6. Tensor Bundles and Tensor Fields

In this chapter we will be concerned with the study of tensors on manifolds. The first section is preliminary.

6.1 New Vector Bundles from old ones

The basic idea here is that given any vector bundles and a "functorial" construction we will obtain new vector bundles. Let $\xi = (E, M, \pi)$ be a C^∞ vector bundle M a C^∞ manifold and $f: M' \to M$ a C^∞ map. We define the induced vector bundle or pullback of ξ $f^*\xi = (E', M', \pi')$ as that C^∞ vector bundle whose total space E' is the subset of $M' \times E$ consisting of all pairs $(p', v) \in M' \times E$ such that

$$f(p') = \pi(v)$$

The projection map π' is defined by $\pi'(p, v) = p'$. If we define $\tilde{f}: E \to E$ by $\tilde{f}(p', v) = v$, then the diagram

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow[f]{} & M \end{array}$$

commutes. So as sets $\pi'^{-1}(p') = \pi^{-1}(f(p'))$, and we define $\pi'^{-1}(p')$ as a vector space by giving it the same vector space structure as $\pi^{-1}(f(p'))$. We need to check that there is a natural way to define $f^*\xi$ as a C^∞ manifold with the local triviality condition fulfilled. Let $(U, \tilde{\phi})$ denote the standard coordinate chart on E. Put $U' = f^{-1}(U)$ and define

$$h': \; U' \times \mathbb{R}^k \to M' \times \pi^{-1}(U)$$

$$h'(p', y) = (p', h(f(p'), y))$$

where $h = g^{-1}$ is given by (4.9). Notice that $f(p') = \pi \circ h(f(p') \circ y)$, so in fact we have $h': U' \times \mathbb{R}^k \to \pi'^{-1}(U')$. It is easy to check that h' is bijective. We define the topology and differential structure on E' which makes h' a diffeomorphism. The standard chart on E' then becomes according to (4.9) $(U, \tilde{\phi}')$ where $\tilde{\phi}' = (\phi' \times id) \circ h'^{-1}$ and (U', ϕ') is a chart on M'. We need to check that in $U' \cap U'' \neq \phi$ $\phi'' \circ \phi'^{-1}$: $R^{m'+k} \to R^{m'+k}$ ($m' = \dim M'$) is a diffeomorphism. But this follows easily since the coordinate representatives of h' and h" are the identity maps. Moreover, if x'^i, $i = 1, \ldots, m'$ are coordinates for (U', ϕ') and (x'^i, v^a) $a = 1, \ldots, k$ coordinates for E', then $\pi \circ (x'^i, v^a) = (x'^i)$ is a C^∞ submersion. The proof that E' is Hausdorff and has a countable basis is exactly as in theorem 4.4. We arrive at

Theorem 6.1: If $\xi = (E, M, \pi)$ is a C^∞ vector bundle over M and f: $M' \to M$ a C^∞ map of C^∞ manifolds, then the induced bundle $f^*\xi$ has a natural structure as a C^∞ vector bundle over M'.

Exercise: Let ξ_1 and ξ be two C^∞ vector bundles related by a bundle map $(\tilde{f}, f)$: $\xi_1 \to \xi$ such that $\tilde{f}$ is an isomorphism on fibres. Show that ξ_1 is isomorphic to the induced bundle $f^*\xi$.

Now let $\xi_1 = (E_1, M_1, \pi_1)$ and $\xi_2 = (E_2, M_2, \pi_2)$ be two vector bundles. We define the direct product $\xi_1 \times \xi_2$ as the vector bundle with projection map $\pi_1 \times \pi_2$: $E_1 \times E_2 \to M_1 \times M_2$ and fibre

$$(\pi_1 \times \pi_2)^{-1}(p_1, p_2) = \pi_1^{-1}(p_1) \times \pi_2^{-1}(p_2) = \pi_1^{-1}(p_1) \oplus \pi_2^{-1}(p_2)$$

It is easy to show that $\xi_1 \times \xi_2$ is a vector bundle over $M_1 \times M_2$. Now let $M_1 = M_2 = M$ and $\xi_1 = (E_1, M, \pi_1)$ and $\xi_2 = (E_2, M, \pi_2)$ two vector bundles over M. Consider the diagonal map

$$\Delta : M \to M \times M$$

defined by $\Delta(p) = (p, p)$ for all $p \in M$. The induced bundle $\Delta^*(\xi_1 \times \xi_2)$ over M is called the Whitney sum of ξ_1 and ξ_2 and is written as $\xi_1 \oplus \xi_2$.

