fields. We define the Lie derivative of a differential 1-form as

$$(4.16) \qquad (\underset{X}{\pounds}\, \omega)_p \equiv \lim_{t\to 0} \frac{(\phi_t^* \omega)_p - \omega_p}{t}$$

and for vector fields

$$(4.17) \qquad (\underset{X}{\pounds}\, Y)_p = \lim_{t\to 0} \frac{Y_p - (\phi_{t*}Y)_p}{t}$$

Recall that $(F_* X)_p = F_* Y_{F^{-1}(p)}$ so

$(\phi_{t*}Y)_p = \phi_{t*}Y_{\phi^{-1}_t(p)} = \phi_{t*}Y_{\phi_{-t}}(p)$. We should check that the above definitions are well defined and C^∞, but this will be clear from the coordinate expressions we shall give shortly. First we have

<u>Lemma 4.4</u> Let ω, ω' and Y, Y' be C^∞ differential 1-forms and C^∞ vector fields on a C^∞ manifold M, and suppose that their

Lie derivatives exist, then

1) $\underset{X}{\pounds}(Y + Y') = \underset{X}{\pounds} Y + \underset{X}{\pounds} Y'$

2) $\underset{X}{\pounds}(\omega + \omega') = \underset{X}{\pounds}\omega + \underset{X}{\pounds}\omega'$

3) $\underset{X}{\pounds} fY = (Xf)Y + f\underset{X}{\pounds} Y$

4) $\underset{X}{\pounds} f\omega = (Xf)\omega + f\underset{X}{\pounds}\omega$

5) $\underset{X}{\pounds}\omega(Y) = (\underset{X}{\pounds}\omega)(Y) + \omega(\underset{X}{\pounds}Y)$

where $f \varepsilon C^\infty(M)$.

<u>Proof</u>: 1) and 2) are trivial to verify.

3) $(\underset{X}{\pounds} fY)_p = \lim_{t\to 0} \frac{1}{t}[(fY)_p - (\phi_{t*}fY)_p]$

$$= \lim_{t\to 0} \frac{1}{t}[f(p)Y_p - \phi_{t*}(fY)_{\phi_{-t}(p)}]$$

$$= \lim_{t\to 0} \frac{1}{t}[f(p)Y_p - f(\phi_{-t}(p))\ \phi_{t*}Y_{\phi_{-t}(p)}],$$

add and subtract $\frac{1}{t} f(p)\phi_{t*}Y_{\phi_{-t}(p)}$ to give

$$= \lim_{t\to 0} \frac{f(p)}{t}[Y_p - \phi_{t*}Y_{\phi_{-t}(p)}] + \lim_{t\to 0}[f(p) - f(\phi_t(p)]\phi_{t*}Y_{\phi_{-t}(p)}$$

then changing $t \longrightarrow -t$ in the second term gives

$$= f(p)(\underset{X}{\pounds}Y)(p) + (Xf)(p)\ Y_p$$

Where we have used $\lim_{t\to 0} \phi_{t*} Y_{\phi_{-t}(p)} = \lim_{t\to 0} (\phi_{t*}Y)_p = Y_p$

This proves 3).

4) is proved similarly and is left as an exercise.

5) $$\underset{X}{\pounds}\omega(Y)_p = \lim_{t\to 0} \frac{1}{t} [\omega_{\phi_t(p)}(Y_{\phi_t(p)}) - \omega_p(Y_p)]$$

$$= \lim_{t\to 0} \frac{1}{t} [(\phi_t^*\omega)_p((\phi_{-t*}Y_p) - \omega_p(Y_p)$$

$$= \lim_{t\to 0} \frac{1}{t} [(\phi_t^*\omega)_p((\phi_{-t*}Y_p) - (\phi_t^*\omega)_p(Y_p) + (\phi_t^*\omega)_p(Y_p) - \omega_p(Y_p)]$$

$$= \omega(\underset{X}{\pounds}Y)_p + \underset{X}{\pounds}\omega(Y)_p$$

We now give the Lie derivatives in coordinate form

<u>Theorem 4.12</u>: Let $\omega = \sum_{i=1}^{n} \omega_i(x)dx^i$, and $Y = \sum_{i=1}^{n} Y^i(x)\frac{\partial}{\partial x^i}$,

$X = \sum_{i=1}^{n} X^i(x)\frac{\partial}{\partial x^i}$ be a C^∞ differential 1-form and C^∞ vector fields, respectively, on M in the local coordinates $(x^1,\ldots,x^n)$. Then we have

1) $\underset{X}{\pounds}\, \omega = \sum_{i,j} \left(X^j \frac{\partial \omega_i}{\partial x^j} + \omega_j \frac{\partial X^j}{\partial x^i} \right)$

