I- Topological Preliminaries.

In this chapter we collect some basic notions and results of topology. It is suggested that the reader review a standard book like Hocking and Young.

Definition 1.1:

A topological space is a set S, together with a family of subsets, called open sets of S, satisfying:

(i) The union of an arbitrary number of open sets is open.
(ii) The intersection of any finite number of open sets is open.
(iii) Both S and ϕ are open, where ϕ denotes the empty set.

From now on, for topological space we will write top space, or ζ-space if an explicit reference to the topology ζ is needed. If S is a ζ-space and $p \varepsilon S$, any open set (i.e. any member of ζ), that contains p will be called a neighborhood of p, and will be denoted by U_p, V_p, etc.

Example 1.1:

a) In any set S, we can define the discrete topology, in which every subset is open, and the indiscrete topology, in which the only open sets are S and ϕ.

b) The set with two elements $\{0,1\}$, with the discrete topology is called $\mathbb{Z}_2$. With the topology $\{\phi,\{0\},\{0,1\}\}$ is called the Sierpinsky Space.

Is clear from these examples, that in a given set may be defined different topologies. Now, in order to define a topology there is no need to specify all the open sets from the beginning, but only a "sufficient" number of them. More precisely:

Theorem 1.1:

Let S be a set, and $\{B_\alpha\}$ any family of subsets of S that satisfies:

(i) $\bigcup_\alpha B_\alpha = S$

(ii) If $p \epsilon B_\alpha \cap B_\beta$, then there exists B_γ such that $p \epsilon B_\gamma$ and $B_\gamma \subset B_\alpha \cap B_\beta$.

Then, the family ζ consisting of all the unions of members of $\{B_\alpha\}$, (including empty unions), is a topology for S.

Exercise: Prove this theorem.

In any ζ-space, a family of open sets satisfying the conditions stated in this theorem is called a basis for ζ. Given two topologies ζ_1, ζ_2 on the same space S with basis $\{O_\alpha\}$ and $\{R_\alpha\}$ respectively, the topology ζ_1, is said to be smaller or finer than ζ_2 if every R_α can be expressed as a union of the O_α's. We write this as $\zeta_2 \geqslant \zeta_1$. (It is also said that ζ_2 is larger than ζ_1). If also $\zeta_1 \geqslant \zeta_2$ we say that ζ_1 and ζ_2 are equivalent The discrete and indiscrete topologies are respectively the smallest and largest topologies that can be defined on a given set.

Example 1.2:

a) ζ itself is a basis for ζ in any ζ-space.

b) In $\mathbb{R}^n$ the family of subsets $\{B(x;r) \mid x\varepsilon\mathbb{R}^n, r > 0\}$ with $B(x;r) = \{x\varepsilon\mathbb{R}^n \mid (\sum_{i=1}^{n} (x^i - y^i)^2)^{1/2} < r\}$ is a basis for the usual Euclidean topology in $\mathbb{R}^n$. (see theorem 1.6 and an exercise below).

Definition 1.2 :

Let S be a space and $X \subset S$.

(i) X is called closed if $S - X$ is open.

(ii) The closure $\bar{X}$ of X is the intersection of all closed sets which contain X.

(iii) X is called dense in S if $\bar{X} = S$.

(iv) The interior X° of X is the union of all open sets contained in X.

(v) The boundary ∂X of X is defined as $\partial X = \bar{X} - X^\circ$

(vi) A point $p\varepsilon S$ is said to be a limit point of X, if every neighborhood of p contains at least one point in X distinct from p. The set of limit points of X is denoted by X'. A limit point is also called an accumulation point. Note: The terminology is not standard. In some books limit point is defined slightly differently.

Exercise : Show that

(a) The intersection (union) of an arbitrary (finite) number of closed sets is closed (hint: De Morgan's laws).

(b) X is closed (open) if and only if $X = \bar{X}$ $(X = X^\circ)$

(c) X is open if and only if every $p \in X$ has a neighborhood $U \subset X$.

Note that open and closed are not exclusive, nor exhaustive concepts. In any space S, both S and ϕ are open and closed, and in $\mathbb{R}^n$ a subset of the form $\{x \in \mathbb{R}' \mid 0 \leq x < 1\}$ is neither open nor closed.

Theorem 1.2:

If S is a space and $X \subset S$, then $\bar{X} = X \cup X'$

proof:

a) $\bar{X} \subset X \cup X'$: We consider only the case $p \in \bar{X}$, $p \notin X$.
Suppose that $p \notin X'$, then there exists a neigborhood U_p of p such that $U_p \cap X = \phi$, so we have a closed set $(\zeta - U_p)$ containing X and $p \notin (\zeta - U_p)$ contradicting that $p \in \bar{X}$.
b) $X \cup X' \subset \bar{X}$: If $p \in X$ then $p \in X$. If $p \in X'$, let C be a closed set containing X and suppose $p \notin C$. Then $S-C$ is a neighborhood of p not intersecting X. This is absurd since $p \in X'$.

Example 1.3:

a) In Sierpinsky space, $\{\bar{1}\} = \{1\}$, $\{1\}^\circ = \phi$, $\{\bar{0}\} = \{0,1\}$ $\{0\}^\circ = \{0\}$, $\{0\}' = 1$. Note that $\{0\}$ is dense.

b) In $\mathbb{R}^1$, the set $\mathbb{Q}$ of rationals is a dense subset. In fact, a numerable dense subset (see a book of analysis).

If S is a space and $X \subset S$, we can construct a topology on X by defining a subset $U \subset X$ to be open if it is of the form $U = A \cap X$ with A open in S. This topology on X is called the subspace or relative topology. Note that "U open in "X" does not imply "U open in S".

Now let $S_1, S_2, \ldots, S_n$ be a finite number of topological spaces. We can define a topology in the cartesian product $S_1 \times S_2 \times \ldots \times S_n$ by taking the family of subsets of the form $U_1 \times U_2 \times \ldots \times U_n$, with U_i open in S_i as a basis. The topology generated by this basis in $S_1 \times S_2 \times \ldots S_n$ is called the product topology.

Finally, let S be a space and let ~ be an equivalence relation on S. We can turn the set S/~ of equivalence classes into a topological space as follows: Let

$$\rho : S \to S/\sim \, : \, \rho \to [\rho] = \{q \varepsilon S \mid \rho \sim q\}$$

be the natural projection. We define a subset $U \subset S/\sim$ to be open, if $\rho^{-1}(U)$ is open in S. S/~ With this topology is called the quotient space and the topology is called the quotient topology.

The third case illustrates a general procedure to construct topologies out of given ones. Through the action of mappings. As will be seen, the other two cases (subspace and product), can also be discussed from a similar point of view.

Definition 1.3:

Let S and S' be spaces, and $f : S \longrightarrow S'$ a map. f is said to be:

(i) Continuous, if for every open set $U' \subset S'$, the inverse image $f^{-1}(U) = \{p \in S \mid f(p) \in U'\}$ is open in S.

(ii) Open, (closed), if the image of each open (closed) set in S, is open (closed) in S'.

(iii) Homeomorphism, if it is an ~~injection~~ bijection, and both f and f^{-1} are continuous.

Exercise: Prove that in the definition of continuous map, the word "open" may be replaced by "closed". Show that the composition of continuous maps (of homeomorphisms) is continuous (a homeomorphism).

Note that continuous, open, and closed are independent notions. We also call the reader's attention to the fact that whenever we talk about a mapping $f: X \to Y$ between two sets X and Y, it must be considered into Y, unless we explicitly call it a surjection or onto. In particular, we could have defined an homeomorphism to be an injection and not a bijection. If S and S' are spaces, and a bijective homeomorphism $h: S \to S'$ exists, we will say that S and S' are homeomorphic and write $S \simeq S'$.

Now, let S be a space, Q an arbitrary set and consider a surjection $f: S \to Q$. We define a topology in Q by choosing a subset $Y \subset Q$ to be open, when ever $f^{-1}(Y)$ is open in S. This topology is called the identification topology in Q determined by f. It is the largest topology for which the map f is continuous.

As an example of the identification topology we have the quotient topology discussed before. The subspace and product topologies can also be defined by mappings but in a different way:

If S is a space and $X \subset S$, consider the <u>inclusion map</u> $u: X \to S: p \to u(p)$ $p \in X$. The subspace topology is the smallest topology for which the inclusion map is continuous.

In the cartesian product $S_1 x \ldots x S_n$ of the n spaces S_i, the map $\pi_k: S_1 x \ldots x S_n \to S_k: (p_1, \ldots, p_n) \to p_k$, is called the <u>projection onto the k-th factor</u>. The product topology is the smallest topology for which the n projections π_k are continuous.

To discuss other topologies between the two extremes, discrete and indiscrete, we need to impose additional conditions on Definition 1.1. For the developments in the following chapters we are interested in the invariance properties of the resulting topologies; this means that sometimes we will ask:

(1) If the additional properties are inherited by subspaces, or if they are transmitted to cartesian products (finite) and to quotient spaces.

(2) Under what kind of mappings these new properties are invariants. In this context, the most important properties are those which are preserved under bijective homeomorphisms, called <u>topological invariants</u>.

<u>Definition 1.4:</u>

A ζ-space is said to have <u>countable basis</u> (or to be <u>2° countable</u>), if ζ has a basis with countably many elements.