Another important concept is that of a subbundle. Suppose $\xi_1 = (E_1, M, \pi_1)$ and $\xi_2 = (E_2, M, \pi_2)$ are vector bundles over M and let $(\tilde{f}, id)$: $\xi_1 \to \xi_2$ be a bundle map such that $\tilde{f}$ is a monomorphism on fibres, then ξ_1 is called a <u>subbundle</u> of ξ_2. Notice that f on fibres simply identifies $\pi_1^{-1}(p)$ as a sub-vector space of $\pi_2^{-1}(p)$, $p \in M$.

<u>Lemma 6.1</u>: Let $\xi_i = (E_i, M, \pi_i)$ i = 1, 1 be subbundles of the vector bundle $\eta = (E, M, \pi)$. Suppose that for all $p \in M$ as vector spaces

$$\pi^{-1}(p) = \pi_1^{-1}(p) \oplus \pi_2^{-1}(p)$$

Then η is isomorphic to the Whitney sum $\xi_1 \oplus \xi_2$.

Proof: We define the bundle map $(\tilde{f}, id)$: $\xi_1 \oplus \xi_2 \to \eta$ by $\tilde{f}(p, v_1 \oplus v_2) = (p, v_1 + v_2)$. This is clearly an isomorphism on fibres and so $\xi_1 \oplus \xi_2$ and η are isomorphic by lemma 4.3.

We now come to the main theorem of this section. Let $\xi_i = (E_i, M, \pi_i)$ $i = 1, \ldots, r$ be vector bundles over M and let F: $V \times \ldots \times V \to V$ be a covariant contravariant, or mixed functor in k arguments. We construct a new vector space $F(\xi_i) = (E, M, \pi)$ over M with fibre over p

$$F_p = F(\pi_1^{-1}(p), \ldots, \pi_k^{-1}(p))$$

Define E as the dijoint union

$$E = \bigcup_{p \in M} F_p$$

and π: $E \to M$ by $\pi(F_p) = p$.

We will now construct coordinates on $F(\xi_i)$ with the local triviality condition fulfilled. On ξ_i we have local diffeomorphisms.

$$h_i: \ U \times \mathbb{R}^{k_i} \to \pi_i^{-1}(U)$$

which provide isomorphisms $h_i(p) = \mathbb{R}^{k_i} \to \pi_i^{-1}(p)$. Then $F(h_1(p), \ldots, h_r(p))$: $F(\mathbb{R}^{k_i}, \ldots, \mathbb{R}^{k_r}) \to F_p$ is an isomorphism by an exercise of chapter 5. This defines a bijection $h: U \times \mathcal{F}(\mathbb{R}^{k_i}, \dots, \mathbb{R}^{k_r}) \to \pi^{-1}(U)$

$$h(p,y) = \mathcal{F}(h_1(p), \dots, h_r(p))(y)$$

$y \in \mathcal{F}(\mathbb{R}^{k_i}, \dots, \mathbb{R}^{k_r})$. We put the topology and differential structure on E which makes h a diffeomorphism for all coordinate neighborhoods $U \subset M$. The standard coordinate chart $(U, \tilde{\phi})$ is then constructed exactly as in (4.9), i.e. $\tilde{\phi} = (\phi \times id) \circ h^{-1}$ where (U, ϕ) is a coordinate chart on M. We need to check that in overlapping neighborhoods $U \cap U'$ $\tilde{\phi}' \circ \tilde{\phi}^{-1}$ is a diffeomorphism. But this follows since $h^{-1} \circ h'$: $U \cap U' \times \mathcal{F}(\mathbb{R}^{k_1}, \dots, \mathbb{R}^{k_r}) \to U \cap U' \times \mathcal{F}(\mathbb{R}^{k_1}, \dots, \mathbb{R}^{k_r})$ is a diffeomorphism. It is easy to check that π is a C^∞ submersion and that the topology on E is Hausdorff and has a countable basis. We arrive at