2) $\underset{X}{\pounds}\, Y = \sum_j \left(\sum_i \frac{\partial Y^j}{\partial x^i} X^i - \frac{\partial X^j}{\partial x^i} Y^i \right) \frac{\partial}{\partial x^j}$

<u>Proof:</u> 1) We first compute

$$\underset{X}{\pounds}\, dx^i = \lim_{t \to 0} \frac{1}{t} [\phi_t^* \, dx^i - dx^i]$$

$$= \lim_{t \to 0} \frac{1}{t} [d(x^i \circ \phi_t) - dx^i]$$

$\phi_t^* dx^i(X) = dx^i(\phi_{t*}X) = \phi_{t*}X(x^i) = X \phi_t^* x^i = X(x^i \circ \phi_t) = d(x^i \circ \phi_t)(X)$

$$= \lim_{t \to 0} \frac{1}{t} \left[\sum_{j=1}^{n} \frac{\partial (x^i \circ \phi_t)}{\partial x^j} dx^j - dx^i \right] = \lim_{t \to 0} \frac{1}{t} \left[\sum_j \left(\frac{\partial (x^i \circ \phi_t)}{\partial x^j} - \delta^i_j \right] dx^j \right.$$

but

$$\left(\frac{\partial (x^i \circ \phi_t)}{\partial x^j} \right) - \delta^i_j \Big) = \left(\frac{\partial (x^i \circ \phi_t)}{\partial x^j} - \frac{\partial (x^i \circ \phi_0)}{\partial x^j} \right)$$

and

$$\lim_{t \to 0} \frac{1}{t} \left(\frac{\partial (x^i \circ \phi_t)}{\partial x^j} - \frac{\partial (x^i \circ \phi_0)}{\partial x^j} \right) = \frac{\partial}{\partial x^j} \lim_{t \to 0} \left(\frac{(x^i \circ \phi_t) - (x^i \circ \phi_0)}{t} \right)$$

$$= \frac{\partial}{\partial x^j} X(x^i) = \frac{\partial}{\partial x^j} X^i$$

The interchange of derivative and limit above is justified, since $x^i \circ \phi_t = x^i(\phi_t)$ is C^∞ and the interchange just says $\dfrac{\partial(x^i \circ \phi_t)}{\partial x^j\, \partial t} = \dfrac{\partial(x^i \circ \phi_t)}{\partial t\, \partial x^j}$. Putting all this together we get

$$\underset{X}{\pounds}\, dx^i = \sum_j \frac{\partial X^i}{\partial x^j}\, dx^j$$

then

$$\underset{X}{\pounds}\, \omega = \sum_i \left[(\underset{X}{\pounds}\, \omega_i(x))\, dx^i + \omega_i(x)\, (\underset{X}{\pounds}\, dx^i) \right] \quad \text{by 4) of lemma 4.4}$$

$$= \sum_i (X\, \omega_i)\, dx^i + \omega_i \sum_j \frac{\partial X^i}{\partial x^j}\, dx^j]$$

$$= \sum_{i,j} X^j \frac{\partial \omega_i}{\partial x^j}\, dx^i + \sum_{i,j} \omega_i \frac{\partial X^i}{\partial x^j}\, dx^j \; ,$$

which gives 1) by interchanging i and j in the last sum.

2) We use 5) of lemma 4.4:

$$0 = \underset{X}{\pounds}\, \delta^i_j = \underset{X}{\pounds}\, dx^i\left(\frac{\partial}{\partial x^j}\right) = \left(\underset{X}{\pounds}\, dx^i\right)\left(\frac{\partial}{\partial x^j}\right) + dx^i\left(\underset{X}{\pounds}\, \frac{\partial}{\partial x^j}\right) .$$

This gives

$$dx^i\left(\underset{X}{\pounds}\, \frac{\partial}{\partial x^j}\right) = -\left(\underset{X}{\pounds}\, dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = -\sum_k \frac{\partial X^i}{\partial x^k}\, dx^k\left(\frac{\partial}{\partial x^j}\right) = -\frac{\partial X^i}{\partial x^j} \; ,$$

thus

$$\underset{X}{£} \frac{\partial}{\partial x^j} = - \sum_{i=1}^{n} \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i}$$

Then

$$\underset{X}{£} Y = \sum_i \underset{X}{£} \left(Y^i \frac{\partial}{\partial x^i} \right) = \sum_i \left[\left(\underset{X}{£} Y^i \right) \frac{\partial}{\partial x^i} + Y^i \underset{X}{£} \frac{\partial}{\partial x^i} \right]$$

$$= \sum_i [(X\ Y^i) \frac{\partial}{\partial x^k} - Y^i \sum_j \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j}]$$

$$= \sum_{i,j} X^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} - \sum_{i,j} Y^i \frac{\partial X^j}{\partial x^i} \frac{\partial}{\partial x^j}$$

interchanging i and j in the first sum gives 2).