It is easy to show that 2° countability is invariant under continuous open surjections, that every subspace of a 2° countable space is 2°countable and that a finite cartesian product is 2°countable if and only if each factor is 2° countable.

If S is a space, a family $\{V_\alpha\}$ of subsets of S is said to be a *cover* of S, if $S = \bigcup_\alpha V_\alpha$. A subfamily of $\{V_\alpha\}$ that is also a cover of S is called a *subcover*. If each member of a cover is open then it is said to be an *open cover*

Definition 1.5:

A space is called Lindelof, if each open cover contains a countable subcover.

We are now ready to state the main property of 2° countable spaces:

Theorem 1.3:

Every 2° countable space is Lindelof.

Proof: Let $\{B_i\}$ be a countable basis for the space S, and $\{U_\alpha\}$ an open cover of S. Each U_α , being open, is a union of B_i's , so that, choosing for each i one U_{α_i} such that U_{α_i} $U_{\alpha_i} \supset B_i$ the result follows, since a basis is a cover.

With respect to the invariance properties of the Lindelof property we have:

Theorem 1.4.

The Lindelof property is invariant under continuous surjections.

Proof: Let $f:S \to S'$ be a continuous surjection, and $\{U_\alpha\}$ an open cover of S'. Then $\{f^{-1}(U_\alpha)\}$ is an open covering of S and if S is Lindelof there exists a countable subcover $\{f^{-1}(U_{\alpha_i})\}$ then $\{U_{\alpha_i}\}$ is a countable subcover of $\{U_\alpha\}$.

It follows from this that if a cartesian product is Lindelof, then each factor must be Lindelof, but the converse is false in general.

Exercise:

Show that if S is Lindelof, and $X \subset S$ is closed, then, (as a subspace), X is Lindelof. (This result is false in general for open subsets).

Another property closely related to 2° countability is separability.

Definition 1.6:

A space is said to be separable if it contains a countable dense subset.

It is easy to show that if $f:S \to S'$ is continuous, and $p \in S$ is a limit point of $X \subset S$, then $f(p)$ is a limit point of $f(X)$. From this it follows that if X is dense in S, then $f(X)$ is dense in $f(S)$, because $f(S) = F(\bar{X}) = f(X) \cup f(X') \subset \overline{F(X)}$. Hence, separability is invariant under continuous surjections.

The reader can also show that if $G \subset S$ is open, and

$H \subset S$ is dense, then $(X \cap G)$ is dense in G. (there are counterexamples which show that this is not true in general if G is closed), so separability is inherited by open subspaces.

Finally we mention that separability of a finite cartesian product is equivalent to separability of each factor.

The relation of separability and 2° countability is:

<u>Theorem 1.5:</u> If S is 2°countable, then every subspace of S is separable.

Proof: As 2°countability is inherited by every subspace, we need consider only the case in which the subspace is S itself: Take a point of each member of a countable basis to form a countable subset X. Every open set is a union of members of the basis, so X is dense.

The converse of this theorem is false in general, except for <u>metric spaces</u>:

<u>Definition 1.7:</u>

A <u>metric</u> on a set M is a function $d: M \times M \to \mathbb{R}$ with the properties, $x, y, z \in M$:

(i) $d(x,y) \geq 0$.

(ii) $d(x,y) = 0$ if and only if $x = y$

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,z) \leq d(x,y) + d(y,z)$

Associated with this metric, we can define a topology (a metric topology) on M in a natural way. First we define a spherical neighborhood $B(x;r)$ with center $x \varepsilon M$ and radius $r > 0$ to be the set of all points $y \varepsilon M$ with $d(x,y) < r$.

Theorem 1.6: The family of spherical neighborhoods $\{B(x;r) \mid x \varepsilon M, r>0\}$ forms a basis for a topology on M.

Proof: It is clear that $\bigcup_{x,\sigma} B(x;r) = M$. Now let $x \varepsilon B(x_1;r_1) \cap B(x_2;r_2)$ and $r = \min\{r_1 = d(x,x_1), r_2 = d(x,x_2)\}$. Then $r > 0$, since the statements $a \varepsilon B(\xi;r)$ and $d(\xi,a) < r$ are equivalent so $x \varepsilon B(x;r)$. Now, if $y \varepsilon B(x;r)$ then $d(y,x_1) \leq d(y,x) + d(x,x_1) < r + d(x,x_1) \leq (r_1 - d(x,x_1)) + d(x,x_1) = r$, that is, $y \varepsilon B(x_1,r_1)$ and similarly $y \varepsilon B(x_2;r_2)$ so that $y \varepsilon B(x_1;r_1) \cap B(x_2;r_2)$.

The topology generated by this basis is called the topology induced by the given metric.

Exercise: In $\mathbb{R}^n$, the set of all ordered n-tuples of real numbers $(x_1,\dots,x_n)$, show that

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$$

is a metric. This is called the Euclidean metric and the resulting topology is called the Euclidean or usual topology in $\mathbb{R}^n$.

Two metrics d, d' in the same space M are said to be equivalent , if the topologies ζ, ζ' induced by each one are equivalent.

Exercise. Show that in a metric space, the properties of 2°countability, separability and Lindelof are all equivalent.

In metric spaces, topological concepts can be phrased in the ε, δ , terms of classical analysis. For example, if M and M' are metric spaces with metrics d and d' respectively, a map $f: M \to M'$ is continuous if and only if $x \in M$, $\varepsilon > 0$, $\delta > 0$ such that $d(x,y) < \delta$ implies $d'(f(x), f(y)) < \varepsilon$.

With respect to invariance properties we have:

(1) Metrizability is a topological invariant.

(2) Every subspace of a metric space is a metric space.

(3) A finite cartesian product is metriable if and only if each factor is metrizable.

In every set X we can define a metric by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \qquad x, y \in X .$$

The topology induced by this metric is the discrete topology. On the other hand, the reader can verify that no metric can induce the topology in the Sierpinsky space. It is then of interest to ask when a given topology can be induced by a metric. One of the most useful results in this context (Urysohn) ,states that a 2° countable space is metrizable if and only if we can "separate" points and closed sets by means of disjoint open sets. (In the 2°countable Sierpinsky space, the point 0 and the point {1} , cannot be so separated). This property, called regularity, belongs to the list of the called <u>separation axioms</u>.

<u>Definition 1.8:</u> A topological space is said to be:

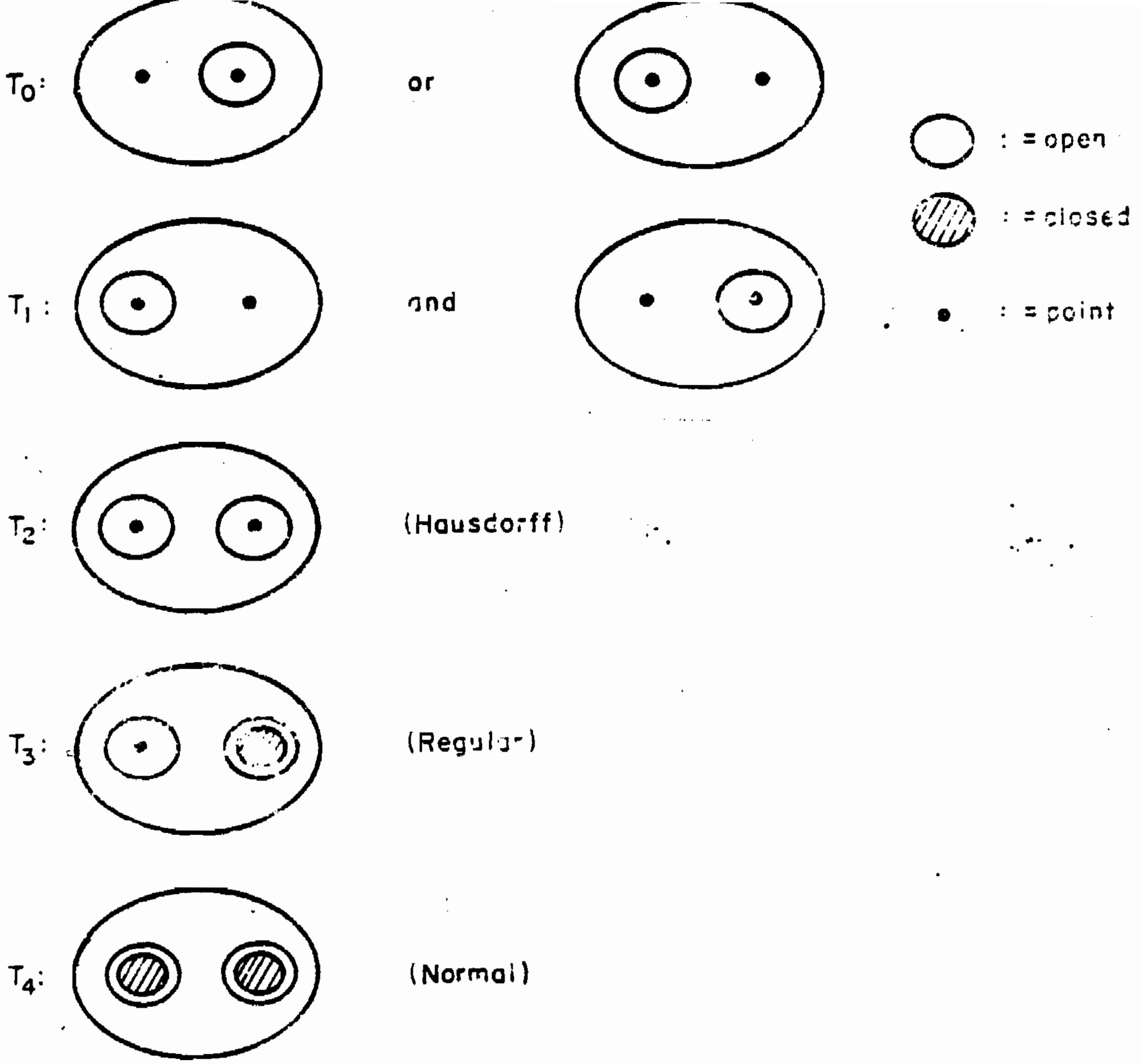
<u>T_0</u> : If for each pair of distinct points, at least one of them has a neighborhood not containing the other.

