

# CONSTANT SCALAR CURVATURE SASAKI METRICS AND PROJECTIVE BUNDLES

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ABSTRACT. In this paper we consider the Boothby-Wang construction over twist 1 stage 3 Bott orbifolds given in terms of the log pair  $(S_n, \Delta_m)$ . We give explicit constant scalar curvature (CSC) Sasaki metrics either directly from CSC Kähler orbifold metrics or by using the weighted extremal approach of Apostolov and Calderbank. The Sasaki 7-manifold (orbifolds) are finitely covered by compact simply connected manifold (orbifolds) with the rational homology of the 2-fold connected sum of  $S^2 \times S^5$ .

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## INTRODUCTION

Arguably the foremost problem in Sasaki geometry is that of determining which isotopy classes of Sasakian structures admit Sasaki metrics of constant scalar curvature (CSCS). The Sasaki version of the Yau-Tian-Donaldson conjecture says roughly that

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CSCS metrics correspond to the affine variety satisfying some type of K-stability requirement. Collins and Székelyhidi [CS18] proved that a CSC Sasaki metric implies the K-semistability of corresponding affine Kähler variety. Recently, following the important work in the Kähler category by Chen and Cheng [CC21a, CC21b], He and Li [He18, HL21] have related the existence of CSC Sasaki metrics to the reduced properness of the Mabuchi K-energy.

Complementary to these existence results, it is important to explore explicit examples. Of particular interest to us are those Sasaki manifolds (orbifolds) that admit a transverse admissible construction [ACGTF08a]. A special case of such Sasaki orbifolds consist of principle  $S^1$  orbibundles over Bott manifolds (orbifolds) [GK94, BCTF19] to be described in the first Section. In this paper we discuss two fairly explicit methods for constructing constant scalar curvature Sasaki metrics on a class of 7-manifolds having the rational cohomology of the 2-fold connected sum of  $S^2 \times S^5$ . The first method is the standard Boothby-Wang construction on twist one stage 3 Bott orbifolds (KS orbifolds). The second method applies the weighted extremal approach of Apostolov and Calderbank [AC21]. Both methods employ the admissible construction at the Kähler level (see [ACGTF08a]) either directly or indirectly. Specifically, in Section 4 we prove Theorem 4.17 which together with Remark 4.18 give

**Main Theorem.** *Let  $M^7$  be a Boothby-Wang constructed manifold with maximal symmetry over a polarized KS orbifold. Then there always exists a positive constant scalar curvature Sasaki metric in the Sasaki cone.*

In terms of K-stability we have the following corollary of the Main Theorem and Corollary 1.1 of [CS18]

**Corollary.** *Let  $\mathcal{S}$  be a Boothby-Wang constructed Sasaki manifold over the KS orbifold  $(S_n, \Delta_m)$ . Then there exists a Reeb vector field  $\xi$  in the Sasaki cone such that the corresponding polarized affine variety  $(Y, \xi)$  is K-semistable, equivalently the Sasakian structure  $(\xi, \eta, \Phi, g)$  is K-semistable.*

In the Gorenstein case, a much stronger statement is known, namely the Sasaki version of the famous Chen-Donaldson-Sun [CDS15a, CDS15b, CDS15c] result by Collins and Székelyhidi [CS19]. In particular, the recent classification of all smooth Fano threefolds that admit Kähler-Einstein metrics [ACC<sup>+</sup>21] gives a classification of all regular Sasaki-Einstein manifolds over smooth KS threefolds.

It is interesting to compare the Sasaki-Einstein solutions obtained from the Boothby-Wang construction from the orbifold Kähler-Einstein solutions of Section 4.4 with those described in [BTF19]. Both types live on simply connected 7-manifolds with the rational cohomology of the 2-fold connected sum of  $S^2 \times S^5$  with torsion in  $H^4$ . However, they are of a somewhat different nature. Those in [BTF19] have holomorphic twist 2, whereas, those in this paper have holomorphic twist 1. Assuming they have the same torsion group one may ask whether they are homotopy equivalent, have the same diffeomorphism type, and/or belong to inequivalent CR structures, contact structures, etc.

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## 1. THE PROJECTIVE BUNDLES

We view our projective bundle as a special type of stage 3 Bott manifold, namely the Bott tower  $M_3(0, n_1, n_2)$  where  $n_1, n_2 \in \mathbb{Z}$ . We recall the definition of a *stage  $k$  Bott manifold* of the *Bott tower of height  $n$* :

$$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots \rightarrow M_2 \xrightarrow{\pi_2} M_1 = \mathbb{C}\mathbb{P}^1 \xrightarrow{\pi_1} \{pt\}$$

where  $M_k$  is defined inductively as the total space of the projective bundle

$$\mathbb{P}(\mathbb{1} \oplus L_k) \xrightarrow{\pi_k} M_{k-1}$$

with fiber  $\mathbb{C}\mathbb{P}^1$ , and some holomorphic line bundle  $L_k$  on  $M_{k-1}$ . We refer to [GK94, BCTF19] for details. In this paper we treat twist 1 stage 3 Bott manifolds  $M_3(0, n_1, n_2)$  and orbifolds. Explicitly these are ruled manifolds of the form

$$(1) \quad S_{\mathbf{n}} = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)) \longrightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$$

with  $n_1, n_2 \in \mathbb{Z}^*$ . To this we add an orbifold structure given in terms of the log pair  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  where

$$(2) \quad \Delta_{\mathbf{m}} = \left(1 - \frac{1}{m_0}\right)e_0 + \left(1 - \frac{1}{m_\infty}\right)e_\infty$$

and  $e_0, e_\infty$  are defined by the zero and infinity sections of the bundle. A motivation of our study comes from the 1986 paper of Koiso and Sakane [KS86] where they study Kähler-Einstein metrics on projectivizations of the form  $\mathbb{P}(\mathbb{1} \oplus L_1 \otimes_{ext} L_2)$  over  $N \times N$  where  $N$  has a Kähler-Einstein metric. For this reason we refer to our log pairs  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  as *KS orbifolds*. The underlying complex manifolds are smooth toric varieties whose Kähler geometry was studied in [BCTF19]. The positive integers  $m_0, m_\infty$  are called *ramification indices*. For each choice of orbifold Kähler form  $\omega_{\mathbf{n}, \mathbf{m}}$ , defining an integer Kähler class on  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  we have a principal  $S^1$  orbibundle

$$S^1 \longrightarrow M^7 \longrightarrow (S_{\mathbf{n}}, \Delta_{\mathbf{m}})$$

with an induced Sasakian structure  $\mathcal{S}_{\mathbf{n}, \mathbf{m}}$ ; moreover, when the class  $[\omega_{\mathbf{n}, \mathbf{m}}]$  is primitive in  $H_{orb}^2((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$ ,  $M^7$  is simply connected. While there are only 2 equivalence classes of Fano KS manifolds, it is clear from Proposition 1.1 below that log Fano KS orbifolds are abundant.

In [BTF] we described a functor from the category of  $S_{\mathbf{w}}^3$  Sasaki joins to the category of Bott orbifolds. However, as noted there, this functor is not surjective on objects. One purpose of this paper is to study this lack of surjectivity in the particular case of twist 1 stage 3 Bott orbifolds. A key notion of Sasaki joins is that of *cone decomposability* which can be thought of as the Sasaki version of de Rham decomposability in the Kähler case [BHLTF18]. Cone decomposability is an invariant of the underlying Sasaki CR structure. We shall see how our KS orbifolds give rise to both cone decomposable Sasakian structures as well as Sasakian structures that are likely cone indecomposable. See Remark 3.10 below.

Note that the ordinary Bott manifold  $M_3(0, n_1, n_2)$  corresponds to the trivial orbifold  $(S_{\mathbf{n}}, \emptyset)$ . We denote by  $KS^{orb}$  the set of manifolds with these orbifold structures, that is,

$$KS^{orb} = \{(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \mid n_1, n_2 \in \mathbb{Z}^*, m_0, m_\infty \in \mathbb{Z}^+\}.$$

Such orbifolds are the objects of a groupoid whose morphisms are biholomorphisms that intertwine the orbifold structures. Actually we are interested in *polarized* KS orbifolds that are polarized by a (primitive) orbifold Kähler class  $[\omega_{\mathbf{n},\mathbf{m}}] \in H_{orb}^2(S_{\mathbf{n}}, \mathbb{Z})$ . These form the objects of our groupoid  $\mathcal{KS}$  whose morphisms are biholomorphisms that intertwine the orbifold structures and intertwine their orbifold Kähler classes. We remark that since the objects of  $\mathcal{KS}$  are themselves orbifolds, we are working with a 2-category. The action of the Coexter group  $\text{Sym}_2 \times \mathbb{Z}_2^3$  on  $KS_{orb}$  is generated by the transposition  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \mapsto (S_{n_2, n_1}, \Delta_{\mathbf{m}})$  and the fiber inversion sending  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \mapsto (S_{-\mathbf{n}}, \Delta_{m_\infty, m_0})$ . Note the interchange of ramification indices  $m_0, m_\infty$  induced by the fiber inversion map.

Of special interest is the full subgroupoid  $\mathcal{KS}^{mon}$  of monotone KS orbifolds which we now describe. Consider the orbifold canonical divisor or dually the orbifold first Chern class. In particular, when is the orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  log Fano, equivalently when does  $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  lie in the Kähler cone? From Section 1.3 of [BTF19] we have

$$\begin{aligned} c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) &= \left(2 - \frac{n_1}{m_\infty}\right)y_1 + \left(2 - \frac{n_2}{m_\infty}\right)y_2 + \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right)y_3 \\ (3) \qquad \qquad \qquad &= \left(2 + \frac{n_1}{m_0}\right)x_1 + \left(2 + \frac{n_2}{m_0}\right)x_2 + \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right)x_3. \end{aligned}$$

with respect to the invariant bases  $\{x_1, x_2, x_3\}$  and  $\{y_1, y_2, y_3\}$ , respectively. For  $i = 1, 2$  here  $x_i = y = i$  is the class of the Fubini-Study metric on the  $i$ th factor of the product  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  pulled back to  $S_{\mathbf{n}}$ , while  $x_3(y_3)$  is the Poincaré dual of  $e_\infty(e_0)$ , respectively. From this we arrive at a special case of Lemma 1.2 of [BTF19]

**Proposition 1.1.** *The orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  in  $KS^{orb}$  is log Fano if and only if the inequalities*

$$\frac{n_1}{m_\infty} < 2, \quad \frac{n_2}{m_\infty} < 2, \quad -\frac{n_1}{m_0} < 2, \quad -\frac{n_2}{m_0} < 2$$

hold.