<u>Theorem 6.2:</u> For C^∞ vector bundles $\xi_i (i=1, \dots r)$ over M, $E = \mathcal{F}(\xi_i)$ ($\mathcal{F}$ a functor) has a natural differential structure which makes it a C^∞ vector bundle over M.

We can apply this theorem to construct many important new vector bundles. For example, if ξ and η are vector bundles on M then by applying the tensor product functor $\xi \otimes \eta$ and $\eta \otimes \xi$ are vector bundles on M. Similarly, by applying the duality functor ξ^* and η^* are vector bundles on M, thus, so are $\xi \otimes \eta^*$, $\xi^* \otimes \eta^*$, etc. Another useful vector bundle is obtained by applying the Hom functor, so the $\text{Hom}(\eta,\xi)$ is a vector bundle on M. We can also apply the symmetric and antisymmetric tensor product functor to a vector bundle η to obtain $S^k(\eta)$ and $\Lambda^k(\eta)$ as new vector bundles. We will devote separate sections to the study of the varios <u>tensor bundles on M</u> obtained by taking tensor products (full, symmmetric, antisymmetric) of the tangent and cotangent bundles on M.

An interesting application of this is the following: Let $i: N \to M$ be an immersion. Then by theorem 6.1 we can construct the induced bundle $i^*T(M)$

over N. It is easy to see that we can identify T(N) as a subbundle of $i^*T(M)$. Thus we can construct the quotient bundle $i^*T(M)/T(N)$. We call this the <u>normal bundle</u> $\nu(N)$ over N. Actually as vector bundles we have an isomorphism.

(6.1) $\quad i^*T(M) \cong T(N) \oplus \nu(N),$

But there is no canonical isomorphism unless we specify some additional structure such as a Riemannian metric.

<u>Exercise</u>: Construct the Whitney sum of two vector bundles functorially using theorem 6.2.

<u>Exercise</u>: Let $(f, id): \eta \to \xi$ be a bundle map. (This is sometimes called a strong bundle map). We say that $\tilde{f}$ has constant rank if $\tilde{f}_p = \tilde{f} | \pi_\eta^{-1}(p)$ has constant rank for all $p \in M$, i.e. rank $\tilde{f}_p$ is independent of p. Show that in this case $\ker \tilde{f} = \bigcup_{p \in M} \ker \tilde{f}_p$ and $\operatorname{Im} \tilde{f} = \bigcup_{p \in M} \operatorname{Im} \tilde{f}_p$ are subbundles of η and ξ respectively. Further, show that $\operatorname{Im} \tilde{f}$ is isomorphic to $\operatorname{coker} \tilde{f} := \eta/\ker \tilde{f}$.

<u>6.2 Tensor Bundles</u>

We consider the mixed functor $T^r_s(T_p(M)) = \overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{r \text{ times}} \otimes \overbrace{T^*_p(M) \otimes \ldots \otimes T^*_p(M)}^{s \text{ times}}$ which associates to each tangent space the above tensor product. By theorem 6.2 we have a C^∞ vector bundle $T^r_s(M)$ called the <u>tensor bundle of type (r,s)</u> given by

$$T^r_s(M) = \bigcup_{p \in M} T^r_s(T_p(M))$$

$$T^r_s(M) = \overbrace{T(M) \otimes \ldots \otimes T(M)}^{r \text{ times}} \otimes \overbrace{T^*(M) \otimes \ldots \otimes T^*(M)}^{s \text{ times}}$$

The tensor bundle $T^r_0(M)$ is referred to (oddly enough) as the <u>contravariant</u> tensor bundle of order r and $T^0_s(M)$ the <u>covariant</u> tensor bundle of order s.

