Complex Manifolds - Riemann Surfaces

A *complex manifold* is a Hausdorff space M with a countable basis for its topology together with ~~a~~ maximal atlas $\mathcal{A}$ such that the charts ~~M~~ (U_i, ϕ_i) in $\mathcal{A}$ satisfy

1) $\phi_i : U_i \longrightarrow \mathbb{R}^{2n} = \mathbb{C}^n$ is a homeomorphism onto an open set of $\mathbb{C}^n$

2) If $U_i \cap U_j \neq \phi$, then the transition functions

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$$

are *biholomorphic* (or *bianalytic*) diffeomorphisms

This last statement means that the maps $\phi_i \circ \phi_j^{-1}$ ~~are~~ (and the inverse $\phi_j \circ \phi_i^{-1}$) are analytic functions of the variables $(z_1, \ldots, z_n)$ on $\mathbb{C}^n$.

A *Riemann Surface* is a complex manifold of complex dimension one, i.e. $n=1$ above.

The simplest example of a compact Riemann surface is the Riemann Sphere

Example 2: The Riemann sphere: (or the extended complex plane or one dimensional complex projective space $\mathbb{P}^1(\mathbb{C})$): Consider the one-point compactification $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of the complex plane. The best way of thinking of this is the stereographic projection of S^2 (example 3.6 pg 3.7-9). Now $\mathbb{C} = \mathbb{R}^2 \simeq S^2 - \{(0,0,1)\}$ by stereographic projection. Thus adding the "ideal point" ∞ corresponds to adding the "north pole" $(0,0,1)$ to the sphere (minus $\{0,0,1\}$), so that $\hat{\mathbb{C}} \simeq S^2$. However, the atlas given in example 3.6 is not quite right since it is not analytic. In terms of the coordinates used there, we have the map $\mathbb{R}^2 - \{(0,0)\} \xrightarrow{f} \mathbb{R}^2 - \{(0,0)\}$ given by $y_i \xmapsto{f} \frac{y_i}{\|y\|^2}$ where $\|y\|^2 = \sum_{i=1}^{2} (y_i)^2$. Let us define the complex coordinate by $z = y_1 + iy_2$. Then our map becomes $z \xmapsto{f} \frac{z}{|z|^2} = \frac{1}{\bar{z}}$ where $|z|^2 = y_1^2 + y_2^2 = z\bar{z}$. Thus, if we compose this map with the complex conjugation map $\sigma: \mathbb{C} \longrightarrow \mathbb{C}$ defined by $\sigma(z) = \bar{z}$, then we have

$$\sigma\circ f(z) = \sigma\left(\frac{1}{z}\right) = \frac{1}{z}, \text{ ie. the map } z \longmapsto \frac{1}{z}$$

this map is bianalytic from $\mathbb{C}-\{0\}$ to itself. Hence S^2 is a complex manifold (called the Riemann sphere).

Complex Torus

<u>Example 2:</u> The two real dimensional Torus:= ~~[illegible]~~

Begin with the complex plane $\mathbb{C} = \mathbb{R}^2$ and fix a complex number τ with $\operatorname{Im}\tau > 0$. Consider the group of transformations on $\mathbb{C}$ $\Gamma = \{\gamma_{n,m}(z) = z+n+m\tau : n,m \in \mathbb{Z}\}$. Γ is a discrete transformation group which acts properly discontinuously on $\mathbb{C}$ (Notice that Γ is a discrete subgroup of the complex Lie group $\mathbb{C}$ under addition). Thus $\mathbb{C}/\Gamma$ is a C^∞ manifold. But also it is a complex manifold since the maps $\gamma_{n,m} : \mathbb{C} \longrightarrow \mathbb{C}$ are clearly bianalytic. Thus the charts we constructed on M/Γ will be bianalytic since the chart on $\mathbb{C}$ is. Thus $\mathbb{C}/\Gamma$ is a complex manifold of complex dimension 1 — a Riemann surface. Furthermore, the map sending $(\lambda+\mu\tau)$ to $(e^{2\pi i\lambda}, e^{2\pi i\mu})$ is a homeomorphism of $\mathbb{C}/\Gamma$ onto the Torus ~~[illegible]~~ $S^1\times S^1$.

Fundamental domain