We obtain a primitive Kähler class  $\frac{c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})}{\mathcal{I}_{\mathbf{n}, \mathbf{m}}}$  in  $H_{orb}^2((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$  with *Fano index* given by

$$(4) \qquad \qquad \mathcal{I}_{\mathbf{n}, \mathbf{m}} = \gcd(2mv_0v_\infty - n_1v_0, 2mv_0v_\infty - n_2v_0, v_0 + v_\infty)$$

where  $\mathbf{m} = (m_0, m_\infty) = m(v_0, v_\infty) = m\mathbf{v}$  and  $m = \gcd(m_0, m_\infty)$ . Here we have used the identity  $x_3 = y_3 - n_1x_1 - n_2x_2$ . The orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  is log Fano if and only if

$$c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \frac{1}{mv_0v_\infty} \left[ \left(2mv_0v_\infty - n_1v_0\right)y_1 + \left(2mv_0v_\infty - n_2v_0\right)y_2 + \left(v_0 + v_\infty\right)y_3 \right]$$

is positive.

The Poincaré dual to  $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  is the orbifold anti-canonical divisor which is an ample  $\mathbb{Q}$ -divisor on  $S_{\mathbf{n}}$  when  $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  is positive. The primitive class  $\frac{c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})}{\mathcal{I}_{\mathbf{n}, \mathbf{m}}}$  can be represented by an orbifold Kähler form  $\omega_{\mathbf{n}, \mathbf{m}}$  on  $S_{\mathbf{n}}$ , or equivalently an orbifold Kähler metric  $g_{\mathbf{n}, \mathbf{m}}$ .

## 2. THE ORBIFOLD BOOTHBY-WANG CONSTRUCTION

The well known Boothby-Wang Theorem [BW58] that associates regular contact structures to the total space of  $S^1$  bundles over symplectic manifolds generalizes easily to the orbifold category. See for example Theorem 7.1.3 of [BG08]. Here as in this theorem we are interested in Sasakian structures over Kählerian structures. Given an integral Kähler class  $[\omega_{\mathbf{n},\mathbf{m}}] \in H_{orb}^{1,1}((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$  we shall always choose a Kähler form  $\omega_{\mathbf{n},\mathbf{m}}$  with maximal symmetry. In this case the total space  $M$  of the corresponding principal  $S^1$  orbibundle has a maximal family  $\mathfrak{t}^+$  of Sasakian structures, called the *Sasaki cone*. Moreover, choosing a connection 1-form  $\eta$  on  $M$  such that  $d\eta = \omega_{\mathbf{n},\mathbf{m}}$  determines a natural Sasakian structure  $\mathcal{S}_{\mathbf{n},\mathbf{m}} = (\xi, \eta, \Phi, g)$  in  $\mathfrak{t}^+$ . We refer to this construction as the *orbifold Boothby-Wang construction*. Generally, unlike  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ , the total space  $M$  has an orbifold structure whose underlying topological space is singular with finite cyclic singularities. The general problem of finding conditions when  $M$  is smooth is quite subtle [Qui71, Kol07, KL21]: however, Kegel and Lange show that  $M$  is a smooth manifold if and only if  $H_{orb}^r(M, \mathbb{Z}) = 0$  for all  $r > 2n + 1$ . More precisely  $M$  is smooth if and only if multiplication by the Euler class  $H_{orb}^r(\mathcal{Z}, \mathbb{Z}) \xrightarrow{\cup_e} H_{orb}^{r+2}(\mathcal{Z}, \mathbb{Z})$  in the Gysin sequence of the orbibundle  $M \rightarrow \mathcal{Z}$  is an isomorphism for all  $r > 2n + 1$ . As we shall see in Section 3.5 precise conditions can be obtained in certain special cases (see also [BG08]).

The question arises as to when there exists a constant scalar curvature Sasaki metric in  $\mathfrak{t}^+$  and how many. In the Gorenstein case this was answered by Futaki, Ono, and Wang [FOW09], but the general CSC case remains open. Some partial results have been obtained by Legendre [Leg11], and by the authors [BTF16, BTF21a] and references therein.

**Remark 2.1.** In categorical language we work with the groupoid  $\mathcal{SKS}$  whose objects  $\{M_{\mathbf{n},\mathbf{m}}^7\}$  are simply connected quasiregular Sasaki 7-orbifolds whose projective algebraic base is a Koiso-Sakane orbifold, and whose morphisms are orbifold biholomorphisms. Here the objects are classes of Sasakian or Koiso-Sakane structures, thus again, they are 2-categories. We are mainly interested in certain subgroupoids, namely, the full subgroupoid  $\mathcal{SKS}^F$  of positive Sasakian structures whose base Koiso-Sakane orbifold is log Fano. Actually, we consider the full subgroupoid  $\mathcal{SKS}^{MF}$  whose Kähler form  $\omega_{\mathbf{n},\mathbf{m}}$  lies in  $\frac{c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})}{\mathcal{I}_{\mathbf{n},\mathbf{m}}}$ , that is the Koiso-Sakane base orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  is monotone log Fano. We also define the subgroupoids  $\mathcal{SKS}_{\mathbf{m}}, \mathcal{SKS}_{\mathbf{m}}^F, \mathcal{SKS}_{\mathbf{m}}^{MF}$  consisting of the corresponding objects with fixed  $\mathbf{m}$ . The objects of the groupoid  $\mathcal{SKS}_{\mathbf{m}}$  are Sasaki classes on the 7-manifolds  $M_{\mathbf{m}}^7$  whose cohomology we describe in the next section. The morphisms are those induced by diffeomorphisms that intertwine the corresponding Sasakian structures.

## 3. THE TOPOLOGY OF THE ORBIFOLDS

In this section we describe both the orbifold topology of  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  and topology the total space of the principal  $S^1$  orbibundles over the log Fano Koiso-Sakane orbifolds  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ . First we recall the cohomology ring of the complex manifold  $S_{\mathbf{n}}$  which is well known and can be found in [BCTF19] and references therein.

$$(5) \quad H^*(S_{\mathbf{n}}, \mathbb{Z}) = \mathbb{Z}[y_1, y_2, y_3] / (y_1^2, y_2^2, y_3(-n_1 y_1 - n_2 y_2 + y_3))$$

We remark that the addition of orbifold structures on the invariant divisors does not effect the cohomology  $H^*(S_{\mathbf{n}}, \mathbb{Z})$ ; however, as we shall see it strongly effects the cohomology of the Sasakian 7-manifolds.

**3.1. The Orbifold Cohomology Groups.** Next we compute the orbifold cohomology groups.

**Lemma 3.1.**

$$H_{orb}^r((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = 0 \\ \mathbb{Z}^3 & \text{if } r = 2 \\ \mathbb{Z}^3 \oplus \mathbb{Z}_{m^0}^2 \oplus \mathbb{Z}_{m^\infty}^2 & \text{if } r = 4 \\ \mathbb{Z} \oplus \mathbb{Z}_{m^0}^3 \oplus \mathbb{Z}_{m^\infty}^3 & \text{if } r = 6 \\ \mathbb{Z}_{m^0}^3 \oplus \mathbb{Z}_{m^\infty}^3 & \text{if } r = 8, 10, \dots \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

*Proof.* Using Lemma 4.3.7 of [BG08] we compute the Leray spectral sequence of the classifying map

$$p : \mathbf{B}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \longrightarrow S_{\mathbf{n}}.$$

The Leray sheaf of  $p$  is the derived functor sheaf  $R^s p_* \mathbb{Z}$ , that is, the sheaf associated to the presheaf  $U \mapsto H^s(p^{-1}(U), \mathbb{Z})$ . For  $s > 0$  the stalks of  $R^s p_* \mathbb{Z}$  at points of  $U$  vanish if  $U$  lies in the regular locus of  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  which is the complement of the union of the zero  $e_0$  and infinity  $e_\infty$  sections of the natural projection  $S_{\mathbf{n}} \longrightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ . At points of  $e_0$  and  $e_\infty$  the fibers of  $p$  are the Eilenberg-MacLane spaces  $K(\mathbb{Z}_{m^0}, 1)$  and  $K(\mathbb{Z}_{m^\infty}, 1)$ , respectively. So at points of  $e_0(e_\infty)$  the stalks are the group cohomology  $H^s(\mathbb{Z}_{m^0}, \mathbb{Z})(H^s(\mathbb{Z}_{m^\infty}, \mathbb{Z}))$ . This is  $\mathbb{Z}$  for  $s = 0$  and  $\mathbb{Z}_{m^0}(\mathbb{Z}_{m^\infty})$  at points of  $e_0(e_\infty)$  when  $s > 0$  is even; it vanishes when  $s$  is odd. The  $E_2$  term of the Leray spectral sequence of the map  $p$  is

$$E_2^{r,s} = H^r(S_{\mathbf{n}}, R^s p_* \mathbb{Z})$$

and by Leray's theorem this converges to the orbifold cohomology  $H_{orb}^{r+s}((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$ . Now  $E_2^{r,0} = H^r(S_{\mathbf{n}}, \mathbb{Z})$  and  $E_2^{r,s} = 0$  for  $r$  or  $s$  odd. For  $r = 0$  the only continuous section of  $R^s p_* \mathbb{Z}$  is the 0 section which implies that  $E_2^{0,s} = 0$  for all  $s$ . Now we have  $E_2^{2r,2s} = 0$  for  $r > 2$  and

$$\begin{aligned} E_2^{2,2s} &= H^2(S_{\mathbf{n}}, R^{2s} p) = H^2(e_0, \mathbb{Z}_{m^0}) \oplus H^2(e_\infty, \mathbb{Z}_{m^\infty}) = \mathbb{Z}_{m^0}^2 \oplus \mathbb{Z}_{m^\infty}^2, \\ E_2^{4,2s} &= H^4(S_{\mathbf{n}}, R^{2s} p) = H^4(e_0, \mathbb{Z}_{m^0}) \oplus H^4(e_\infty, \mathbb{Z}_{m^\infty}) = \mathbb{Z}_{m^0} \oplus \mathbb{Z}_{m^\infty}. \end{aligned}$$