Now given a chart (U, x) on M and the bases $(\frac{\partial}{\partial x^i})_p$ and $(dx^i)_p$ for $T_p(M)$ and $T^*_p(M)$, respectively, then a basis for $T^r_s(T_p(M))$ is

$(\frac{\partial}{\partial x^{i_1}})_p \otimes \cdots \otimes (\frac{\partial}{\partial x^{i_r}})_p \otimes (dx^{j_1})_p \otimes \cdots \otimes (dx^{j_s})_p$. Any tensor $T \in T^r_s(T_p(M))$ can be written in this basis as

$$T = \sum_{\substack{i_1 \cdots i_r \\ j_1 \cdots j_s}} T^{\ell_1 \cdots i_r}_{j_1 \cdots j_s} (\frac{\partial}{\partial x^{i_1}})_p \otimes \cdots \otimes (\frac{\partial}{\partial x^{i_r}}) \otimes (dx^{j_1})_p \otimes \cdots \otimes (dx^{j_s})_p \qquad (6.3)$$

Now a change of coordinate system at $p \in M$ induces a change of basis in $T_p(M)$. More generally let $F: M \to N$ be any C^∞ map given in local coordinates by

$$x'^i = (\psi \circ F \circ \phi^{-1})^i (x'_1 \cdots x^n)$$

Then from theorems (4.2) and (4.3) we have

$$F_*(\frac{\partial}{\partial x^i})_p = \sum_j (\frac{\partial x'^j}{\partial x^i})_{\phi(p)} (\frac{\partial}{\partial x'^j})_{F(p)}$$

and

$$F^*(dx'^i)_{F(p)} \sum (\frac{\partial x'^i}{\partial x^j})_{\phi(p)} (dx^j)_p$$

Now suppose F is a diffeomorphism so that F^{-1} exists, then we can define a map F_* on the tensor space $T^r_s(T_p(M))$. We have

$$F_*(F_{*j}(F^{-1})^*): \quad T^r_s(T_p(M)) \to T^r_s(T_{F(p)}(N))$$

Defined by (writing F_* for $F_*(F_*,(F^{-1})^*)$)

$$F_* = F_* \otimes \cdots \otimes F_* \otimes (F^{-1})^* \otimes \cdots \otimes (F^{-1})^*$$

For any two diffeomorphisms $M \xrightarrow{F} N \xrightarrow{G} L$ we write

$$F_* G_* = F_* G_* \otimes \cdots \otimes F_* G_* \otimes (F^{-1})^* (G^{-1})^* \otimes \cdots \otimes (F^{-1})(G^{-1})^*$$

then it is easy to check that (show it)

$$(FG)_* = F_* G_*$$

and identities map to identities. $F_*(F_{*j}(F^{-1})^*)$ is a covariant functor. We have in general for $T \in T^r_s(T_p(M))$

$$F_*T = \sum_{\substack{i_1\cdots j_r \\ j_1\cdots j_s}} T^{i_1\cdots i_r}_{j_1\cdots j_s} \sum_{\substack{k_1\cdots k_r \\ \ell_1\cdots \ell_s}} \left(\frac{\partial x'^{k_1}}{\partial x^{i_1}}\right)_{\phi(p)}\cdots\left(\frac{\partial x'^{k_r}}{\partial x^{\ell_r}}\right)_{\phi(p)}\left(\frac{\partial x^{j_1}}{\partial x'^{\ell_r}}\right)_{\phi(p)}\cdots\left(\frac{\partial x^{j_s}}{\partial x'^{\ell_s}}\right)_{\phi(p)}$$

$$\times \left(\frac{\partial}{\partial x'^{k_1}}\right)_{F(p)} \otimes \cdots \otimes \left(\frac{\partial}{\partial x'^{k_r}}\right)_{F(p)} \otimes (dx'^{\ell_1})_{F(p)} \otimes \cdots \otimes (dx'^{\ell_s})_{F(p)} \qquad (6.4)$$