<u>T_1</u> : If for each pair of distinct points, each one of them has a neighborhood not containing the other.

<u>T_2</u> : Or <u>Hausdorff</u> , If each pair of distinct points have non intersecting neighborhoods.

<u>T_3</u> : Or <u>regular</u> . If it is T_1 and for every closed set A, and every point $p \notin A$, there exists a neighborhood U_p of p and an open set $V \supset A$, such that $U_p \cap V = \phi$.

<u>T_4</u> : Or <u>normal,</u> If it is T_1 and for every pair A,B of disjoint closed sets, there are open sets $U \supset A$ and $V \supset B$ such that $U \cap V = \phi$.



In the following, we will seldom encounter spaces which satisfy separation axioms weaker than Hausdorff, but as examples consider first the Sierpinsky space, which is T_0 but not T_1, and second, any infinite set X with topology $\tau = \{\phi\} \cup \{A \subset X \mid X-A$ is finite$\}$, is a T_1 space which is not Hausdorff.

Hausdorff topologies have, among others, the following important features:

(1) Every finite set is closed.

(2) If p is a limit point of a set X, then any neighborhood of p contains infinitely many points of X.

<u>Exercise:</u> Prove these properties.

A topological property is called <u>hereditary</u> if it holds for every subspace of a top space. The property of being T_1, T_2 or T_3 are all hereditary. To see this we first notice that the T_1 condition can be rephrased as: a space is T_1 if and only if all its single point sets are closed. For fixing q every $p \neq q$ has a neighborhood $U_p \subset T$ containing p such that $q \notin U_p$. Thus $T - \{q\} = \cup_p U_p$ is open; hence $\{q\}$ is closed. The converse statement is trivial. So if $A \subset T$ is a subset in a T_1 space with $q \in A$, then $\{q\}$ is closed in A since $A - \{q\} = A \cap T - \{q\}$ is open in A.

<u>Lemma 1.1</u>: The T_3 condition is hereditary.

Proof: Let T be regular and $X \subset T$ a subspace. Suppose A is closed in X with $p \in X - A$. There is a closed subspace $B \subset T$ such that $B \cap X = A$. But $p \notin B$, so there are neighborhoods $U_p \subset T$ of p and V of B which do not intersect. But then $\overset{\prime}{U}_p = U_p \cap X$ and $\overset{\prime}{V} = V \cap X$ satisfy $\overset{\prime}{V} \supset A$ and $\overset{\prime}{U}_p \cap \overset{\prime}{V} = \emptyset$. Q. E. D.

<u>Exercise</u>. Show that Hausdorff is an hereditary property, and that the product of Hausdorff spaces is Hausdorff.

<u>Exercise</u>: Show that T_i implies T_{i-1}, $i=1,\ldots,4$.

<u>Exercise</u>: Show that a metric space is regular.

<u>Definition 1.9</u>: A topological space is said to be <u>compact</u> if every open cover has a finite subcover. A top space is <u>locally compact</u> if every point has a neighborhood whose closure is compact.

<u>Theorem 1.6</u>: (Heine - Borel) A subset of R^n is compact if and only if it is closed and bounded.

<u>Exercise</u>: Review the proof of this theorem.

Remark. Boundedness is a metric space concept.

Theorem 1.7: R^n is locally compact.

Proof: Every point $x \in R^n$ has a spherical neighborhood $S(x,r)$ and $\overline{S(x,r)}$ is compact by the Heine-Borel theorem.

Theorem 1.8: If S is compact and $f: S \longrightarrow T$ is a continuous surjection, then T is compact. If S is locally compact and $f: S \longrightarrow T$ is a homeomorphism onto T, then T is locally compact.

Proof: To prove the first statement, let $\{C_\alpha\}$ be an open cover for T, then $\{f^{-1}(C_\alpha)\}$ is an open cover for S since f is a continuous surjection. Moreover, $\{f^{-1}(C_\alpha)\}$ has a finite subcover $\{V_{\alpha_i}\}$ and $\{f(V_{\alpha_i})\}$ is a finite subcover for T. For the second statement, we have that every $p \in S$ has a neighborhood U such that $\overline{U}$ is compact. Since f is a homeomorphism f(U) is open in T and contains f(p). By the first statement $f(\overline{U})$ is compact. But by the surjectivity of f, f(p) is an arbitrary point of T.

Remark. In the second statement we need the condition that f be a homeomorphism and not just continuous. (See Hocking and Young for a counterexample).

We give an example of a bijective continuous map which is not a homeomorphism.

Example 1.4: Let S be the nonnegative reals and define $f: S \longrightarrow S^1$ (unit circle) by

$$\theta = f(x) = \frac{2\pi x^2}{1+x^2}$$

f is a bijection and continuous; however, it is not a homeomorphism since f^{-1} is not continuous. For if it were we would violate theorem 1.8 since S^1 is compact and the nonnegative reals is not. Indeed for $\varepsilon > 0$ small $\theta \frac{(2\pi - \varepsilon)^{1/2}}{\varepsilon}$ is very large.

Exercise: Show that the product of two compact spaces is compact.

Exercise: Show that a compact Hausdorff space is regular.

We will strengthen this last exercise to

Theorem 1.9: A locally compact Hausdorff space is regular.

We will need a definition: Let T be a top space with topology ζ. Define the set $T^* = T \cup \{\infty\}$ where ∞ is a "point" not in T. On T* we put the topology ζ^* defined as follows: the members of ζ^* are all sets which are either

(i) Open sets of ζ

(ii) $T^* - C$ where C is a compact subset of T.

T* is called the <u>one-point compactification of T</u>.

Exercise: Show that ζ^* defines a topology on T*.

Lemma 1.2: Let T be a locally compact Hausdorff space (which is not compact) then T* is a compact Hausdorff space. Moreover, T is a subspace of T*, T* − T is a single point, and $T^* = \bar{T}$.

Proof: T is a subspace of T*, since for any open set U of T*, $U \cap T$ is open. Moreover, since T is not compact $T^* - C$ contains ∞ and intersects T nontrivially. Thus ∞ is a limit point of T and $T^* = \bar{T}$. T* is Hausdorff since if $p \in T$ and $q = \infty$ then we can find a neighborhood U of p such that $\bar{U}$ is compact and does not contain q. Then U and $T^* - \bar{V}$ are disjoint neighborhoods of p and q respectively. To show that T* is compact, Let $\mathcal{U}$ be an open cover. $\mathcal{U}$ contains a open set of the form $T^* - C$. Let $\{U'_\alpha\}$ be all members of $\mathcal{U}$ −

different from $T^* - C$. Put $V_\alpha = U'_\alpha \cap T$. $\{V_\alpha\}$ is an open cover which covers C, i. e. $C \subset \cap_\alpha V_\alpha$. Since C is compact there is a finite subcover $\{V_{\alpha_i}\}$ and $\cup V_{\alpha_i} \cup T^* - C$ is a finite subcover for $\mathcal{U}$. Q. E. D.

Proof of theorem 1.9: If T is a locally compact Hausdorff space, T^* is a compact Hausdorff space and by a previous exercise is regular. But then T being a subspace of a regular space is regular, by lemma 1.1. Q. E. D.

Example 1.5: The one-point compactification of the real line $\mathbb{R}$ is homeomorphic to the circle S^1. Similarly the one-point compactification of $\mathbb{R}^2$ is homeomorphic to the two sphere S^2. The explicit homeomorphisms will be given in the next chapter- stereographic projections.

Definition 1.10: Let T be a top space and p and q any points of T. Let $I_{a,b}$ be the closed interval $[a,b] \subset \mathbb{R}$. A path from p to q is a continuous map $f: I_{a,b} \longrightarrow T$ such that $f(a) = p$ and $f(b) = q$. T is said to be path connected if for every pair of points p, q, of T there is a path from p to q.

Remark. Without loss of generality we can take a=0, b=1, so that I is the unit interval. Why?.

Definition 1.11: A top space T is connected if it cannot be written as the union of two disjoint (nonempty) open sets.

Proposition 1.1: If T is path connected, then T is connected.

Exercise: Prove this proposition.

Remark. The converse statement is not true (see a topology book for a counter example).

Example 1.6: $\mathbb{R}^n$ is path connected and thus connected.

Proof: Let x and y be any points of $\mathbb{R}^n$ and define $f: I \longrightarrow R$ by

$$f(t) = ty + (1-t)\ x$$

which is easily seen to be continuous.

Theorem 1.10: The continuous image of a path connected (connected) space is path connected (connected).