One easily sees that this spectral sequence collapses whose limit is the orbifold cohomology  $H_{orb}^r((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$  which implies the result.  $\square$

In addition Lemma 3.1 implies

**Lemma 3.2.**  $\pi_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \mathbb{1}$ .

*Proof.* From the homotopy sequence of the orbibundle

$$\mathbb{C}\mathbb{P}^1[\mathbf{v}]/\mathbb{Z}_m \longrightarrow (S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \longrightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$$

one easily sees that  $\pi_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  is Abelian. But this implies

$$\pi_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \approx H_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z},$$

and this vanishes by Lemma 3.1 and universal coefficients.  $\square$

**3.2. The Cohomology of the Branched Covers.** We consider the total space  $M_{\mathbf{n},\mathbf{m}}^7$  of an  $S^1$  orbibundle over a KS orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  with an orbifold Kähler form  $\omega_{\mathbf{n},\mathbf{m}}$ . Note that in our case  $S_{\mathbf{n}}$  is a smooth projective variety so  $M_{\mathbf{n},\mathbf{m}}^7$  is a Seifert bundle as defined in the foundational paper [OW75] of Orlik and Wagreich (see also the more recent unpublished description by Kollár [Kol04]). A fundamental result of Orlik and Wagreich says that every  $\mathbb{C}^*$  Seifert bundle is a branched cover over a principal  $\mathbb{C}^*$  bundle. In our case this branching occurs precisely along the orbifold divisor  $\Delta_{\mathbf{m}}$ . The goal is to understand the integral cohomology of  $M_{\mathbf{n},\mathbf{m}}^7$ . However, before treating the orbifold case, we need to consider the regular case.

**3.3. The Regular Case.** For simplicity we treat only the case where the Kähler class is primitive, i.e.

$$(6) \quad [\omega_{\mathbf{n}}] = c_1 y_1 + c_2 y_2 + c_3 y_3 \text{ with } c_i \in \mathbb{Z}^+, \quad \gcd(c_1, c_2, c_3) = 1.$$

From Section 3.3 of [BCTF19] this is a Kähler class for all positive  $c_i$  if  $n_1 n_2 > 0$ , whereas, if  $n_1 n_2 < 0$ , it is a Kähler class for  $c_1 > 0$ ,  $c_2 > -n_2 c_3$ ,  $c_3 > 0$ .

One sees from the long exact homotopy sequence of the fibration

$$(7) \quad S^1 \longrightarrow M_{\mathbf{n}}^7 \longrightarrow S_{\mathbf{n}}$$

that

**Lemma 3.3.** *Let  $M_{\mathbf{n}}^7$  denote the total space of the principal  $S^1$  bundle over a KS manifold  $S_{\mathbf{n}}$  defined by a primitive Kähler class  $[\omega_{\mathbf{n}}] \in H^2(S_{\mathbf{n}}, \mathbb{Z})$ . Then  $M_{\mathbf{n}}^7$  has the rational cohomology of the 2-fold connected sum  $2\#(S^2 \times S^5)$ , and we have*

$$(1) \quad \pi_1(M_{\mathbf{n}}^7) = \mathbb{1},$$

$$(2) \quad \pi_2(M_{\mathbf{n}}^7) = \mathbb{Z}^2.$$

Thus, by the Hurewicz Theorem we also have  $H_2(M_{\mathbf{n}}^7, \mathbb{Z}) = \mathbb{Z}^2$  which implies that  $H^3(M_{\mathbf{n}}^7, \mathbb{Z})_{\text{tor}} = 0$ . One can also see from the Leray-Serre spectral sequence of the fibration (7) that  $M_{\mathbf{n}}^7$  has the rational cohomology of the 2-fold connected sum  $2\#(S^2 \times S^5)$ , and we also have  $H^3(M_{\mathbf{n}}^7, \mathbb{Z}) = 0$ . We now show that all such simply connected spaces have nonzero torsion in  $H^4$ . There will be zero torsion in  $H^4$  if and only if the differential  $d_2 : E_2^{2,1} \longrightarrow E_2^{4,0}$  is invertible over  $\mathbb{Z}$ . This differential is represented by the matrix

$$(8) \quad C = \begin{pmatrix} c_2 & c_3 & 0 \\ c_1 & 0 & c_3 \\ 0 & c_1 + n_1 c_3 & c_2 + n_2 c_3 \end{pmatrix}$$

with respect to the basis  $\{\alpha \otimes y_1, \alpha \otimes y_2, \alpha \otimes y_3\}$  of  $E_2^{2,1}$  and the basis  $\{y_1 y_2, y_1 y_3, y_2 y_3\}$  of  $E_2^{4,0}$ . But then we have

$$(9) \quad -\det C = c_3 [c_2 (c_1 + n_1 c_3) + c_1 (c_2 + n_2 c_3)]$$

which implies that the order of  $H^4(M_{\mathbf{n}}^7, \mathbb{Z})$  is  $c_3 [c_2 (c_1 + n_1 c_3) + c_1 (c_2 + n_2 c_3)]$ . This is always greater than 1, so  $H^4(M_{\mathbf{n}}^7, \mathbb{Z})_{\text{tor}} \neq 0$ . Summarizing we have

**Theorem 3.4.** *Let  $M_{\mathbf{n}}^7$  denote the total space of the principal  $S^1$  bundle over a KS manifold  $S_{\mathbf{n}}$  defined by a primitive Kähler class  $[\omega_{\mathbf{n}}] = c_1x_1 + c_2x_2 + c_3x_3$ . Then*

$$H^r(M_{\mathbf{n}}^7, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = 0, 7 \\ \mathbb{Z}^2 & \text{if } r = 2, 5 \\ G_{reg} & \text{if } r = 4 \\ 0 & \text{otherwise.} \end{cases}$$

where  $G_{reg}$  is an Abelian group of order  $c_3[c_2(c_1 + n_1c_3) + c_1(c_2 + n_2c_3)]$ . Moreover,  $G_{reg}$  is never the identity.

**Example 3.1.** The simplest case is the standard product  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  with Kähler class

$$[\omega] = c_1x_1 + c_2x_2 + c_3x_3, \quad \gcd(c_1, c_2, c_3) = 1$$

and  $n_1 = n_2 = 0$  which gives  $|G_{reg}| = 2c_1c_2c_3$ . Taking  $c_1 = c_2 = c_3 = 1$  gives the homogeneous space

$$M^7 = (SU(2) \times SU(2) \times SU(2))/U(1) \times U(1)$$

with its Sasaki-Einstein metric (see for example [BFGK91]). Here  $G_{reg} = \mathbb{Z}_2$ .

**Example 3.2.** Monotone Fano case. Up to equivalence there are two monotone Fano cases polarized by the first Chern class, namely,  $\mathbf{n} = (1, -1)$  and  $\mathbf{n} = (1, 1)$ . From Equation (3) we have  $c_1 = 1, c_2 = 3, c_3 = 2$  when  $\mathbf{n} = (1, -1)$ , and  $c_1 = 1, c_2 = 1, c_3 = 2$  when  $\mathbf{n} = (1, 1)$ . These give  $|G_{reg}| = 20$  and  $|G_{reg}| = 12$ , respectively. Note that  $M_{1,-1}^7$  is conjecturely<sup>1</sup> cone indecomposable; whereas,  $M_{1,1}^7$  is cone decomposable. The latter is described by Theorem 2.5 in [BTF] with  $\mathbf{l}_1 = (1, 1), \mathbf{w}_1 = (1, 1), \mathbf{l}_2 = (1, 2)$ , and  $\mathbf{w}_2 = (3, 1)$ .

**3.4. The General Case; Branched Covers.** Returning to the orbifold case, we consider the Leray spectral sequence of the quotient map  $M_{\mathbf{n}, \mathbf{m}}^7 \xrightarrow{\pi} (S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  viewed as an  $S^1$ -Seifert bundle over  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ . Since the fibers are  $S^1$  we only have the  $R^1\pi_*\mathbb{Z}_M$  direct image sheaf, the  $E_2$  term of the Leray spectral sequence

$$(10) \quad E_2^{p,q} = H^p(S_{\mathbf{n}}, R^q\pi_*\mathbb{Z})$$

satisfies  $E_2^{p,q} = 0$  when  $q \neq 0, 1$ , and this converges to  $H^{p+q}(M_{\mathbf{n}, \mathbf{m}}^7, \mathbb{Z})$ . So the only differentials are those induced by

$$(11) \quad d_2 : E_2^{0,1} = H^0(S_{\mathbf{n}}, R^1\pi_*\mathbb{Z}_M) \longrightarrow E_2^{2,0} = H^2(S_{\mathbf{n}}, \mathbb{Z}) \approx \mathbb{Z}^3.$$

We note that the sheaf injection

$$(12) \quad R^1\pi_*\mathbb{Z}_M \longrightarrow \mathbb{Z}_{S_{\mathbf{n}}}$$

is multiplication by 1 on the regular locus, multiplication by  $m_0$  on  $e_0$  and by  $m_{\infty}$  on  $e_{\infty}$ . So the image of the map (11) is the order  $\mu = \text{lcm}(m_0, m_{\infty})$  times a primitive orbifold Kähler class  $[\omega_{\mathbf{n}, \mathbf{m}}] \in H_{orb}^2((S_{\mathbf{n}}, \Delta_{\mathbf{m}}), \mathbb{Z})$ . To compute the cohomology of such spaces we consider the  $\mathbb{C}^*$  Seifert bundle over  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  which is the cone  $C(M_{\mathbf{n}, \mathbf{m}}^7) = Y_{\mathbf{n}, \mathbf{m}} = M_{\mathbf{n}, \mathbf{m}}^7 \times \mathbb{R}^+$  so that  $Y_{\mathbf{n}, \mathbf{m}}|_{r=1} = M_{\mathbf{n}, \mathbf{m}}^7$ . Now  $Y_{\mathbf{n}, \mathbf{m}}$  is a branched cover of  $Y_{\mathbf{n}, \mathbf{m}}/\mathbb{Z}_{\mu}$  with branching locus  $\Delta_{\mathbf{m}}$ . This follows from [Kol04] by identifying the orbifold class