In particular for a change of coordinate in M(=N), F=id so F_* is $(\mathrm{id})_*$. Thus if $T^{i_1\cdots i_r}_{j_1\cdots j_s}$ are the components of T in the coordinates $x' = (x^1_r, \dots, x^n)$ then (6.4) gives

$$T'^{k_1\cdots k_r}_{1\cdots s} = \sum_{\substack{i_1\cdots i_r \\ j_1\cdots j_s}} T^{i_1\cdots i_r}_{j_1\cdots j_s} \left(\frac{\partial x'^{k_1}}{\partial x^{i_1}}\right)_{x(p)}\cdots\left(\frac{\partial x'^{k_r}}{\partial x^{i_r}}\right)_{x(p)}\left(\frac{\partial x^{j_i}}{\partial x'^{\ell_i}}\right)_{x(p)}\cdots\left(\frac{\partial x^{j_s}}{\partial x'^{\ell_s}}\right)_{x(p)} \qquad (6.3)$$

Remark. In order to define F_*: $T^r_s(T_p(M) \to T^r_s(T_{F(p)}(N))$ we need not only that F be C^∞ but that it be invertible and that F^{-1} be C^∞, i.e. F be a diffeomorphism (local). However, if F is only C^∞ we can still define F_*: $T^r_0(T_p(M)) \to T^r_0(T_{F(p)}(N))$ for contravariant tensors, since now $F_* = F_* \otimes \cdots \otimes F_*$. Similarly, we can define $F^* = F^* \otimes \cdots \otimes F^*$: $T^0_s(T_{F(p)}(N) \to T^0_s(T_p(M)$.

Now if (U, x) is a local coordinate chart on M, the corresponding standard coordinate chart on $T^r_s(M)$ is (U,ϕ), $\phi = (x, t^{i_1\cdots i_r}_{j_1\cdots j_s})$. We can then follow the proof of proposition 4.6 and arrive at

Proposition 6.1: Let F: M → N be a diffeomorphism. Then F_*: $T^r_s(M) \to T^r_s(N)$ is a diffeomorphism and (F_*, F) is a bundle map, and $T^r_s(M)$ and $T^r_s(N)$ are isomorphic.

Exercise: State and prove the analogue of proposition 4.6 for the contravariant tensor bundle $T^r_0(M)$.

A C^∞ (local) global section of $T^r_s(M)$ is called a (local) global tensor field of type (r,s). In practice tensor fields often arise in terms of C^∞ (M) multilinear maps. So we will prove

Proposition 6.2: Tensor fields of type $(0,s)$ can be identified with the set of $C^\infty(M)$-multilinear C^∞ maps $\underline{T(M)} \times \ldots \times \underline{T(M)} \to C^\infty(M)$.

Proof: Since Hom is a contravariant functor, there is by theorem 6.1 a C^∞ vector bundle $\mathrm{Hom}(\overbrace{T\times\ldots\times T}^{s \text{ times}},\mathbb{R})$ over M. Moreover, it follows from lemma 4.3 and an exercise of chapter 5 that this bundle is isomorphic (strongly, i.e. identity map on M) as vector bundles to $T_s(M)$. Furthermore, the C^∞ sections of $\mathrm{Hom}(T\times\ldots\times T, \mathbb{R})$ are $C^\infty(M)$-multilinear C^∞ maps $A: \underline{T(M)}\times\ldots\times\underline{T(M)} \to C^\infty(M)$. We need only show that any such map is necessarily a C^∞ section of $\mathrm{Hom}(T\times\ldots\times T,\mathbb{R})$. Define the C^∞ section $T: M \to \mathrm{Hom}(\overbrace{T\times\ldots\times T}^{s \text{ times}}, \mathbb{R})$ by