Proof: Let $F: S \longrightarrow T$ be continuous with $F(S) = T$. If S is path connected there is an continuous map $f: I = I_{0,1} \longrightarrow S$ with $f(0) = x$, $f(1) = y$ for any two points $x, y \in S$. But then $F \circ f: I \longrightarrow T$ is continuous with $F \circ f(o) = F(x)$ and $F \circ f(1) = F(y)$. So T is path connected since F is surjective. If S is connected, assume that T is not. Then there are nonempty open sets U and V of T such that $U \cap V = \emptyset$. But then by the surjectivity and continuity of f, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty open sets of S. Moreover, they are disjoint, which contradicts the connectivity of S. Q.E.D.

Example 1.7: $\mathbb{R}^n - \{0\}$, $n > 1$, is path connected and thus connected. Consider example 1.6. If x,y are any points of $\mathbb{R}^n - \{0\}$ such that $f(t) \neq 0$ for all $t \in I$, then f is a path in $\mathbb{R}^n - \{0\}$ to y. If there is a $t_0 \in \overset{\circ}{I}$ such that $f(t_0) = 0$ consider another $z \in \mathbb{R}^n - \{0\}$ which is not on the path in $\mathbb{R}^n$ which joins x and y. Then if $g(t)=tz+(1-t)\ x$ we must have $g(t)\neq 0$ for all $t\varepsilon I$, for if not it is easy to see that x, y, z would be collinear. Similarly, $h(t) = ty+(1-t)z$ never vanishes. Thus $h\circ g(t)$ never vanishes and is a path in $\mathbb{R}^n - \{0\}$ from x to y.

Exercise: Show that the unit sphere $S^n = \{x\varepsilon\ \mathbb{R}^{n+1} : |x| = 1$, where $|x| = d(x,0\ \}$ (d is the usual matric on $\mathbb{R}^{n+1}$) is path connected.

Let C be a subset of S. Then C is called a path component (component) if C is path connected (connected) and is not a proper subset of another path connected (connected) subset of S.

Definition 1.12: Let f, g: $S \longrightarrow T$ be continuous maps of the top space S into the top space T. f is said to be homotopic to g if there is a continuous map $H: S \times I \longrightarrow T$ which satisfies

$$H\ (p,0) = f(p)$$

$$H\ (p,1) = g(p)$$

for every $p\ \varepsilon\ S$. The map H is called a homotopy between f and g. A space S is said to be contractible to a point p_o (or just contractible) if the identity map id: $S \longrightarrow S$ defined by $id(p) = p$ for all $p\ \varepsilon\ S$ is homotopic to the constant map $c: S \longrightarrow S$ defined by $c(p) = p_o$.

Example 1.8: $\mathbb{R}^n$ is contractible to a point, say $0\ \varepsilon\ \mathbb{R}^n$. Consider the identity map $id(x) = x$, $x\ \varepsilon\ \mathbb{R}^n$ and the zero map $0(x) = 0$.

Define $H: \mathbb{R}^n \times I \longrightarrow \mathbb{R}^n$ by $H(x,t) = (1-t)x$, We have $H(x,0) = x = id(x)$ and $H(x,1) = 0 = 0(x)$. It is easy to verify that H is continuous. Thus H is a homotopy between id and 0.

Remark: A contractible space is clearly path connected since $H_p: I \longrightarrow S$ defined by $H_p(t) = H(p,t)$ = homotopy between the identity and constant maps is a path from p to p_o.

Definition 1.13: Let f, g: $I \longrightarrow S$ be two paths in S. f is said to be path homotopic to g if $f(0)=g(0)= p_o$, $f(1)=g(1)=p_1$, and there is a continuous map $H: I \times I$ --- S such that $H(S,o)=f(S)$, $H(s,1)= g(s)$ and $H(o,t)= p_o$, $H(1,t) = p_1$. H is then called a path homotopy between f and g.

Remark. The first two conditions on H say that f and g are homotopic. The second two conditions say that for each $t \in I$, the map $s \longmapsto F(s,t)$ is a path from p_o to p_1. Thus path homotopy is a stronger condition than homotopy.

Let $C^o(S,T)$ be the set of continuous maps from S into T. For any f, g $\in C^o(S,T)$ we write $f \sim g$ if f is homotopic to g. Similarly, let $P^o(S)$ denote the set of paths in S. For any f, g $\in P^o(S)$ we write $f \sim_p g$ if f is path homotopic to g.

Lemma 1.3: The relations $\sim$ and $\sim_p$ are equivalence relations on $C^o(S,T)$ and $P^o(S)$ respectively.

Proof: We verify the transitivity property only. The remainder is left as an exercise. Suppose $f \sim g$ and $g \sim h$. We show that

$f \sim h$.

Let F and G be homotopies for f, g and g, h respectively. Define $H: S \times I \longrightarrow T$ by

$$H(p,t) = \begin{cases} F(p, 2t) & \text{for} \quad t \in [0, \frac{1}{2}] \\ G(p, 2t-1) & \text{for} \quad t \in [\frac{1}{2}, 1] \end{cases}$$

It is easy to check that H is well-defined. Now H is continuos on the closed subsets $S \times [0, \frac{1}{2}]$ and $S \times [\frac{1}{2}, 1]$ of $S \times I$. That H is continuous follows from lemma 1.3 given below. Now suppose F and G are path homotopies, then $H(0,t) = p_0$ for all $t \in [0,1]$ and $H(1,t) = p_1$ for all $t \in [0,1]$. So H is a path homotopy.

Exercise: Finish the proof of this lemma.

Lemma 1.4: (the pasting lemma). Let $S = A \cup B$ where A and B are closed subsets of S. Let $f: A \longrightarrow T$ and $g: B \longrightarrow T$ be continuous, and suppose that $f(x) = g(x)$ for all $x \in A \cap B$.
Define $h: S \longrightarrow T$ by

$$h(x) = \begin{cases} f(x) & \text{for } x \in A \\ g(x) & \text{for } x \in B \end{cases}$$

Then h is continuous.

Proof: Since $f(x) = g(x)$ for $x \in A \cap B$, h is well-defined. Let U be open in T, then $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$ is open in S. Q.E.D.

Let $f \in C^0(S,T)$ or $P^0(S)$, then we denote by $[f]$ the corresponding homotopy or path homotopy class. We denote by $[C^0(S,T)]$ the set of homotopy classes of maps from S to T, and by $[P^0(S)]$ the set of path homotopy classes of paths in S.

<u>Exercise</u>: Show that if T is contractible $[C^{o}(S,T)]$ consists of a single element. Show that if T is path connected and S is contractible then $[C^{o}(S,T)]$ consists of a single element.

On $[P^{o}(S)]$ we define a composition as follows: Let f be a path from p_o to p_1 and g a path from p_1 to p_2, we define the <u>composition</u> $f * g$ of f and g to be the path

$$f * g\ (t) = \begin{cases} f(2t) & \text{for } t \in [\ 0, \frac{1}{2}] \\ g(2t-1) & \text{for } t \in [\ \frac{1}{2}, 1] \end{cases}$$

That $f * g$ is a path from p_o to p_2 follows from lemma 1.3. Now we must show that $*$ makes sense on classes $[f]$ so that we can define $[f] * [g] = [f+g]$. Suppose that $f \sim_p f'$ and $g \sim_p g'$ and let F and G be the corresponding path homotopies. Define H by

$$H(s,t) = \begin{cases} F(2s,t) & \text{for } s \ \in [0,\frac{1}{2}] \\ G(2s-1,t) & \text{for } s \ \in [\frac{1}{2},1] \end{cases}$$

H is well-defined since $F(1,t) = p_1 = G(0,t)$ for all t, and continuous by lemma 1.3. Moreover, $H(s,0) = F(2s,0) = f(2s)$, $H(s,1) = f'(2s)$ for $s \in [0,\frac{1}{2}]$ and $H[s,0] = G(2s-1,0) = g(2s-1)$, $H(s,1) = g'(2s-1)$ for $s \in [\frac{1}{2},1]$. Thus $f * g \sim f' * g'$. But also $H(0,t) = F(0,t) = P_o$ and $H(1,t) = G(1,t) = p_2$, so $f * g \sim_p f' * g'$.

<u>Theorem 1.11</u>: The operation $*$ on $[P^{o}(S)]$ satisfies the following

1) (Associativity) If $[f] * ([g] * [h])$ is defined, then so is $([f] * [g]) * [h]$ and they are equal.

2) (Right and left identities) For $p \in S$ let e_p: $I \longrightarrow S$ be the constant path defined by $e_p(t) = p$ for all $t \in I$. If f is a path from p_o to p_1 then $[f]*[e_{p_1}]=[1]$ and $[e_{p_o}]*[1]=[1]$. e_{p_1} and e_{p_2} are called the right and left identities respectively.

3) (Inverse) Let f be a path from p_o to p_1. Define $\bar{f}(t)=f(1-t)$. It is the reverse path of f. Then $[f]*[\bar{f}] = e_{p_o}$ and $[\bar{f}]*[f]= e_{p_1}$.

The proof of this theorem is involved but not difficult and can be found in any book covering homotopy theory (e.g. Munkres).

Remark. The properties of this theorem define on $[P^o(S)]$ the structure of a <u>groupoid</u>. (See MacLane and Birkhoff).