<sup>1</sup>See Remark 3.10 below

$[\omega_{\mathbf{n},\mathbf{m}}]$  with the rational first Chern class  $c_1(Y/X)$  in [Kol04] with  $X = S_{\mathbf{n}}$ . Now generally the total space  $M_{\mathbf{n},\mathbf{m}}^7$  is a Sasaki orbifold whose underlying topological space is a compactly generated Hausdorff space which allows us to apply the theory of ramified covers [Smi83, AP06]. The space  $M_{\mathbf{n},\mathbf{m}}^7$  is simply connected, and a  $\mu$ -fold cover of the total space of the ordinary  $S^1$  bundle defined by the primitive integral class  $\mu[\omega_{\mathbf{n},\mathbf{m}}]$ . Thus, we have the commutative diagram of Seifert orbibundles:

$$(13) \quad \begin{array}{ccccc} S^1 & \longrightarrow & M_{\mathbf{n},\mathbf{m}}^7 & \xrightarrow{f} & (S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \\ \downarrow & & \downarrow \pi & & \downarrow \\ S^1/\mathbb{Z}_{\mu} & \longrightarrow & M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu} & \xrightarrow{f_{\mu}} & (S_{\mathbf{n}}, \emptyset). \end{array}$$

Note that since  $M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu}$  is the total space of a principal  $S^1$  bundle over the simply connected smooth projective algebraic variety  $S_{\mathbf{n}}$  defined by a primitive integral class  $\mu[\omega_{\mathbf{n},\mathbf{m}}]$ , it is a simply connected smooth 7-manifold. Moreover, the covering map  $\pi$  induces a map of the corresponding Leray spectral sequences, and since the fiber is an  $S^1$  the only nonzero higher direct image sheaf is  $R^1\pi_*\mathbb{Z}$ . This gives the natural isomorphism  $R^1\pi_*\mathbb{Q}_M \approx \mathbb{Q}_{S_{\mathbf{n}}}$  which then implies

**Lemma 3.5.** *There is an isomorphism  $H^*(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Q}) \approx H^*(M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu}, \mathbb{Q})$ .*

However, we would also like information about integral cohomology groups. For this we consider the **transfer homomorphism** for ramified covers [Smi83]. Following [AP06] we apply this to our branched cover  $\pi : M_{\mathbf{n},\mathbf{m}}^7 \longrightarrow M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu}$ . Theorem 5.4 of [AP06] says that the transfer homomorphism

$$\tau : H^*(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Z}) \longrightarrow H^*(M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu}, \mathbb{Z})$$

induces multiplication by  $\mu$  in  $H^*(M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu}, \mathbb{Z})$ . We set  $G_{reg} = H^4(M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu}, \mathbb{Z})$  and apply this to  $* = 4$ . If  $\gcd(|G|, \mu) = 1$  we see that

$$\tau \circ \pi^* : G_{reg} = H^4(M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu}, \mathbb{Z}) \longrightarrow G_{reg} = H^4(M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu}, \mathbb{Z})$$

is an isomorphism which implies that  $\pi^* : H^4(M_{\mathbf{n},\mathbf{m}}^7/\mathbb{Z}_{\mu}, \mathbb{Z}) \longrightarrow H^4(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Z})$  is injective. Thus,  $\pi^*(G_{reg})$  is a subgroup of  $H^4(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Z})$  in this case. When  $\gcd(|G_{reg}|, \mu) \neq 1$  the homomorphism  $\tau \circ \pi^*$  has a non-trivial kernel. This kernel consists of all the prime factors  $\mathbb{Z}_{p_i^{r_1}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$  of  $G_{reg}$  such that  $p_i$  is a prime in  $\mu$ . Denote this group by  $G_{reg}^{\mu}$ . In this case  $H^4(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Z})$  contains the subgroup of  $G_{reg}$  containing only those prime factors whose primes  $p$  are not in  $\mu$ , that is the factor group  $G_{reg}/G_{reg}^{\mu}$ . Summarizing we have

**Lemma 3.6.**  *$H^4(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Z})$  contains  $G_{reg}/G_{reg}^{\mu}$*

We also have

**Lemma 3.7.**  *$H^3(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Z}) = 0$ .*

*Proof.* The only non-zero term of total degree 3 in (10) is

$$E_2^{2,1} = H^2(S_{\mathbf{n}}, R^1\pi_*\mathbb{Z})$$

where we have a differential

$$d_2 : E_2^{2,1} = H^2(S_{\mathbf{n}}, R^1\pi_*\mathbb{Z}) \longrightarrow E_2^{4,0} = H^4(S_{\mathbf{n}}, \mathbb{Z}) = \mathbb{Z}^3.$$

This differential gives torsion in  $H^4(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Z})$  implying that  $H^4(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Q})$  vanishes which in turn implies that  $E_r^{2,1} = 0$  for  $r > 2$  which proves the lemma.  $\square$

**Theorem 3.8.** *Let  $M_{\mathbf{n},\mathbf{m}}^7$  denote the total space of the principal  $S^1$  orbibundle over a KS orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  defined by a primitive orbifold Kähler class  $[\omega_{\mathbf{n},\mathbf{m}}] = c_1x_1 + c_2x_2 + c_3x_3$  such that  $\mu[\omega_{\mathbf{n},\mathbf{m}}]$  is a primitive class in  $H^2(S_{\mathbf{n}}, \mathbb{Z})$ . Then*

$$H^r(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = 0, 7 \\ \mathbb{Z}^2 & \text{if } r = 2, 5 \\ G & \text{if } r = 4 \\ 0 & \text{otherwise.} \end{cases}$$

where  $G$  is an Abelian group that contains  $G_{reg}/G_{reg}^\mu$ . In particular, if  $\gcd(|G_{reg}|, \mu) = 1$  then  $G$  contains  $G_{reg}$ .

*Proof.* Since  $[\omega_{\mathbf{n},\mathbf{m}}]$  is a primitive orbifold class and  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  has  $\pi_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \mathbb{1}$ , then the orbifold  $M_{\mathbf{n},\mathbf{m}}^7$  has  $\pi_1^{orb}(M_{\mathbf{n},\mathbf{m}}^7) = \mathbb{1}$  by the long exact homotopy sequence of the classifying spaces. The remainder of the theorem then follows from lemmas 3.5, 3.6, 3.7.  $\square$

**Remark 3.9.** It follows from Theorem 3.4 that  $G_{reg}$  is never the identity; although, we can have  $G_{reg} = G_{reg}^\mu$  so that  $G_{reg}/G_{reg}^\mu = \mathbb{1}$  in this case. However, this does not imply that  $H^4(M_{\mathbf{n},\mathbf{m}}^7, \mathbb{Z})$  is the identity.

**Remark 3.10.** Let  $M$  be a Boothby-Wang Sasaki manifold over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ . The set of KS orbifolds  $\mathcal{KS}_0$  splits into a disjoint union two types; those that have a polarization such that the corresponding Boothby-Wang constructed Sasaki orbifold is represented by a quasi-regular ray in the  $\mathbf{w}$ -cone of a join of the form  $M \star_1 S_{\mathbf{w}}^3$ , and those that do not have such a polarization. The former are characterized by the condition  $n_1n_2 > 0$  (see Theorem 4.3 below) and the corresponding Sasaki manifolds are called *cone decomposable* (see Definitions 3.1 and 4.1 in [BHLTF18]). It remains an open question whether a Boothby-Wang constructed Sasaki orbifold over the latter type, with  $n_1n_2 < 0$ , is necessarily *cone indecomposable* or simply not represented by a quasi-regular ray in the  $\mathbf{w}$ -cone of a join of the form  $M \star_1 S_{\mathbf{w}}^3$ .

**3.5. When  $M^7$  is an  $S_{\mathbf{w}}^3$ -join.** Generally, we are interested in how  $G$  depends on  $\mathbf{m}$  and  $\mathbf{n}$ . Unfortunately, Theorem 3.8 doesn't give us much useful information about this dependence. However, we can determine this dependence in a particular cone decomposable case, namely when the space  $M_{\mathbf{n},\mathbf{m}}^7$  can be represented as a join of the form  $M_{1,\mathbf{w}} = M \star_1 S_{\mathbf{w}}^3$  where  $M$  is a principal  $S^1$  bundle over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ . Let  $M^5$  denote the total space of the Boothby-Wang bundle over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  determined by the Kähler class  $k_1y_1 + k_2y_2$  with  $k_1, k_2 \in \mathbb{Z}^+$ . When  $k_1$  and  $k_2$  are relatively prime,  $M^5$  is diffeomorphic to  $S^2 \times S^3$ . More generally  $S^2 \times S^3$  is a  $k$ -fold cover of  $M^5$ , namely we have  $M^5 \approx (S^2 \times S^3)/\mathbb{Z}_k$  where  $k = \gcd(k_1, k_2)$ . Moreover, if  $\gcd(l_\infty, w_0) = \gcd(l_\infty, w_\infty) = 1$ ,  $M_{1,\mathbf{w}}$  will be smooth, and we have an  $S^1$  bundle

$$S^1 \longrightarrow M^5 \times S_{\mathbf{w}}^3 \longrightarrow M^5 \star_1 S_{\mathbf{w}}^3$$

which gives the homotopy exact sequence

$$(14) \quad 0 \longrightarrow \pi_2(M^5) \longrightarrow \pi_2(M^5 \star_1 S_{\mathbf{w}}^3) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_k \longrightarrow \pi_1(M^5 \star_1 S_{\mathbf{w}}^3) \longrightarrow 1.$$

