NOTES

1. Covering Manifolds

Definition 1.1. *Let $\tilde{M}$ and M be connected smooth manifolds. A smooth surjection $\pi : \tilde{M} \longrightarrow M$ is called a* **covering map** *if each point $p \in M$ has a connected open neighborhood U such that $\pi^{-1}(U) = \sqcup_\alpha U_\alpha$ and $\pi|_{U_\alpha} : U_\alpha \longrightarrow U$ is a diffeomorphism for each α. The manifold $\tilde{M}$ is called a* **covering manifold** *of M.*

We also say that the open set U is *evenly covered*. Compare this definition with the 'topological definition'. Clearly a smooth covering map is in particular a topological covering map. In these notes by a covering map we always mean a smooth covering map. The U_α are precisely the connected components of $\pi^{-1}(U)$. Note that since $\tilde{M}$ has a countable basis for its topology we can take the index set in Definition 1.1 to be countable.
Example 1: $\tilde{M} = \mathbb{R}$ and $M = S^1$. The map $\pi(t) = e^{2\pi i t}$ is a smooth covering map.
Example 2: The natural projection $\pi : S^n \longrightarrow \mathbb{RP}^n$ is a smooth covering map.
Exercise: Show that a smooth covering map is a submersion.

Definition 1.2. *A diffeomorphism $h : \tilde{M} \longrightarrow \tilde{M}$ that satisfies $\pi \circ h = \pi$ is called a* **deck transformation**.

Deck transformations are sometimes called *covering transformations*. It is easy to see that the set Γ of all deck transformations of $\tilde{M}$ form a group (Show it).
Exercise: Show that h maps $\pi^{-1}(p)$ to itself, and that the either $hU_\alpha \cap U_\beta = \emptyset$ or $hU_\alpha = U_\beta$.

Lemma 1.3. *The group of deck transformations Γ acts freely on $\tilde{M}$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} & \tilde{M} \arrow[d, "\pi"] \\ \tilde{M} \arrow[ur, "h"] \arrow[r, "\pi"] & M \end{tikzcd}$$

which says that h is a lift of π. Another obvious lift of π is the identity map $\mathrm{id} : \tilde{M} \longrightarrow \tilde{M}$. By a lemma from topology any two lifts that coincide at one point are identical. So $hx = x$ for some $x \in \tilde{M}$ implies $h = \mathrm{id}$.

$\square$

Now let $p \in M$ and fix an element $x_0 \in \pi^{-1}(p)$. Define a map $\psi : \Gamma \longrightarrow \pi^{-1}(p)$ by $\psi(h) = hx_0$. It follows from Lemma 1.3 that this map is injective. This implies that Γ is countable since $\pi^{-1}(p)$ is countable, so we give Γ the discrete topology.