<u>Definition 1.14</u>: A path $f \in P^o(S)$ such that $f(0)=f(1)=P_o$ is called a <u>loop</u> at p_o. Denote by $\Omega(S, p_o)$ the set of loops at p_o. The set of path homotopy classes of loops at p_o with the operation $*$ is called <u>the fundamental group</u> of S at p_o. It is denoted by $\pi_1(S,p_o)$.

Remark. The operation $*$ indeed defines $\pi_1(S,p_o)$ as a group. For the properties of theorem 1.1 now become that for a group, since for every $f, g \in \Omega(S,p_o)$, $f(0) = f(1)=g(0)= g(1)$. We recall the definition of a group. A set G is called a <u>group</u> if it has an operation (called group multiplication) $G \times G \longrightarrow G$ such that

1) (Associativity) $f(gh) = (fg)h$ for all $f, g, h \in G$

2) (Identity) There is an element $e \in G$ such that $ge=eg=g$ for all $g \in G$.

3) (Inverse) For every $g \in G$, there is an element $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

Notice that the identity element in $\pi_1(S,p_o)$ is identified with the path homotopy class of the constant loop.

π_1 is also called the first homotopy group, implying that there are "higher order" homotopy groups.

<u>Lemma 1.5</u>: Let $\alpha\colon I \longrightarrow S$ be a path from p_o to p_1. The map $\alpha_*\colon \pi_1(S,p_o) \longrightarrow \pi_1(S,p_1)$ defined by

$$\alpha_*([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

is a group isomorphism.

Proof: First we show that α_* is well-defined. Indeed, since $f(1) = \alpha(1) = p_1$ we can compose to get

$$f * \alpha(t) = \begin{cases} f(2t) & \text{for } t \in [0,\frac{1}{2}] \\ \alpha(2t-1) & \text{for } t \in [\frac{1}{2},1] \end{cases}$$

Moreover, $f * \alpha(0) = f(0) = p_o$ and $f * \alpha(1) = \alpha(1)\ p_1$, so by the pasting lemma $f * \alpha$ is a path from p_o to p_1. But since $\bar{\alpha}(1) = \alpha(0) = p_o$ and $f * \alpha(0) = f(0) = p_o$ we can compose again to get a continuous map $\bar{\alpha} * f * \alpha \in \Omega(S, p_1)$ since $\bar{\alpha} * f * \alpha(0) = \bar{\alpha}(0) = \alpha(1) = p_1$ and $\bar{\alpha} * f * \alpha(1) = \alpha(1) = p_1$. Now α_* is a group homomorphism since $\alpha_* ([f]) * \alpha([g]) = [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha]$ $= [\bar{\alpha}] * ([f] * [g]) * [\alpha] = \alpha_*([f] * [g])$.

Furthermore, α_* is a bijection since if $\bar{\alpha}$ is the reverse path of α then $\bar{\alpha}_* (\alpha_*[f]) = [\alpha] * \alpha_* ([f]) * [\bar{\alpha}]$ $= [\alpha] * [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] = [f]$

Similarly, $\alpha_* \circ \bar{\alpha}_* [h] = [h]$ for $h \varepsilon \Omega (S, p_1)$.

Corollary: If S is path connected, then for any two points p_o, p, ε S $\pi_1 (S, p_o)$ is isomorphic to $\pi_1(S, p_1)$. Thus if S is path connected we can refer to the fundamental group $\pi_1(S)$.

Remark. Clearly, for any top space S we can refer to the fundamental group of any path component of S.

Definition 1.15: A top space S is called simply connected if it is path connected and $\pi_1(S, p_o)$ is the trivial one-element group for some $p_o \varepsilon S$ and hence for every $p \varepsilon S_1$, i. e $\pi_1(S) = \{e\}$.

Example 1.9: $\mathbb{R}^n$ is simply connected. If $f \varepsilon \Omega (\mathbb{R}^n, x_o)$, then the straight line homotopy

$$H(s,t) = t\, x_o + (1-t)\, f(s)$$

is a path homotopy between f and the constant loop C defined by $C(x) = x_o$ for all $x \varepsilon \mathbb{R}^n$. Thus $\pi_1(\mathbb{R}^n)$ has only one element, e.

More generally we have

Theorem 1.12: If S is contractible, S is simply connected.

Proof: S is path connected by the corollary to lemma 1.2. Now let f, g ε Ω' (S, p_o) be arbitrary. There is homotopy $H:S \times I \longrightarrow S$ between id and the constant map c. Define the map $G:I \times I \longrightarrow S$ by

$$G(s,t) = \begin{cases} H\ (f(s),\ 2t) & \text{for } t\ \varepsilon\ [\,0,\ \frac{1}{2}] \\ H\ (g(s),\ 2-2t) & \text{for } t\ \varepsilon\ [\frac{1}{2},\ 1] \end{cases}$$

G is well-defined since $H(f(s),\ 1) = c = H(g(s),1)$, and continuous by lemma 1.3. Moreover, $G(s,0) = H(f(s),0) = id \circ f(s) = f(s)$, $G(s,1) = H(g(s),\ 0) = g(s)$. So G is a homotopy between f and g. But $G(0,t) = G(1,t)$, so for each fixed t, $G(s,t)\ \varepsilon\ \Omega\ (S,p_o)$. However, we do not know that $G(0,t) = p_o$, so G is not a path homotopy. Define h(t) by $h(t) = G(0,1-t)$, i. e. h(t) is the reverse loop of $G(0,t)$. Now consider $h*f*\bar{h}$. We first show that $h*f*\bar{h} \sim_p g$. Define the map $F:\ I \times I \longrightarrow S$ by

$$H(t,r) = \begin{cases} h(2t) & 0 \leq t \leq \frac{1-r}{2} \\ G(\frac{4t-2r-2}{3r+1},\ r) & \frac{1-r}{2} \leq t \leq \frac{r-3}{4} \\ \bar{h}(4t-3) & \frac{r-3}{4} \leq t \leq 1 \end{cases}$$

This is well defined and continuous since $h(1-r) = G(0,r) = G(1,r) = \bar{h}(r)$. Moreover, you can check that $H(t,0) = h*f*\bar{h}(t)$, $H(t,1) = g(t)$, and $H(0,r) = G(0,1) = g(0) = p_o = H(0,1)$. Now without loss of generality we can take g to be the constant loop $g(t) = p_o$ at p_o with $[g] = e$. But by lemma 1.4 $h_*:\ \pi_1(S,\ p_o) \longrightarrow \pi_1(S,p_o)$ is a group isomorphism. So it sends e into e, i. e. $[f] \sim_p h_*(e) = e$. Q.E.D.

Now let G be a group and a topological space, then G is called a topological group if the group multiplication $\phi : G \times G \longrightarrow G$ is continuous in the topology on G, i. e. $\phi(g,h) = gh^{-1}$ is continuous. By taking $g=e$ this implies that $h \longrightarrow h^{-1}$ is continuous so ϕ is a homeomorphism.

Example: 1.10: Let S be a bounded metric space, i. e. $\sup_{x,y\in S}(d(x,y) < \infty$. Examples would be compact metric spaces or bounded subspaces of $\mathbb{R}^n$. Denote by H the set of all homeomorphisms $f: S \longrightarrow S$. Put a metric on H by defining a distance function $\rho(f,g) = \sup_{x\in S} d(f(x), g(x))$. Thus H is a metric space. Define group multiplication on H as composition. It is easy to check that this turns H into a topological group.

Exercise: Show that the reals (usual topology) under addition and the positive reals under multiplication are topological groups.

2. Topological Manifolds.

I will first give a definition of a manifold which is common in the literature and then give a more easily understood definition. Actually the two definitions are completely equivalent, although this will not be proved here.

Definition 2.1: A manifold M of dimension n is a top space which satisfies the following:

i) M is Hausdorff

ii) M has a countable basis of open sets.

iii) Every point $p \in M$ has a neighborhood U which is homeomorphic to a spherical neighborhood of R^n, i.e. M is locally Euclidean.

Actually there are more general definitions of a manifold. OF particular interest is the generalization to the infinite dimensional situation (See Lang). We can also drop the Hausdorff requirement i) in the definition and talk about non Hausdorff manifolds. These will appear briefly later on. Unless explicitly stated as not necessarily Hausdorff, a manifold will be assumed to be Hausdorff.

It follows from iii) and the facts that R^n is locally compact and local compactness is invariant under homeomorphisms, that M is locally compact. But it is known (Hocking and Young) that every locally compact Hausdorff space with a countable basis is metrizable. Thus we can replace definition 2.1 by:

Thus we will use iii) of definition 2.1 interchangebly with 1) and 2) above. In fact 1) above is the most convenient.

Another easy consequence of the definition is

Proposition 2.2: An open subset U of a manifold is a manifold.

Proof: For $p \varepsilon U \subset M$ there is a V containing p homeomorphic to an open subset of $\mathbb{R}^n$ by ϕ. Then $\phi(U \cap V)$ is open in $\mathbb{R}^n$ and contains a spherical neighborhood whose inverse image under ϕ is open contained in U and contains p.

Such open subsets are called open submanifolds.

Theorem 2.1:

If M' is homeomorphic to a manifold M, then M' is a manifold.