So when  $k = 1$  so  $M^5 = S^2 \times S^3$ , the join  $M_{1,\mathbf{w}}^7$  will also be simply connected with  $\pi_2(M_{1,\mathbf{w}}^7) = \mathbb{Z}^2$ . More generally, the topological analysis proceeds as in Section 4 of [BTF16], see also [BTF19]. We consider the commutative diagram of fibrations

$$(15) \quad \begin{array}{ccccc} (S^2 \times S^3)/\mathbb{Z}_k \times S_{\mathbf{w}}^3 & \longrightarrow & M_{1,\mathbf{w}} & \longrightarrow & \mathbf{B}S^1 \\ \downarrow = & & \downarrow & & \downarrow \psi \\ (S^2 \times S^3)/\mathbb{Z}_k \times S_{\mathbf{w}}^3 & \longrightarrow & \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbf{B}\mathbb{C}P^1[\mathbf{w}] & \longrightarrow & \mathbf{B}S^1 \times \mathbf{B}S^1 \end{array}$$

where  $\mathbf{B}G$  is the classifying space of a group  $G$  or Haefliger's classifying space [Hae84] of an orbifold if  $G$  is an orbifold. Note that  $H^2((S^2 \times S^3)/\mathbb{Z}_k, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_k$ . The diagram gives a map of spectral sequences, and we note that the lower fibration is a product of well understood fibrations. The  $E_2$  term of the top fibration is

$$(16) \quad E_2^{p,q} = H^p(\mathbf{B}S^1, H^q(S^2 \times S^3 \times S_{\mathbf{w}}^3, \mathbb{Z})) \approx \mathbb{Z}[s] \otimes \Lambda[\alpha, \beta, \gamma],$$

where  $\alpha$  is a 2-class,  $\beta, \gamma$  are 3-classes, and  $s_1, s_2$  are the positive generators of  $H^*(\mathbf{B}S^1, \mathbb{Z})$ . By the Leray-Serre Theorem this converges to  $H^{p+q}(M_{1,\mathbf{w}}, \mathbb{Z})$ . The non-vanishing differentials of the bottom product fibration are, first  $d_4(\beta) = s_1^2$  and second  $d_4(\gamma) = w_0 w_\infty s_2^2$  and those induced by naturality. As described in [BTF16] we also have  $\psi^* s_1 = l_\infty s$  and  $\psi^* s_2 = -l_0 s$ . So by naturality the differentials of the top fibration are  $d_4(\beta) = l_\infty^2 s^2$  and  $d_4(\gamma) = w_0 w_\infty l_0^2 s^2$ . Now  $M_{1,\mathbf{w}}$  is smooth if and only if  $\gcd(l_\infty, w_0 w_\infty) = 1$ . This gives  $H^4(M_{1,\mathbf{w}}, \mathbb{Z}) = \mathbb{Z}l_\infty^2 \times \mathbb{Z}w_0 w_\infty l_0^2 = \mathbb{Z}w_0 w_\infty l_0^2 l_\infty^2$ .

#### 4. ADMISSIBLE PROJECTIVE BUNDLES

*Admissible projective bundles* were described in general [ACGTF08a]. Here we will restrict ourselves to a specific type of these, namely projective bundles of the form

$$S_{\mathbf{n}} = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)) \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$$

that satisfy the following conditions:

- For  $i = 1, 2$  let  $(\pm\omega_i, \pm g_i)$  be Kähler metrics with constant scalar curvature  $\pm 2s_i = \pm 4/n_i$  on  $\mathbb{C}P^1$ . [It is assumed that  $n_i \neq 0$ .] The  $\pm$  means that either  $+g_i$  or  $-g_i$  is positive definite.
- $\mathcal{O}(n_1, n_2) \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$  is the holomorphic line bundle over  $\mathbb{C}P^1 \times \mathbb{C}P^1$  which satisfy

$$c_1(\mathcal{O}(n_1, n_2)) = \frac{1}{2\pi}[\omega_1 + \omega_2] = n_1 x_1 + n_2 x_2$$

On admissible projective bundles such as the ones defined above we can now construct the *admissible metrics*, [ACGTF08a]. Here we recover the main points of this construction.

Consider the standard circle action on  $S_{\mathbf{n}} = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)) \longrightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ . It extends to a holomorphic  $\mathbb{C}^*$  action. The open and dense set  $S_{\mathbf{n}}^0$  of stable points with respect to the latter action has the structure of a principal circle bundle over the stable quotient. The hermitian norm on the fibers induces via a Legendre transform a function

$\mathfrak{z} : S_{\mathbf{n}}^0 \rightarrow (-1, 1)$  whose extension to  $S_{\mathbf{n}}$  consists of the critical manifolds  $e_0 := \mathfrak{z}^{-1}(1) = \mathbb{P}(\mathbb{1} \oplus 0)$  and  $e_{\infty} := \mathfrak{z}^{-1}(-1) = \mathbb{P}(0 \oplus \mathcal{O}(n_1, n_2))$ . Letting  $\theta$  be a connection one form for the Hermitian metric on  $S_{\mathbf{n}}^0$ , with curvature  $d\theta = \omega_1 + \omega_2$ , an admissible Kähler metric and form are given (up to scale) by the respective formulas

$$(17) \quad g = \frac{1+r_1\mathfrak{z}}{r_1}g_1 + \frac{1+r_2\mathfrak{z}}{r_2}g_2 + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2, \quad \omega = \frac{1+r_1\mathfrak{z}}{r_1}\omega_1 + \frac{1+r_2\mathfrak{z}}{r_2}\omega_2 + d\mathfrak{z} \wedge \theta,$$

valid on  $S_{\mathbf{n}}^0$ . Here  $\Theta$  is a smooth function with domain containing  $(-1, 1)$  and  $r_i$ ,  $i = 1, 2$  are real numbers of the same sign as  $g_i$  and satisfying  $0 < |r_i| < 1$ . The complex structure yielding this Kähler structure is given by the pullback of the base complex structure along with the requirement  $Jd\mathfrak{z} = \Theta\theta$ . The function  $\mathfrak{z}$  is hamiltonian with  $K = J \text{grad } \mathfrak{z}$  a Killing vector field. In fact,  $\mathfrak{z}$  is the moment map on  $S_{\mathbf{n}}$  for the circle action, decomposing  $M$  into the free orbits  $S_{\mathbf{n}}^0 = \mathfrak{z}^{-1}((-1, 1))$  and the special orbits  $\mathfrak{z}^{-1}(\pm 1)$ . Finally,  $\theta$  satisfies  $\theta(K) = 1$ .

Now  $g$  is a (positive definite) Kähler metric which extends smoothly to all of  $S_{\mathbf{n}}$  if and only if  $\Theta$  satisfies the following positivity and boundary conditions

$$(18) \quad (i) \Theta(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \quad (ii) \Theta(\pm 1) = 0, \quad (iii) \Theta'(\pm 1) = \mp 2.$$

The Kähler class  $\Omega_{\mathbf{r}} = [\omega]$  of an admissible metric as in (17) is also called *admissible* and is uniquely determined by the parameters  $r_1, r_2$ , once the data associated with  $M$  (i.e.  $s_i = 2/n_i$ ,  $g_i$  etc.) is fixed. Indeed, we have

$$\Omega_{\mathbf{r}} = \frac{[\omega_1]}{r_1} + \frac{[\omega_2]}{r_2} + 2\pi\Xi,$$

where  $\Xi$  is the Poincare dual of  $e_0 + e_{\infty}$ . For a more thorough description of  $\Xi$ , please consult Section 1.3 of [ACGTF08a]. Note that on  $S_{\mathbf{n}}$  any Kähler class is admissible up to scale. Using that  $y_3 - x_3 = n_1x_1 + n_2x_2$ , we can also write

$$\begin{aligned} \Omega_{\mathbf{r}}/2\pi &= \frac{n_1}{r_1}x_1 + \frac{n_2}{r_2}x_2 + x_3 + y_3 \\ &= \frac{n_1(1+r_1)}{r_1}x_1 + \frac{n_2(1+r_2)}{r_2}x_2 + 2x_3 \\ &= \frac{n_1(1-r_1)}{r_1}x_1 + \frac{n_2(1-r_2)}{r_2}x_2 + 2y_3. \end{aligned}$$

Define a function  $F(\mathfrak{z})$  by the formula  $\Theta(\mathfrak{z}) = F(\mathfrak{z})/p_c(\mathfrak{z})$ , where  $p_c(\mathfrak{z}) = (1+r_1\mathfrak{z})(1+r_2\mathfrak{z})$ . Since  $p_c(\mathfrak{z})$  is positive for  $-1 \leq \mathfrak{z} \leq 1$ , conditions (18) are equivalent to the following conditions on  $F(\mathfrak{z})$ .

$$(19) \quad (i) F(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \quad (ii) F(\pm 1) = 0, \quad (iii) F'(\pm 1) = \mp 2p_c(\pm 1).$$

**4.1. Orbifolds.** Now we allow our admissible metrics to compactify as orbifold metrics on the log pair

$$(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = (\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)), \Delta_{\mathbf{m}}) \longrightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1,$$

where

$$\Delta_{\mathbf{m}} = (1 - 1/m_0)D_1 + (1 - 1/m_{\infty})D_2 = (1 - 1/m_0)e_0 + (1 - 1/m_{\infty})e_{\infty},$$

and  $m_0, m_\infty \in \mathbb{Z}^+$ . Then (18) generalizes to

$$(20) \quad \begin{aligned} (i) \quad & \Theta(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \\ (ii) \quad & \Theta(\pm 1) = 0, \\ (iii) \quad & \Theta'(-1) = 2/m_\infty, \quad \text{and} \quad \Theta'(1) = -2/m_0. \end{aligned}$$

and, with  $\Theta(\mathfrak{z}) = F(\mathfrak{z})/p_c(\mathfrak{z})$  as above, we get that this is equivalent to

$$(21) \quad \begin{aligned} (i) \quad & F(\mathfrak{z}) > 0, \quad -1 < \mathfrak{z} < 1, \\ (ii) \quad & F(\pm 1) = 0, \\ (iii) \quad & F'(-1) = 2p_c(-1)/m_\infty, \quad \text{and} \quad F'(1) = -2p_c(1)/m_0. \end{aligned}$$

Note that this does not change the expression for  $\Omega_r$  above whereas from (3) we already know that the adjusted Chern class is

$$\begin{aligned} c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) &= \left(2 - \frac{n_1}{m_\infty}\right)x_1 + \left(2 - \frac{n_2}{m_\infty}\right)x_2 + \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right)y_3 \\ &= \left(2 + \frac{n_1}{m_0}\right)x_1 + \left(2 + \frac{n_2}{m_0}\right)x_2 + \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right)x_3. \end{aligned}$$

**4.2. Connection with  $S_{\mathbf{w}}^3$ -joins.** Consider  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = (\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)), \Delta_{\mathbf{m}}) \longrightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ , polarized with a primitive orbifold Kähler class  $[\omega_{\mathbf{n}, \mathbf{m}}]$ . As mentioned above, this will be the rescale of some admissible Kähler class  $\Omega_{\mathbf{r}}$  determined by  $\mathbf{r} = (r_1, r_2)$ . If  $n_1 n_2 > 0$ , we will say that  $[\omega_{\mathbf{n}, \mathbf{m}}]$  is *diagonally admissible* if and only if  $r_1 = r_2$ . Note that if  $n_1 n_2 < 0$ , it is never possible to have  $r_1 = r_2$ , since the sign of  $r_i$  is equal to the sign of  $n_i$ . Note that diagonally admissible is equivalent to being admissible (up to scale) as defined (more narrowly) in Section 2.5.1 of [BTF21a] with  $N = \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  and  $[\omega_N] = \frac{n_1}{n}x_1 + \frac{n_2}{n}x_2$ , where  $n = (\text{sign of } n_i) \gcd(|n_1|, |n_2|)$ .