$$T(p)(X_1(p), \ldots, X_s(p)) = A(X_1, \ldots, X_s)(p)$$

for $X_i \in \underline{T(M)}$, $i = 1, \ldots, s$. We must show that $T(p)$ is well defined, i.e. if $X_i(p) = Y_i(p)$ for each $i=1, \ldots, s$, then $A(X_1, \ldots, X_s)(p) = A(Y_1, \ldots, Y_s)(p)$ or equivalently if $X_i(p) = 0$ for all $i = 1, \ldots, s$, then $A(X_1, \ldots, X_s)(p) = 0$, $X_i \in \underline{T(M)}$. We prove this for the case $s = 1$ as the general case is exactly analogous. Let (U,x) be a coordinate chart about $p \in M$ and let $X \in \underline{T(M)}$ such that $X(p) = 0$, then on U we have

$$X = \sum_i X^i \frac{\partial}{\partial x^i}, \quad X^i(p) = 0 \text{ on } U$$

Let $f \in C^\infty(M)$ such that $f = 1$ on some neighborhood V of p and $\mathrm{supp}\, f \subset U$. Then $f \frac{\partial}{\partial x^i}$ are global vector fields on M and putting $Y = fX$ we have

$$A(Y) = \sum_{i=1}^{n} X^i A\left(f \frac{\partial}{\partial x^i}\right)$$

Evaluating at p, gives $A(Y)(p) = 0$. But $A(Y) = A(fX) = f\,A(X)$. So $A(X)(p) = A(Y)(p) = 0$. Q.E.D.

Exercise: What is wrong with the following: $A(X)(p) = \sum_i X^i(p)\, A\left(\frac{\partial}{\partial x^i}\right)(p) = 0$ since $X(p) = 0$?

Exercise: Let $F: N \to M$ be C^∞. Show that $\mathcal{F}^* = F^* \otimes \ldots \otimes F^* : T_s(M) \to T_s(N)$ is C^∞. Show that if $T \in T_s(M)$ then $\mathcal{F}^*T \in T_s(N)$.

Exercise: Identify tensor fields of type (1, 1) with the $C^\infty(M)$-multilinear C^∞ maps $\underline{T(M)} \to \underline{T(M)}$. Generalize this to tensor fields of general type (r, s).

It should be clear that if $(U_0 x)$ is local coordinate chart for M, then a tensor field T can be written on U as

$$T = \Sigma T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}, \quad T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \in C^\infty(U) \qquad (6.6)$$

and that the transformation formulas (6.4) and (6.5) hold where now we allow p to vary over all points of $U \cap V$, so we write $\frac{\partial x'^i}{\partial x^j}$ instead of $(\frac{\partial x'^i}{\partial x^j})x(p)$ and dx'^i instead of $(dx'^i)F(p)$, etc. I will not bother to write down these formulas.

Recall the contraction map $C^i_j : T^r_s(T_p(M)) \to T^{r-1}_{s-1}(T_p(M))$ defined in Chapter 5. Since this map is functorial (show it) it passes to a strong bundle map $C^i_j : T^r_s(M) \to T^{r-1}_{s-1}(M)$ by taking C^∞ sections C^i_j passes to a C^∞ map on C^∞ sections. We thus have a tensor field CT with components in local coordinates given by

$$(CT)^{i_1 \ldots i_{r-1}}_{j_1 \ldots j_{s-1}} = \sum_k T^{i_1 \ldots k \ldots i_r}_{j_1 \ldots k \ldots j_s}$$

where the upper (lower) k appears in the i^{th} (j^{th}) place.

Example 6.2: Consider the tensor field of type (1, 1) $\delta: \underline{T(M)} \times \underline{T^*(M)} \to C^\infty(M)$ defined by

$$\delta(X,\omega) = \omega(X)$$

Look at this in local coordinates (U, x) writing

$$X = \sum_i X^i \frac{\partial}{\partial x^i}, \qquad \omega = \sum_i \omega_i \, dx^i$$

$$\delta = \sum_{i,j} = \delta^i_j \frac{\partial}{\partial x^i} \otimes dx^j$$

Then $\delta(X, \omega) = \Sigma\ \delta^i_j\ X^j \omega_i = \Sigma\ X^i\ \omega_i$, so the component δ^i_j of δ is just the familiar Kronecker delta.

We say that a tensor field T is <u>invariant</u> if for two arbitrary coordinate charts $(U, x)_j(U', x')$ with $U \cap U' = \phi$, the components of T with respect to x and x' are the same.