Proof: Let $U \subset M$, $U' \subset M'$ and consider the commutative diagram

$$\begin{tikzcd}
U' \arrow[rr, "f"] \arrow[dr, "\phi\circ f"'] & & U \arrow[dl, "\phi"] \\
& R^n &
\end{tikzcd}$$

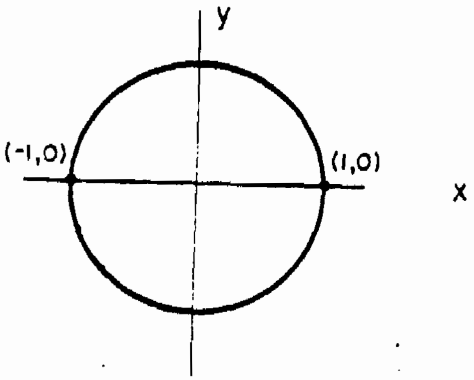
$\phi\circ f$ is homeomorphism

This shows that M' is locally Euclidean. We need to show that M' is Hausdorff and has a countable basis. Let $f: M \longrightarrow M'$ be a homeomorphism onto and let $x, y \varepsilon M$ with $x \neq y$, then there exists neigborhoods U_x, U_y, of x,y, respectively with $U_x \cap U_y = \{\phi\}$.

Now $f(U_x)$ and $f(U_y)$ are open and contain $f(x)$ and $f(y)$, respectively. Moreover $f(U_x) \cap f(U_y) = \{\phi\}$, since if it were not, it's members would necessarily be the image of members of $U_x \cap U_y$. But this is empty. We leave as an exercise to show that M' also has a countable basis.

Let's now look at some nontrivial examples of manifolds:

Example 2.1 circle, S^1: The map $f: (0,2\pi) \longrightarrow S^1$ defined by $f(\theta) = (\cos\theta, \sin\theta)$ is a homeomorphism of the open interval $(0,2\pi)$ onto $S^1-(1,0)$. Also the map $g: (-\pi,\pi) \longrightarrow S^1$ is a homeomorphism of the open interval $(-\pi,\pi)$ onto $S^1-(-1,0)$, where $g(\theta) = (\cos\theta, \sin\theta)$. Thus all points of S^1 have neighborhoods which are homeomorphic to $\mathbb{R}^1$. S^1 is then a manifold, since it inherits a natural metric from $\mathbb{R}^2$. The figure below illustrates the missing points.

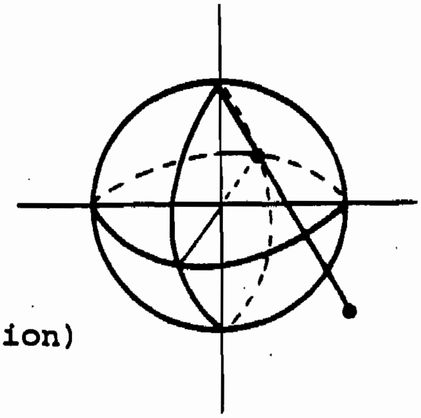


Notice that intervals on the circle $-\varepsilon+\theta_0<\theta<\theta_0+\varepsilon$ are open in the metric topology induced from R^2 since the ball $S(p_0,r)$ where $p_0 = (\cos\theta_0, \sin\theta_0)$, $r = \sin\varepsilon$ is an open set of R^2. Such intervals form a basis for the metric topology on S^1.

There is another set of homeomorphisms which generalizes easily to spheres, given by the stereographic projections.

Example 2.2: Spheres, S^n: We know that

$$S^n = \{\underset{\sim}{x} \varepsilon R^{n+1}: x_1^2 + \ldots + x_{n+1}^2 = 1\}$$



We define the mappings (stereographic projection)

1) $y: S^n \longrightarrow R^n$ by $y^i(\underset{\sim}{x}) = \dfrac{x_i}{1-x_{n+1}}$

2) $y': S^n \longrightarrow R^n$ by $y^i(\underset{\sim}{x}) = \dfrac{x_i}{1+x_{n+1}}$

Notice that the domain of 1) is $S^n - (0,\ldots,1)$ whereas that of 2) is $S^n - (0,\ldots,-1)$. It is not difficult to show that these are indeed homeomorphisms.

Exercise: Show it!

Thus S^n is locally homeomorphic to R^n. In a way exactly analogous to the circle, S^n with its relative topology from R^{n+1} becomes a separable metric space. Thus S^n is a manifold.

Example 2.3: Torus: $T^2 = S^1 \times S^1$ or more generally $T^n = S^1 \times \ldots \times S^1$. Clearly from the definition of manifold if M_1 and M_2 are manifolds then $M_1 \times M_2$ is a manifold.

Example 2.4: The Möbius strip: Consider the half open rectangle $S = \{(x,y) \varepsilon \mathbb{R}^2: 0 \leq x \leq 1, -1<y<1\}$. Introduce an equivalence relations on $\mathbb{R}$ by saying that (x,y) and (x',y') are equivalent

if $(x,y) = (x',y')$ or $(x,y) = (x'+1,-y')$.
What we are really doing is identifying the points $(0,y)$ with the points $(1,-y)$. The resulting space $S/\sim\, = M^2$ is the Möbius strip. It can be described analytically by the map $f{:}S \longrightarrow \mathbb{R}^3$ defined by

$$f(x,y) = (2\cos 2\pi x + y\cos\pi x\cos 2\pi x,\ 2\sin 2\pi x + y\cos\pi x\sin 2\pi x,\ y\sin\pi x)$$

Notice that $f(0,y) = (2+y,0,0) = f(1,-y)$. The image of f in $\mathbb{R}^3$ is the Möbius strip. The map f on the interior S^0 of S given by $S^0 = \{(x,y)\in R^2 : 0<x<1,\ -1<y<1\}$ provides us with a homeomorphism onto $M^2-(2+y,0,0)$. A similar construction with the interval $0<x<1$ replaced by $-\frac{1}{2}<x<\frac{1}{2}$ provides for the homeomorphism onto a neighborhood of $(2+y,0,0)$.

<u>Exercise</u>: Go through this construction.
Almost everybody has made a Möbius strip with a strip of paper by giving it a half twist and glueing the ends together. This is not quite M^2 constructed above, but it would be if we would have started with $\bar{S}$ instead of S, i.e. $\bar{S} = \{(x,y)\in R^2 .\ 0\leqslant x\leqslant 1,$ $-1\leqslant y\leqslant 1\}$. The result then is the Möbius strip with boundary. This is not quite a manifold as we have defined, but rather a manifold with boundary. We will give the formal definition shortly. Anyway the M^2 constructed above is a manifold. The metric topology being the subspace topology of R^3.
Another example somewhat more difficult to visualize is the projective plane. First we discuss some ideas on quotient spaces. Consider a top. space S, an equivalence relation $\sim$ on S, and the

quotient space $S/\sim$ with the quotient topology. Now in general the projection $\rho:S \longrightarrow S/\sim$ is not open, but in many cases it is and this leads to

Lemma 2.1.: Let $\rho:S \longrightarrow S/\sim$ be the projection map of a top space S to its quotient $S/\sim$ and let $S/\sim$ have the quotient topology. Furthermore, suppose that ρ is open, then if S has a countable basis $\{U_\alpha\}$ so does $S/\sim$.

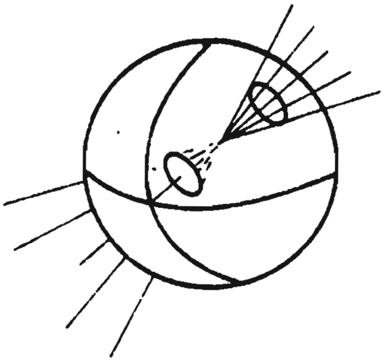
Proof: Let $W \subset S/\sim$, then $\rho^{-1}(W) = \bigcup_i U_{\alpha_i}$ for some subfamily $\{U_{\alpha_i}\}$ of $\{U_\alpha\}$ and $W = \bigcup_i \rho(U_{\alpha_i})$ where each $\rho(U_{\alpha_i})$ is open . Thus $\{\rho(U_\alpha)\}$ is a countable basis for $S/\sim$. (Check this).

Lemma 2.2: Let $\rho:S \longrightarrow S/\sim$ be the natural projection and suppose it is open. Define $E \subset S\times S$ as $E = \{(x,y)\varepsilon S\times S : x\sim y\}$. If E is a closed subspace of $S\times S$, then $S/\sim$ is Hausdorff.

Proof: Let E be closed and $\rho(x)$, $\rho(y)$ be distinct points of $S/\sim$. Then $(x,y)\varepsilon S\times S-E$, which is open. There is thus an open set $U\times V \subset S\times S$ with $(U\times V)\cap E = \{\emptyset\}$. But this says that $\rho(U)\cap\rho(V) = \{\emptyset\}$, and $S/\sim$ is Hausdorff since $\rho(U)$ and $\rho(V)$ are open.

Remark: We need the condition that E above be closed to conclude that $S/\sim$ is Hausdorff. There are known counter examples. In fact the converse of lemma 2.2 is true. (Prove it!).

Example 2.5: The projective plane $P^n(R) = (\mathbb{R}^{n+1}-\{0\})/\sim$ where $x\sim y$ if and only if $y = tx$ for some $t \neq 0$. The equivalence classes $[x]$ are just the lines the through the origin $\{0\}$, and we have also $P^n(\mathbb{R}) = S^n /\sim$ where $x,y\varepsilon S^n$ are equivalent if $y = \pm x$. (Show this!).