Indeed, suppose  $n_1 n_2 > 0$  and  $r_1 = r_2 = r$  in the above setting. Then the admissible metric simplifies to

$$(22) \quad g = \frac{1+r\mathfrak{z}}{r}g_{N_n} + \frac{d\mathfrak{z}^2}{\Theta(\mathfrak{z})} + \Theta(\mathfrak{z})\theta^2, \quad \omega = \frac{1+r\mathfrak{z}}{r}\omega_{N_n} + d\mathfrak{z} \wedge \theta,$$

with admissible Kähler class

$$\begin{aligned} \Omega_{\mathbf{r}}/2\pi &= \frac{n}{r} \left( \frac{n_1}{n}x_1 + \frac{n_2}{n}x_2 \right) + x_3 + y_3 \\ &= \frac{n(1+r)}{r} \left( \frac{n_1}{n}x_1 + \frac{n_2}{n}x_2 \right) + 2x_3 \\ &= \frac{n(1-r)}{r} \left( \frac{n_1}{n}x_1 + \frac{n_2}{n}x_2 \right) + 2y_3, \end{aligned}$$

where  $g_{N_n} = n_1 g_1 + n_2 g_2$ . As such, we can view this metric as an admissible metric on  $S_{\mathbf{n}} = \mathbb{P}(\mathbb{1} \oplus L_n) \longrightarrow N$ , where  $N = \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  and

- $(\pm\omega_{N_n}, \pm g_{N_n})$  is a Kähler metric with constant scalar curvature  $\pm 4s = \pm 4\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$ . [The  $\pm$  still means that either  $+g_{N_n}$  or  $-g_{N_n}$  is positive definite.]
- $L_n \longrightarrow N$  is the holomorphic line bundle over  $N = \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  which satisfy

$$c_1(L_n) = \frac{1}{2\pi}[\omega_{N_n}] = n \left( \frac{n_1}{n}x_1 + \frac{n_2}{n}x_2 \right).$$

This special case, is exactly of the type considered in e.g. Section 5 of [BTF16] or more generally (including cyclic orbifolds  $N$ ) in Section 2.5 of [BTF21a]. Note that the base metric  $(\pm\omega_{N_n}, \pm g_{N_n})$  is only Kähler-Einstein if also  $n_1 = n_2$ .

Now consider the natural Boothby-Wang constructed Sasaki structure on the  $S^1$ -bundle  $M \rightarrow N$  defined by the primitive  $[\omega_N] = (\frac{n_1}{n}x_1 + \frac{n_2}{n}x_2)$ . Following Proposition 4.22 of [BHLTF18] (with the amendment of Lemma 2.12 and Corollary 2.14 of [BTF21a]) we then realize the following connection with  $S_{\mathbf{w}}^3$ -joins.

**Proposition 4.1.** *For  $n_1 n_2 > 0$ ,  $r_1 = r_2 = r$  rational,  $n = (\text{sign of } n_i) \gcd(|n_1|, |n_2|)$ , and  $\gcd(m_1, m_2, |n|) = 1$ , there is a choice of co-prime  $w_0, w_\infty \in \mathbb{Z}^+$  and co-prime  $l_0, l_\infty$  such that, when we form the  $S_{\mathbf{w}}^3$ -join  $M_{l_0, l_\infty, \mathbf{w}} := M \star_{l_0, l_\infty} S_{w_0, w_\infty}^3 = M \star_{\mathbf{1}} S_{\mathbf{w}}^3$ , the quasi-regular quotient of  $M_{l_0, l_\infty, \mathbf{w}}$  by the flow of the Reeb vector field  $\xi_{\mathbf{v}}$  determined by  $(v_0, v_\infty)$  in the  $\mathbf{w}$ -cone (where  $m = \gcd(m_0, m_\infty)$  and  $\mathbf{m} = (m_0, m_\infty) = m\mathbf{v} = m(v_0, v_\infty)$ ), is the log pair  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  with induced (transverse) Kähler class  $m \gcd(\mathfrak{s}, w_0 v_\infty) [\omega_{\mathbf{n}, \mathbf{m}}]$  where  $\mathfrak{s} = \gcd(l_\infty, |w_0 v_\infty - w_\infty v_0|)$  and  $[\omega_{\mathbf{n}, \mathbf{m}}]$  is a orbifold primitive class that is an appropriate rescale of the admissible class  $\Omega_{\mathbf{r}}$  on  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ . The join is smooth if and only if  $\gcd(w_0, l_\infty) = \gcd(w_\infty, l_\infty) = 1$ . In particular, the join is smooth if  $m_0 = m_\infty = 1$ .*

*Proof.* We follow the proof of Proposition 4.22 in [BHLTF18] (including the paragraphs leading up to the proposition), to identify the join by first picking  $\mathbf{w} = (w_0, w_\infty)$  to be the unique positive, integer, and co-prime solution of

$$r = \frac{w_0 m_\infty - w_\infty m_0}{w_0 m_\infty + w_\infty m_0}$$

and then, as the next step, picking the pair  $(l_0, l_\infty)$  as the unique, positive integers, and co-prime solution of

$$l_\infty n = l_0 (w_0 m_\infty - w_\infty m_0).$$

Proposition 4.22 in [BHLTF18] was slightly misleading in implying that the quasi-regular quotient of  $\xi_{\mathbf{v}}$  would always produce a primitive orbifold Kähler class, but, with Lemma 2.12 and Corollary 2.14 of [BTF21a] in hand, we can say that the transverse Kähler class is  $m \gcd(\mathfrak{s} \Upsilon_N, w_0 v_\infty l_0) [\omega_{\mathbf{n}, \mathbf{m}}]$  where  $[\omega_{\mathbf{n}, \mathbf{m}}]$  is a orbifold primitive class that is an appropriate rescale of the admissible class  $\Omega_{\mathbf{r}}$  on  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  and  $\Upsilon_N$  is the orbifold order of  $N$ . Since  $\Upsilon_N = 1$  and  $\gcd(\mathfrak{s}, l_0) = 1$  (recall  $\gcd(l_0, l_\infty) = 1$ ) we get the desired Kähler class.

The rest of the claims follow straight from the proof Proposition 4.22 in [BHLTF18]. In particular, note that  $m = l_\infty / \mathfrak{s}$ , as it should be according to Theorem 3.8 in [BTF16].  $\square$

**Remark 4.2.** Note that when the join is smooth, it is easy to see that  $\gcd(\mathfrak{s}, w_0 v_\infty) = 1$  and so the transverse Kähler class is  $m [\omega_{\mathbf{n}, \mathbf{m}}]$ . In particular, if  $m_0 = m_\infty = 1$ , we do have a primitive (and admissible) transverse Kähler class.

Combining Theorem 2.7 and Lemma 2.12 from [BTF21a] for the "only if" and Proposition 4.1 for the "if", we arrive at the following theorem.

**Theorem 4.3.** *Assume  $\gcd(m_1, m_2, |n_1|, |n_2|) = 1$  and  $[\omega_{\mathbf{n}, \mathbf{m}}]$  is a orbifold primitive class on  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ . There exist a constant  $k \in \mathbb{Z}^+$  such that the polarized orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}, k[\omega_{\mathbf{n}, \mathbf{m}}])$  is the quotient with respect to (the canonical Reeb vector field in) a*

quasi-regular ray in the  $\mathbf{w}$ -cone of a (possibly non-smooth) join of the form  $M \star_1 S_{\mathbf{w}}^3$ , where  $M$  is a Boothby-Wang constructed Sasaki manifold over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ , if and only if  $n_1 n_2 > 0$  and  $[\omega_{\mathbf{n}, \mathbf{m}}]$  is a diagonally admissible Kähler class.

Suppose now that  $n_1 n_2 > 0$ , we are in the log Fano case as in Proposition 1.1, and the polarization is chosen such that  $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \mathcal{I}_{\mathbf{n}, \mathbf{m}}[\omega_{\mathbf{n}, \mathbf{m}}]$ . Since  $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \left(2 - \frac{n_1}{m_\infty}\right)x_1 + \left(2 - \frac{n_2}{m_\infty}\right)x_2 + \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right)y_3$  and a diagonally admissible Kähler class is a rescale of  $\Omega_{\mathbf{r}} = \frac{(1-r)}{r}(n_1 x_1 + n_2 x_2) + 2y_3$  for some  $0 < |r| < 1$  with  $rn_i > 0$ , we see that  $[\omega_{\mathbf{n}, \mathbf{m}}] = c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})/\mathcal{I}_{\mathbf{n}, \mathbf{m}}$  is diagonally admissible if and only if

$$2 - \frac{n_1}{m_\infty} = \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right) \frac{n_1(1-r)}{2r} \quad \text{and} \quad 2 - \frac{n_2}{m_\infty} = \left(\frac{1}{m_0} + \frac{1}{m_\infty}\right) \frac{n_2(1-r)}{2r},$$

i.e.,

$$n_1 \left( \frac{1}{m_\infty} + \left( \frac{1}{m_0} + \frac{1}{m_\infty} \right) \frac{(1-r)}{2r} \right) = 2 = n_2 \left( \frac{1}{m_\infty} + \left( \frac{1}{m_0} + \frac{1}{m_\infty} \right) \frac{(1-r)}{2r} \right).$$

This clearly implies that we must have  $n_1 = n_2$ . On the other hand, if  $n_1 = n_2$  (and still assuming log Fano) we can solve for an appropriate  $r$ . In conclusion,

**Corollary 4.4.** *In the log Fano case, there exists a constant  $k \in \mathbb{Z}^+$  such that the polarized orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}, kc_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}))$  is the quotient with respect to a quasi-regular ray in the  $\mathbf{w}$ -cone of a (possibly non-smooth) join of the form  $M \star_1 S_{\mathbf{w}}^3$ , where  $M$  is a Boothby-Wang constructed Sasaki manifold over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ , if and only if  $n_1 = n_2$ .*

Returning to Proposition 4.1 and Theorem 4.3, we note that in the case where  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}, [\omega_{\mathbf{n}, \mathbf{m}}])$  is the quotient with respect to a quasi-regular ray in the  $\mathbf{w}$ -cone of a join of the form  $M \star_1 S_{\mathbf{w}}^3$  (so assuming  $k = 1$  in Theorem 4.3, i.e., assuming the transverse Kähler class is primitive) we must have that  $m = 1$  and  $\gcd(\mathfrak{s}, w_0 v_\infty) = 1$ . The first equality implies that  $l_\infty = \mathfrak{s} = \gcd(l_\infty, |w_0 v_\infty - w_\infty v_0|)$  and hence  $l_\infty$  is a factor of  $|w_0 v_\infty - w_\infty v_0|$ . Since we also have that  $\gcd(l_\infty, w_0 v_\infty) = 1$ , we must have  $\gcd(l_\infty, w_\infty v_0) = 1$  and in particular  $\gcd(w_0, l_\infty) = \gcd(w_\infty, l_\infty) = 1$ . This means that  $M \star_1 S_{\mathbf{w}}^3$  is smooth. We therefore have the following companion to Theorem 4.3.