<u>Exercise:</u> Show that δ is invariant.

6.3 Symmetric Bundles, Metrics

Since the construction of the symmetric tensor product is functorial (Exercise Chapter 5), it follows from theorem 6.1 that the symmetric tensor products of T(M) or T*(M) are C^∞ vector bundles over M. We define the following vector spaces:

$$S^k_p(M) = \mathcal{S}(\overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{k \text{ times}})$$

$$S^k_p(M)^* = \mathcal{S}(T_p(M)^* \otimes \ldots \otimes T_p(M)^*)$$

It is easy to check that $S^k_p(M)^*$ is in fact dual to $S^k_p(M)$. We now define the k^{th} order contra-and covariant symmetric bundles, respectively by

$$S^k(M) = \bigcup_{p \varepsilon M} S^k_p(M)$$

$$S^k(M)^* = \bigcup_{p \varepsilon M} S^k_p(M)^*$$

C^∞ sections of these bundles are called <u>symmetric ontravariant</u> and <u>symmetric covariant tensor fields</u>, respectively. We mention that multiplication in the symmetric algebras $\sum_{k=0}^{\infty} S^k_p(M)$ and $\sum_{k=0}^{\infty} S^k_p(M)^*$ is denoted by juxtaposition. We leave for the reader the task of writing down a symmetric tensor field in terms of local coordinates and the corresponding transformation formulas under C^∞ maps.

We will be especially interested in the bundle $S^2(M)^*$. Now clearly proposition 6.2 carries over to the symmetric case so we can identify C^∞

sections of $S^2(M)^*$ with symmetric $C^\infty(M)$-multilinear C^∞ maps $\underline{T(M)} \times \ldots \times \underline{T(M)} \to C^\infty(M)$. In particular, a C^∞ section g of $S^2(M)^*$ is identified with a symmetric C^∞ map g: $\underline{T(M)} \times \underline{T(M)} \to C^\infty(M)$. g is said to be nondegenerate if the bilinear form $g(p)(X_p, Y_p)$ is nondegenerate at all points $p \in M$, where $X_p, Y_p \leftarrow T_p(M)$. A nondegenerate C^∞ section g of $S^2(M)^*$ is called a pseudo-Riemannian metric on M. Let sig g: $M \to \mathbb{Z}$ be defined by (sig g)(p) = sig g(p) = number of negative eigenvalues of g(p) viewed as a symmetric bilinear form on $T_p(M)$. We have

Lemma 6.2: sig g is independent of $p \in M$, if M is connected.

Proof: Since g is continuos, sig g is also continuous. But the continuous immage of a connected space is connected. Thus sig g(p) is the same integer for all $p \in M$.

A pseudo-Riemannian manifold (M, g) is a C^∞ manifold, M together with a pseudo-Riemannian metric g. Notice that in principal different components of M can have different signatures. Notice also that if g has signature k, then -g has signature n-k and these two situations describe the same pseudo-Riemannian manifold. That is, (M, g) really refers to g up to a sign only. A pseudo-Riemannian manifold with signature o (or n) is called a Riemannian manifold. This means of course that g is everywhere positive (negative) definite. A pseudo-Riemannian manifold of signature 1 (or n-1) is called a Lorentzian manifold.

Let (U, x) be a local coordinate chart for M and let g be a pseudo-Riemannian metric on M. Then a C^∞ section g of $S^2(M)^*$ on U can be written as

$$g|_{\bar{\bar{u}}} \Sigma g_{ij}\, dx^i dx^j \equiv ds^2 \tag{6.7}$$

$g_{ij} = g_{ji} \in C^\infty(U)$. Classically this is usually called ds^2 and we will use this notation also. Viewed as a multilinear map g: $\underline{T(M)} \times \underline{T(M)} \to C^\infty(M)$, it follows from the proof of proposition 6.2 that we can localize g, i.e. $g(X,X)|_u = g(X|_u, Y|_u)$. Doing this we see that

$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \tag{6.8a}$$