If we show that $\rho:\mathbb{R}^{n+1}-\{0\} \longrightarrow P^n(\mathbb{R})$ is open, then $P^n(\mathbb{R})$ will have a countable basis by the first lemma since $\mathbb{R}^{n-1}-\{0\}$ does. Call $X = \mathbb{R}^{n+1}-\{0\}$, and define $\phi_t:X \longrightarrow X$ by $\phi_t(x) = tx$ for $t = 0$. ϕ_t is a homeomorphism with $\phi_t^{-1} = \phi_{\frac{1}{t}}$. If $U \subset X$, then $[U] = \bigcup_{t\in R-\{0\}} \phi_t(U)$ is open since each $\phi_t(V)$ is open. But this says that for each open $U, \rho(U)$ is open. Thus $P^n(\mathbb{R})$ has a countable basis by the first Lemma.

Next let's show that for $P^n(\mathbb{R})$ the set E defined in the second lemma is closed. Consider the function $f:X\times X \longrightarrow R$ defined as

$$f(x_1,\ldots,x_{n+1};y_1,\ldots,y_{n+1}) = \sum_{i\neq j} (x_iy_j-x_jy_i)^2$$

Clearly, f is continuous. Moreover, if $y_i = tx_i$, then $f = 0$. Conversely, assume $f = 0$ then $(x_iy_j-x_jy_i) = 0$, assuming $x_i,y_j \neq 0 \Rightarrow \frac{x_i}{y_i} = \frac{x_j}{y_j} \Rightarrow y_i = tx_i$, $t \neq 0$. That is, $f = 0$ if and only if $y\sim x$. So $E = \{(x,y):x\sim y\} = f^{-1}(0)$. But $X\times X - E = f^{-1}(R-0)$ is open, since f is continuous; thus E is closed. Hence, by the second lemma $P^n(\mathbb{R})$ is Hausdorff.

We need to show that $P^n(R)$ is locally Euclidean. To do this we define the following open sets of $X: \tilde{U}_i = \{x \varepsilon X: x^i \neq 0\}$, that is we remove from $X = \mathbb{R}^{n+1} - \{0\}$, the n-plane $x^i = 0$. Let $U_i = \rho(\tilde{U}_i)$. These are open since we saw that $\rho: X \longrightarrow P^n(\mathbb{R})$ is open. We define the functions $\phi_i : U_i \longrightarrow \mathbb{R}^n$ by

$$\phi_i([x]) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i+1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right)$$

where x is such that $\rho(x) = [x] \varepsilon U_i$. Clearly, ϕ_i is continuous. Moreover, an easy argument as above shows that $\phi_i(x) = \phi_i(y)$ if and only if $y \sim x$. Thus ϕ_i is a bijection, that is 1-1 onto. Now $\phi_i^{-1} : \mathbb{R}^n \longrightarrow U_i$ is given by $\phi_i^{-1}(z_1, \ldots, z_n) = \rho(z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_n)$. To see this consider

$$\begin{tikzcd}
\mathbb{R}^n \arrow[rr, "\psi_i"] & & X \arrow[dl, "\rho"] \\
& P^n(\mathbb{R}) \arrow[ul, "\phi_i"] &
\end{tikzcd}$$

where $\psi_i(z_1, \ldots, z_n) = (z_1, \ldots, z_{i-1}, 1, z_i, \ldots, z_n)$, and we identify $(z_1, \ldots, z_{i-1}, z_i, \ldots, z_n) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_{n+1}}{x_i} \right)$ Thus ϕ^{-1} is continuous, and thus a homeomorphism. Then $P^n(\mathbb{R})$ is a manifold since $\{U_\alpha\}$ cover $P^n(\mathbb{R})$.

We now return to some simpler examples, before giving a generalization of $P^n(\mathbb{R})$.

<u>Example 2.6:</u> n×m matrices, M^{nm}: obviously M^{nm} is homeomorphic to $\mathbb{R}^{nm}$. A global homeomorphism is $f: M^{nm} \longrightarrow \mathbb{R}^{nm}$ given by

$f(A) = (a_{11},\dots,a_{1m}, a_{21},\dots,a_{2m},\dots,a_{nm})$ where

$$A = \begin{bmatrix} a_{11},\dots,a_{1m} \\ \vdots \quad\quad \vdots \\ a_{n1},\dots,a_{nm} \end{bmatrix}.$$ Thus M^{nm} can be considered

as a metric space with the metric induced by that of $\mathbb{R}^{nm}$. Thus M^n is a manifold. The square matrices M^{nn} will be written as M^n (homeomorphic to $\mathbb{R}^{n^2}$).

Example 2.7: The general linear group. $GL(n,\mathbb{R})$:

$$GL(n,\mathbb{R}) = \left\{ A \varepsilon M^n : \det A \neq 0 \right\}.$$ Note that $\det: M^n \longrightarrow \mathbb{R}$

is continuous, and $\mathbb{R} - \{0\}$ is open, so $GL(n,\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ is open in M^n. It is thus a manifold. It is also a group with group composition given by matrix multiplication. Clearly, this composition is continuous in the metric topology on $GL(n,\mathbb{R})$. Thus $GL(n,R)$ is not only a top group, but also a group manifold, i.e. a top. group whose underlying top. space is a manifold and whose composition is continuous in the metric toplogy.

Example 2.8: Grassman manifolds $G(k,n)$. Consider the manifold M^{kn} of all k×n real matrices and recall the definition of the rank of $A \varepsilon M^{kn}$. This is the dimension of the subspace spanned by the rows of A (or equivalently the columns). Denote by $F(k,n)$ the subset of M^{kn} of all matrices of rank k. $F(k,n)$ is an open submanifold of M^{kn}. Any linear independent set of k-elements of $\mathbb{R}^n$ is called a k-frame in $\mathbb{R}^n$. Thus a k-frame is just a k×n matrix $(X_1,\dots,X_k)$ where $X_i = \begin{pmatrix} x_i^1 \\ \vdots \\ x_i^n \end{pmatrix}$. There is a natural action of $GL(k,\mathbb{R})$ on $F(k,n)$ given by $X_i' = \sum_{j=1}^{k} A_i^j X_j$. Since A is nonsingular X_i' defines another k-frame. The set of all k-planes through the origin of $\mathbb{R}^n$ is denoted by $G(k,n)$ and is called a Grassman manifold. A k-frame thus determines a k-plane. Moreover, two k-frames X_i' and X_i determine the same k-plane if and only if there is an $A \in GL(k,R)$ such that $X_i' = \sum_j A_i^j X_j$. It is easy to check that this defines an equivalence relation ~ on $F(k,n)$ and we have $G(k,n) = F(k,n)/\sim$. Let $\rho: F(k,n) \longrightarrow G(k,n)$ denote the projection. We leave as an exercise to show that ρ is open. We show here that $G(k,n)$ is Hausdorff (assuming ρ is open). Consider the nx2k matrix $M \in F(k,n) \times F(k,n)$ $(k<n)$

$$M = \begin{pmatrix} x_1^1 & \dots & x_1^k & x_1'^1 & \dots & x_1'^k \\ & & \vdots & \vdots & & \\ x_n^1 & \dots & x_n^k & x_k'^1 & \dots & x_k'^k \end{pmatrix}$$

and all $(k+1)\times(k+1)$ minor determinants consisting of k unprimed columns and one primed column. A typical example is

$$\begin{vmatrix} x_1^1 & \dots & x_1^k & x_1'^1 \\ \vdots & & \vdots & \vdots \\ x_{k+1}^1 & \dots & x_{k+1}^k & x_{k+1}'^1 \end{vmatrix}$$

A well known theorem from linear algebra says that such a determinant vanishes if and only one of the columns (or rows) is a linear combination of the others. But since the $X^1_\alpha, \ldots, X^k_\alpha$ are linearly independent the only possibility is

$$X'^1_\alpha = \sum_{j=1}^{k} A^1_j X^j_\alpha \qquad \alpha = 1, \ldots, k+1$$

Running through all such possibilities we find

$$X'^i_\alpha = \sum_{j=1}^{k} A^i_j X^j_\alpha \qquad i = 1, \ldots, k \quad \alpha = 1, \ldots, n$$

Moreover, since $(X'^1_\alpha, \ldots, X'^k_\alpha)$ are themselves linearly independent (A^i_j) must be nonsingular, i.e. $(A^i_j) \varepsilon GL(k,R)$. It follows that $X, X' \varepsilon F(k,n)$ are equivalent if and only if all such minor determinants vanish. Thus the set $E \subset F(k,n) \times F(k,n)$ defined by $E = \{(X, X') \varepsilon F(k,n) \times F(k,n) : X \sim X'\}$ is the zero set of all such $(k+1) \times (k+1)$ minor determinants. Since determinants are continuous, it follows that E is closed, and by lemma 2.2 that $G(k,n)$ is Hausdorff.

Exercise: Show that the quotient projection $\rho : F(k,n) \longrightarrow G(k,n)$ is open and thus that $G(k,n)$ has a countable basis.