**Proposition 4.5.** *Consider the polarized orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}, [\omega_{\mathbf{n}, \mathbf{m}}])$ , where  $[\omega_{\mathbf{n}, \mathbf{m}}]$  is a primitive integer orbifold Kähler class and  $\gcd(m_0, m_\infty, |n_1|, |n_2|) = 1$ .*

*If  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}, [\omega_{\mathbf{n}, \mathbf{m}}])$  is the quotient with respect to (the canonical Reeb vector field in) a quasi-regular ray in the  $\mathbf{w}$ -cone of a join of the form  $M \star_1 S_{\mathbf{w}}^3$ , then this join is smooth. Moreover, in this case  $\gcd(m_0, m_\infty) = 1$ .*

**4.3. Connection with Yamazaki's Fiber joins.** In [Yam99] T. Yamazaki introduced the fiber join for  $K$ -contact structures and in [BTF21b] this was extended to Sasaki structures. In the smooth case ( $m_0 = m_\infty = 1$ ) of KS orbifolds, it follows from Section 5.3 of [BTF21b] that  $S_{\mathbf{n}}$  polarized by a primitive Kähler class, which in turn is an appropriate rescale of  $\Omega_{\mathbf{r}}$ , is the quotient of the regular ray in the  $\mathfrak{t}_{spher}^+$  cone of a Yamazaki fiber join over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ , if and only if there exist  $k_1^1, k_1^2, k_2^1, k_2^2 \in \mathbb{Z}^+$  such that

$$(23) \quad \begin{aligned} k_1^1 - k_2^1 &= n_1 & k_1^2 - k_2^2 &= n_2 \\ \frac{k_1^1 - k_2^1}{k_1^1 + k_2^1} &= r_1 & \frac{k_1^2 - k_2^2}{k_1^2 + k_2^2} &= r_2. \end{aligned}$$

In that case, the corresponding Yamazaki fiber join is given as follows:<sup>2</sup>

Let  $\pi_j : \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$  denote the natural projection to the  $j^t$  factor of the product  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$  and let  $\mathcal{K}$  denote the canonical bundle on  $\mathbb{C}\mathbb{P}^1$ . Let  $L_i$  be a holomorphic line bundle over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^1$  given by  $L_i = k_i^1 \pi_1^* \mathcal{K}^{-\frac{1}{2}} + k_i^2 \pi_2^* \mathcal{K}^{-\frac{1}{2}}$ , for  $k_i^j \in \mathbb{Z}^+$ . The choice of  $L_1$  and  $L_2$  can be given by the matrix  $K = \begin{pmatrix} k_1^1 & k_1^2 \\ k_2^1 & k_2^2 \end{pmatrix}$ . Note that  $c_1(L_i)$  are both in the Kähler cone of  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  and hence  $L_i$  are positive line bundles over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ . Each  $c_1(L_i)$  also defines a principal  $S^1$ -bundle over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ ,  $M_i \rightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ , and we identify  $L_i$  with  $M_i \times_{S^1} \mathbb{C}$ . Then  $M_i \xrightarrow{\pi} \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  has a natural Sasaki structure defined by the Boothby-Wang construction. Consider now  $L_1^* \oplus L_2^*$  and equip each  $L_i^*$  with a Hermitian metric giving us a norm  $d_i : L_i^* \rightarrow \mathbb{R}^{\geq 0}$ . Then, the fiber join,  $M_K = M_1 \star_f M_2$  is defined as the  $S^3$ -bundle over  $S$  whose fibers are given by  $d_1^2 + d_2^2 = 1$ . Now  $M_K$  has a natural  $CR$ -structure  $(\mathcal{D}, J)$  with a family,  $\mathfrak{t}_{sph}^+$ , of compatible Sasaki structures  $\mathcal{S}_{\mathbf{a}} = (\xi_{\mathbf{a}}, \eta_{\mathbf{a}}, \Phi_{\mathbf{a}}, g_{\mathbf{a}})$ , where  $\mathbf{a} = (a_1, a_2) \in (\mathbb{R}^+)^2$  and  $(a_1, a_2) = (1, 1)$  corresponds to the regular Sasaki structure in  $\mathfrak{t}_{sph}^+(\mathcal{D}, J)$ . Note that  $\mathfrak{t}_{sph}^+$  is a proper subcone of the unreduced Sasaki cone  $\mathfrak{t}^+$  of  $(M_K, \mathcal{D}, J)$ .

**Remark 4.6.** In [BTF21b] we did not consider the more general question of determining the quotients of quasi-regular Sasaki structures in  $\mathfrak{t}_{sph}^+$  and the transverse Kähler class. We conjecture that those would indeed be certain polarized KS orbifolds and will explore this in future studies.

**4.4. Ricci Solitons and Kähler-Einstein.** It is well-known that there exists a unique Kähler-Ricci soliton on any toric compact Fano complex orbifold [Don08, TZ00, SZ12, WZ04, Zhu00]. The existence proofs by Wang-Zhu [WZ04], Shi-Zhu [SZ12] and Donaldson [Don08] all use a continuity method and thus do not provide an explicit expression of the Kähler-Ricci soliton. It is therefore interesting to explore cases where explicit descriptions of Kähler-Ricci solitons are possible. Explicit examples can be found in e.g. [DW11, LTF13].

Here we want to explore the Ricci Soliton and more specifically the Kähler-Einstein equations under the constraint of the orbifold endpoint conditions (21). This will yield explicit examples on Fano  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  and is essentially a mild orbifold extension of the work by Koiso and Sakane [KS86, Koi90]. We will follow in their footsteps using the notation of Section 3 of [ACGTF08b].

The admissible metrics are Ricci solitons with  $V = (\frac{c}{2}) \text{grad}_g \mathfrak{z}$  if and only if

$$(24) \quad \rho - \lambda\omega = \mathcal{L}_V \omega,$$

where the Ricci form is

$$\rho = s_1 \omega_1 + s_2 \omega_2 - \frac{1}{2} dd^c \log F = \left( s_1 - \frac{1}{2} \frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})} \right) \omega_1 + \left( s_2 - \frac{1}{2} \frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})} \right) \omega_2 - \frac{1}{2} \left( \frac{F'}{p_c} \right)'(\mathfrak{z}) d\mathfrak{z} \wedge \theta$$

<sup>2</sup>We refer to [BTF21b] for details and a more general description of the Yamazaki fiber joins.

and  $\lambda, c \in \mathbb{R}$ . Obviously,  $c = 0$  corresponds to Kähler-Einstein metrics. Since  $\mathcal{L}_V \omega = (\frac{c}{2}) dd^c \mathfrak{z}$  and  $s_i = 2/n_i$ , equation (24) becomes a pair of ODEs

$$(25) \quad \begin{aligned} \frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})} + c \frac{F(\mathfrak{z})}{p_c(\mathfrak{z})} &= 4/n_1 - 2\lambda(\mathfrak{z} + \frac{1}{r_1}) \\ \frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})} + c \frac{F(\mathfrak{z})}{p_c(\mathfrak{z})} &= 4/n_2 - 2\lambda(\mathfrak{z} + \frac{1}{r_2}), \end{aligned}$$

Note that (25) implies that  $n_1 = n_2$  if and only if  $r_1 = r_2$ .

Note that (ii) and (iii) of (21) together with (25) implies the necessary conditions

$$(26) \quad 2\lambda = \frac{1}{m_0} + \frac{1}{m_\infty}$$

and

$$(27) \quad r_1 = \frac{\frac{1}{m_0} + \frac{1}{m_\infty}}{\frac{4}{n_1} + \frac{1}{m_0} - \frac{1}{m_\infty}}, \quad r_2 = \frac{\frac{1}{m_0} + \frac{1}{m_\infty}}{\frac{4}{n_2} + \frac{1}{m_0} - \frac{1}{m_\infty}}.$$

On the other hand, assuming (26) and (27), the ODEs of (25) are equivalent to the single ODE

$$(28) \quad \frac{F'(\mathfrak{z})}{p_c(\mathfrak{z})} + c \frac{F(\mathfrak{z})}{p_c(\mathfrak{z})} = (\frac{1}{m_\infty} - \frac{1}{m_0}) - (\frac{1}{m_0} + \frac{1}{m_\infty})\mathfrak{z}$$

Further, (28) and (ii) of (21) together imply (iii) of (21). In summary, we will get an admissible Ricci soliton solution as in (24) exactly when (26) and (27) are satisfied and (28) has a solution  $F(\mathfrak{z})$  that satisfies (i) and (ii) of (21).

Notice that since we require  $0 < |r_i| < 1$  and  $r_i n_i > 0$ , (27) has an appropriate solution  $(r_1, r_2)$  iff

$$\frac{n_1}{m_\infty} < 2, \quad \frac{n_2}{m_\infty} < 2, \quad -\frac{n_1}{m_0} < 2, \quad -\frac{n_2}{m_0} < 2.$$

This is exactly the Fano condition in Proposition 1.1. In turn, under this condition, (26) and (27) correspond exactly to  $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = \lambda \Omega_{\mathbf{r}}/2\pi$ .