Notice that this last expression is precisely

$$XY - YX = \sum_j X^j \frac{\partial}{\partial x^j} Y - \sum_i Y^i \frac{\partial}{\partial x^i} X$$

since the 2nd order derivatives cancel each other.

If we define the <u>Lie bracket</u> $[X,Y] = XY - YX$,

Then we have

$$\underset{X}{£} Y = [X,Y] \qquad (4.18)$$

We also have immediately from theorem 4.12,

<u>Corollary</u>: The Lie derivatives $\mathcal{L}_X : \underline{T(M)} \longrightarrow \underline{T(M)}$ and $\mathcal{L}_X : \underline{T^*(M)} \longrightarrow \underline{T^*(M)}$ are well defined C^∞ maps.

In a previous exercise it was seen that the C^∞ sections $\underline{E(M)}$ of any vector bundle has a natural structure as a $C^\infty(M)$ module. This is, of course, true in particular for $\underline{T(M)}$. However, the above corollary gives $\underline{T(M)}$ a more interesting algebraic structure not shared by the more general case $\underline{E(M)}$. Indeed we have $\mathcal{L}_X Y = [X,Y] \in \underline{T(M)}$ for any $X,Y \in \underline{T(M)}$. Thus $\underline{T(M)}$ has a natural multiplication $[\cdot,\cdot] : \underline{T(M)} \times \underline{T(M)} \longrightarrow \underline{T(M)}$. This leads to the following:

<u>Definition 4.4</u>: A (non-associative) algebra $\mathcal{L}$ with the product denoted by $[x,y]$, which satisfies for any $x,y,z \in \mathcal{L}$

1) $[x,y] = -[y,x]$

2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

is called a <u>Lie algebra</u>. Condition 2) is known as the Jacobi identity.

<u>Theorem 4.13</u>: $\underline{T(M)}$ with the product given by the Lie bracket $[X,Y]$ is a Lie algebra.

Proof: We have already seen that $\underline{T(M)}$ is a vector space. We have the product $[X,Y] = XY - YX$. Clearly $[X,Y] = -[Y,X]$. To see 2) of the definition, compute for $f \in C^\infty(M)$

i) $[X,[Y,Z]]f = (X[Y,Z])f - ([X,Z]X)f$

$= X(YZ-ZY)f - (YZ-ZY)Xf$

$= XYZf - XZYf - YZXf + ZYXf$

Similarly,

ii) $[Y,[Z,X]]f = YZXf - YXZf - ZXYf + XZYf$

and

iii) $[Z,[X,Y]]f = ZXYf - ZYXf - XYZf + YXZf$

Adding i) + ii) + iii) gives 0 , verifying the Jacobi identity and proving the theorem.

T(M) with this Lie algebra structure will be denoted by $\mathfrak{X}(M)$, and referred to as the Lie algebra of vector fields on M.

Exercise: Let $\mathcal{U}$ be any associative algebra. Define $\mathcal{U}_L$ to be the space with the new multiplication law $[x,y] = xy-yx$ where $x,y \in \mathcal{U}$. Show that $\mathcal{U}_L$ is a Lie algebra.

Exercise: Let $f,g \in C^\infty(M)$ and $X,Y \in \mathfrak{X}(M)$.

Show that

$$[fX,gY] = fg[X,Y] - g(Yf)X + f(Xg)Y$$

This shows, in particular, that $\mathfrak{X}(M)$ is not $C^\infty(M)$-linear.

Exercise Let $F: M_1 \longrightarrow M_2$ be a C^∞ map. and let $X_i \in \underline{T}(M_1)$, $Y_i \in \underline{T}(M_2)$ with $F_* X_i = Y_i$ $i = 1,2$. Show that $F_*[X_1, X_2] = [Y_1, Y_2]$. This shows that $F_*: \mathfrak{X}(M_1) \longrightarrow \mathfrak{X}(M_2)$ is a Lie algebra homomorphism.

Exercise: Show that $\mathfrak{X}(M)$ is infinite dimensional.

Exercise: Let $X_1, \ldots, X_n$ be a basis for a Lie algebra $\mathfrak{L}$ (finite dimensional), whose product is the Lie bracket. Then $[X_i, X_j]$ must be a linear combination of the X's, namely

$$[X_i, X_j] = \sum_{k=1}^{n} C^k_{ij} X_k$$

Find conditions on the "structure constants" C^k_{ij} which follow from the definition of Lie algebra.

Exercise: Prove part 4) of lemma 4.4.

In a subsequent chapter we will extend $\mathfrak{L}_X$ to a map on tensor fields.