Exercise: Show that $G(k,n)$ is a manifold of dimension $(n-k)k$. Hint. Let $X \varepsilon F(k,n)$ and consider the $k \times k$ submatrix $\tilde{X}$. Let $\tilde{U}$ be the open set of $F(k,n)$ consisting of $k \times n$ matrices X such that $\tilde{X}$ is nonsingular. Put $U = \rho(\tilde{U})$. Show that every $Y \varepsilon \tilde{U}$ is equivalent to a $k \times n$ matrix X such that the $k \times k$ submatrix is the identity. Define a map $\phi : U \longrightarrow M^{k(n-k)} \simeq R^{k(n-k)}$ by deleting the first k columns of the representative X for Y . Show that ϕ is a homeomorphism.

Remark: $G(1,n) = P^n(R)$

Exercise: Show that $G(k,n) = G(n-k,n)$.

Hint. Consider the map which sends a k-plane into its orthogonal complement.

Example 2.9: One-sheeted hyperboloid H_1^2 :

$H_1^2 = \{x \in R^3 : x_1^2 + x_2^2 - x_3^2 = 1\}$. We construct a global homeomorphism $f: H^2 \longrightarrow S^1 \times R^1$ defined by $f(x_1,x_2,x_3) = \left(\frac{x_1}{\sqrt{1+x_3^2}}, \frac{x_2}{\sqrt{1+x_3^2}}, x_3\right)$. Here we consider $S^1 \times \mathbb{R}^1 = \{(x_1,x_2,x_3) \in R^3 : x_1^2 + x_2^2 = 1\}$.

It is easy to see that f is 1-1 and continuous. The inverse function f^{-1} is also continuous. (show it). Thus H_1^2 is homeomorphic to $S^1 \times \mathbb{R}^1$ and is thus a manifold since $S^1 \times \mathbb{R}^1$ is.

Exercise: Construct a global homeomorphism between $S^1 \times \mathbb{R}^1$ and $\mathbb{R}^2 - \{0\}$.

Example 2.10: The two-sheeted hyperboloids H_n^n

Define $H_n^n = \{x \in \mathbb{R}^{n+1} : x_{n+1}^2 = 1 + x_1^2 + \ldots + x_n^2 \}$

H_n^n is not connected since there are two ranges for x_n, namely $x_n \geq 1$ or $x_n \leq -1$. The component $H_n^{n,+} =$ $\{x \in \mathbb{R}^{n+1} : x_{n+1}^2 = 1 + x_1^2 + \ldots + x_n^2 , x_n \geq 1\}$ is connected since it is globally homeomorphic to $\mathbb{R}^n$. This follows since projection map $f: H_n^{n,+} \longrightarrow \mathbb{R}^n$ defined by

$$f\left[x^1, \ldots, x^n , + \sqrt{1+x_1^2 + \ldots + x_n^2}\right]$$

is a homeomorphism. (Check this!). Moreover, the map $p:\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ restricted to $H_n^{n,+} \longrightarrow H_n^{n,-}$ defined by $p(x_1,\ldots,x_n) = (x_1,\ldots,-x_n)$ is a homeomorphism so $H_n^{n,-}$ is connected and homeomorphic to $\mathbb{R}^n$. Thus H_n^n has two connected components each homeomorphic to $\mathbb{R}^n$.

We can easily generalize to hyperboloids of the form $H_m^n = \{x\varepsilon R^{n+1}: x_{n+1}^2+\ldots+x_{m+1}^2 - x_m^2-\ldots-x_1^2 = 1\}$.

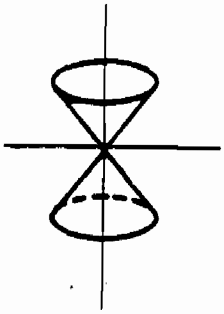
Exercise: Show following example 2.9 that the general one-sheeted hyperboloids H_1^n are homeomorphic to $S^{n-1}\times\mathbb{R}^1$. Why does not an argument similar to that of example 2.10 work?

Considering spaces like S^n and H_m^n we make a definition. Let f be an algebraic function $f: R^n \longrightarrow \mathbb{R}$ the set $f^{-1}(0)$ is called an algebraic variety, $V = f^{-1}(0)$. That is, V is just the set zeros of an algebraic function. (Actually all we need is that f be a polynomial in Cartesian-Coordinates) Clearly S^n and H^n are algebraric varieties. But all algebraic varieties are not manifolds. A familiar example is

Example 2.11: The cone $C^{n,1}$: $n\leq 2$

$$C^{n,1} = \{\underset{\sim}{x}\varepsilon R^{n+1}: \; x_1^2 +\ldots+ x_n^2 - x_{n+1}^2 = 0\}$$

Clearly, $C^{n,1}$ is an algebraic variety. It is not a manifold, however. There is no nbd of $0 \varepsilon C^{n,1}$ homeomorphic to $\mathbb{R}^n$. To see this notice that open sets of $C^{n,1}$ containing zero must be of the form $V_0 = C^{n,1}\cap U_0$ where V_0 is open in $\mathbb{R}^{n+1}$. So we can reduce our considerations to spherical nbds of $0\varepsilon\mathbb{R}^n$. Now I



claim that $\{0\}$ is a cut point of $C^{n,1}$, that is that $C^{n,1}-\{0\}$ is disconnected. Define $C_\pm = \left\{ x \in C^{n,1} : x_{n+1} \begin{matrix} >0 \\ <0 \end{matrix} \right\}$. Clearly, $C^{n,1}-\{0\} = C_+ \cup C_-$. Moreover $C_+ \cap C_- = \{0\}$. Thus $C^{n,1}-\{0\}$ is disconnected. Let V_0 be nbd of $0 \in C^{n,1}$ then $(V_0-\{0\})$ is disconnected. Now suppose that $f: V_0 \longrightarrow \mathbb{R}^n$ is a homeomorphism. Then by a theorem we proved $f(V_0-\{0\})$ must be disconnected. But it is not difficult to show that $\mathbb{R}^n - \{\text{point}\}$ is connected for $n \geq 2$. (See Hocking and Young). Contradiction.

<u>Exercise</u>: Construct a global homeomorphism between $\mathbb{R}^n - \{0\}$ and $S^{n-1} \times \mathbb{R}$.

We can use this result to show that S^n is connected, knowing that $\mathbb{R}^{n+1}-\{0\}$ is for $n > 0$.

We had mentioned earlier the need to generalize our definition of manifold to include boundaries. We now do this. Consider the half-space

$$\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n \geq 0\}$$

Now $\mathbb{H}^n$ is not homeomorphic to $\mathbb{R}^n$ by invariance of the domain. We define a <u>manifold with boundary</u> of dimension n to be an Hausdorff space with a countable basis satisfying.

i) Every point $x \in M$ has a neighborhood V_x that is homeomorphic to either $\mathbb{R}^n$ or $\mathbb{H}^n$.

It can be shown using invariance of the domain that for each $x \varepsilon M$ U_x is either homeomorphic to $\mathbb{R}^n$ or H^n but <u>not</u> both. The set of points locally homeomorphic to $\mathbb{R}^n$ is called the interior of M, while the sét homeomorphic to $\mathbb{H}^n$ is called boundary of M, denoted ∂M. Intuitively, ∂M has dimension n-1. But we do not prove this, here. Now we can write $\mathbb{H}^n = \{x \varepsilon R^n : x_n > 0\} \cup \{x \varepsilon \mathbb{R}^n : x_n = 0\}$. The first is globally homeomorphic to $\mathbb{R}^n$ while the second is globally homeomorphic to $\mathbb{R}^{n-1}$. Thus up to homeomorphism $H^n = R^n \cup R^{n-1}$ and $\partial \mathbb{H}^n = \mathbb{R}^{n-1}$.

<u>Example 2.12</u>: The closed ball $B(0,r) = \{x \varepsilon R^n : y_1^2 + \ldots + y_1^2 \leqslant r^2\}$ is a manifold with boundary. Clearly the interior $B^\circ(0,r)$ is a spherical neighborhood and is thus globally homeomorphic to $\mathbb{R}^n$. The boundary $\partial B(0,r) = S^{n-1}$ which is locally homeomorphic to $\mathbb{R}^{n-1} = \partial \mathbb{H}^n$. We now show that every point $p \varepsilon \partial B(0,r)$ has a nbd. homeomorphic to H^n. First, we can construct a homeomorphism between $S(0,r)$ and the cubical neighborhoods $C^n(0,1) = \{x \varepsilon R^n : |x^i| < 1$ for all $i=1,\ldots,n\}$. This is done by composing the homeomorphism of proposition 2.1 with the homeomorphism $x : C^n(0,1) \longrightarrow R^n$ defined by

$$x^i = \frac{z^i}{1-(z^i)^2}$$

Then $y : S^n(0,r) \longrightarrow C^n(0,1)$ is given by

$$y^i = \frac{r\, z^i}{1-(z^i)^2} \cdot \frac{1}{\left[1 + \sum_j \frac{(z^j)^2}{(1-(z^j)^2}\right]^{1/2}}$$

It is straightforward to check that this map extends to a well-defined continuous map on the closures, i.e. $\bar{y} : \overline{S^n(o,r)} = \mathcal{B}(o,r) \longrightarrow \overline{C^n(0,1)}$. Furthermore $\bar{y}$ is a homeomorphism. Clearly, every point on $\partial C^n(0,1)$ with the possible exceptions of the corners has a neighborhood homeomorphic to H^n. But the points of $\mathcal{B}(o,r)$ corresponding to the corners of $\overline{C^n(0,1)}$ can be handled by rotating $\mathcal{B}(0,r)$ with respect to $\overline{C^n(0,1)}$.