Similarly to the smooth case, we shall see that in the Fano case, there is indeed always an admissible Ricci soliton in the appropriate Kähler class:

Using the integrating factor method, observe that

$$(29) \quad \begin{aligned} F(\mathfrak{z}) &= e^{-c\mathfrak{z}} \int_{-1}^{\mathfrak{z}} e^{ct} \left( (\frac{1}{m_\infty} - \frac{1}{m_0}) - (\frac{1}{m_0} + \frac{1}{m_\infty})t \right) p_c(t) dt \\ &= e^{-c\mathfrak{z}} \int_{-1}^{\mathfrak{z}} e^{ct} (t - t_0) g(t) dt \end{aligned}$$

solves (28) and (ii) of (21) iff  $G(c) = 0$ , where

$$(30) \quad G(k) = e^{kt_0} \int_{-1}^1 e^{k(t-t_0)} (t - t_0) g(t) dt,$$

$t_0 = \frac{m_0 - m_\infty}{m_0 + m_\infty}$ , and  $g(t) = -(\frac{1}{m_0} + \frac{1}{m_\infty})p_c(t)$ .

Note that  $t_0 \in (-1, 1)$  and  $g(t) < 0$  for  $t \in [-1, 1]$ . Thus  $e^{-kt_0}G(k)$  is a strictly decreasing function of  $k$  tending to  $\mp\infty$  as  $k \rightarrow \pm\infty$ , and hence has a unique zero  $c$  (consistent with the uniqueness of Ricci solitons).

To check (i) of (21) we consider another auxiliary function

$$h(\mathfrak{z}) = e^{c\mathfrak{z}}F(\mathfrak{z}) = \int_{-1}^{\mathfrak{z}} e^{ct}(t - t_0)g(t)dt.$$

Note that the sign of  $h(\mathfrak{z})$  equals the sign of  $F(\mathfrak{z})$ . In particular,  $h(\pm 1) = 0$ , and (due to (iii) of (21))  $h$  is positive to the immediate right of  $\mathfrak{z} = -1$  and immediate left of  $\mathfrak{z} = +1$ . Now, since  $h'$  clearly has exactly one zero (namely  $t_0$ ) in  $(-1, 1)$ , it is positive on  $(-1, 1)$ . Therefore (i) of (21) is also satisfied.

Combining the arguments above with Proposition 1.1 we conclude the following.

**Proposition 4.7.** *For  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  the following conditions are equivalent:*

- *The inequalities  $\frac{n_1}{m_\infty} < 2$ ,  $\frac{n_2}{m_\infty} < 2$ ,  $-\frac{n_1}{m_0} < 2$ ,  $-\frac{n_2}{m_0} < 2$  are satisfied;*
- *$(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  is log Fano;*
- *There exist a Kähler-Ricci soliton on  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ .*

*In this case, the Kähler-Ricci soliton  $(g, \omega)$  is admissible and satisfies  $\rho = \mathcal{L}_V\omega$ , with  $\lambda = \frac{1}{2m_0} + \frac{1}{2m_\infty}$  and  $V = (\frac{c}{2})\text{grad}_g\mathfrak{z}$  for a suitable real constant  $c$ . The Kähler-Ricci soliton is Kähler-Einstein iff this  $c$  is equal to zero.*

Of course this can also be seen as an orbifold extension of Theorem 3.1 in [ACGTF08a] (which in turn is essentially due to Koiso [Koi90]) in the case of  $S_{\mathbf{n}} = \mathbb{P}(\mathbb{1} \oplus \mathcal{O}(n_1, n_2)) \longrightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ .

**Remark 4.8.** Note that, assuming we are in the Fano case,  $c$  in the above proposition is a non-zero multiple of the Futaki invariant,  $\mathcal{F}(V)$ , of  $c_1^{\text{orb}}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  applied to  $V$ . This underscores the well-known fact that the existence of a Kähler-Ricci Soliton with respect to a non-trivial holomorphic vector field is an obstruction to the existence of a Kähler-Einstein metric.

4.4.1. *Kähler-Einstein.* Now we turn our attention to Kähler-Einstein examples on  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ . From the work above we see that under the assumption that the log Fano condition and (27) hold,  $\Omega_{\mathbf{r}}$  contains an admissible Kähler-Einstein metric with  $\rho = (\frac{1}{2m_0} + \frac{1}{2m_\infty})\omega$  if and only if  $G$  defined in (30) satisfies that  $G(0) = 0$ , i.e.,

$$(31) \quad \int_{-1}^1 \left( \left( \frac{1}{m_\infty} - \frac{1}{m_0} \right) - \left( \frac{1}{m_0} + \frac{1}{m_\infty} \right) t \right) (1 + r_1 t)(1 + r_2 t) dt = 0.$$

Carrying out the integration in (31) and substituting (27), we have the following

**Proposition 4.9.** *The Bott orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  admits a Kähler-Einstein metric (which happens to be admissible) iff the following two conditions are satisfied*

- (1)  *$(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  is log Fano (i.e.,  $\frac{n_1}{m_\infty} < 2$ ,  $\frac{n_2}{m_\infty} < 2$ ,  $-\frac{n_1}{m_0} < 2$ ,  $-\frac{n_2}{m_0} < 2$ )*
- (2)

$$24(m_0^3 m_\infty^2 - m_0^2 m_\infty^3) - 8(n_1 + n_2)(m_0^3 m_\infty - m_0^2 m_\infty^2 + m_0 m_\infty^3) + 3n_1 n_2 (m_0^3 - m_0^2 m_\infty + m_0 m_\infty^2 - m_\infty^3) = 0$$

**Example 4.1.** If we assume  $n_1 = 1$  and  $n_2 = 2$ , then the equation in Proposition 4.9 rewrites to

$$6(2m_0 m_\infty - m_0 + m_\infty)(m_0^2(-1 + 2m_\infty) - m_\infty^2(1 + 2m_0)) = 0.$$

Clearly  $2m_0m_\infty - m_0 + m_\infty = 0$  has no positive integer solutions  $(m_0, m_\infty)$ . Likewise, assume by contradiction that  $(m_0, m_\infty)$  are positive integer solutions of

$$m_0^2(-1 + 2m_\infty) - m_\infty^2(1 + 2m_0) = 0,$$

i.e.,

$$m_0^2(-1 + 2m_\infty) = m_\infty^2(1 + 2m_0).$$

Since  $\gcd(2m^i \pm 1, m^i) = 1$  for  $i = 0, \infty$  we see that this would imply that  $m_0 = m_\infty$  (since they would have to have the same prime factorization). But  $m_0 = m_\infty$  will clearly never solve the equation.

**Remark 4.10.** The diophantine nature of the equation in Proposition 4.9 makes it hard to spot solutions other than the classic smooth Koiso-Sakane example (see Example 4.2 below). Further, Example 4.1 and Proposition 4.9 tells us that there exist at least one pair  $\mathbf{n} = (n_1, n_2)$  such that for all pairs  $\mathbf{m} = (m_0, m_\infty) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ , the Bott orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  admits no Kähler-Einstein metric. For these reasons we will instead view a choice of  $(r_1, r_2) \in \mathbb{Q}^2$  such that  $0 < |r_i| < 1$  as a pair of parameters that determine a unique KE example on the appropriate corresponding KS orbifold.

From this point of view, note that (31) is equivalent to the equation

$$(32) \quad \frac{m_\infty}{m_0} = \frac{3 + r_1r_2 - r_1 - r_2}{3 + r_1r_2 + r_1 + r_2}.$$

and, assuming (32), equation (27) is equivalent to

$$(33) \quad \frac{n_1}{m_0} = \frac{2r_1(3 + r_1r_2 - r_1 - r_2)}{3 + 2r_1r_2 + r_1^2}, \quad \frac{n_2}{m_0} = \frac{2r_2(3 + r_1r_2 - r_1 - r_2)}{3 + 2r_1r_2 + r_2^2}.$$

We notice that if we pick a rational pair  $(r_1, r_2)$  such that  $0 < |r_i| < 1$ , then there is a unique quadruple of appropriate integers  $(n_1, n_2, m_0, m_\infty)$ , solving (32) and (33), such that  $n_i$  has the same sign as  $r_i$ ,  $m_i > 0$ , and  $\gcd(|n_1|, |n_2|, m_0, m_\infty) = 1$ . This yields a KE example on the corresponding KS orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$  with fibers  $\mathbb{CP}^1[v_0, v_\infty]/\mathbb{Z}_m$ . Here we have used our earlier notation;  $m = \gcd(m_0, m_\infty)$  and  $\mathbf{m} = (m_0, m_\infty) = m\mathbf{v} = m(v_0, v_\infty)$ .

**Example 4.2.** Assume  $r_1 = r$  and  $r_2 = -r$  with  $r \in (0, 1) \cap \mathbb{Q}$ . Then (32) and (33) yields

$$\frac{m_\infty}{m_0} = 1, \quad \frac{n_1}{m_0} = 2r, \quad \frac{n_2}{m_0} = -2r.$$

When  $r = 1/2$  this yields the Koiso-Sakane smooth Kähler-Einstein metric on  $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(1, -1)) \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$  ([KS86]). In general, we write  $r = p/q$  in reduced form with co-prime integers  $0 < p < q$ . Then

$$(n_1, n_2, m_0, m_\infty) = \begin{cases} (2p, -2p, q, q) & \text{if } q \text{ is odd} \\ (p, -p, q/2, q/2) & \text{if } q \text{ is even} \end{cases}$$

This gives us Kähler-Einstein metrics on  $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(2p, -2p))$  with fibers  $\mathbb{CP}^1/\mathbb{Z}_q$  and on  $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(p, -p))$  with fibers  $\mathbb{CP}^1/\mathbb{Z}_{q/2}$ , respectively. Using (4) we see that the index is equal to 2 if  $q$  is odd and equal to 1 if  $q$  is even.

Below we will explore more examples beyond the Koiso-Sakane examples. We set  $r_i = p_i/q_i$  with  $q_i \in \mathbb{Z}^{\geq 2}$  and  $p_i \in \mathbb{Z}$  such that  $0 < |p_i| < q_i$  and  $\gcd(|p_i|, q_i) = 1$ . Further, without loss we can assume that  $r_1$ , hence  $p_1$ , is positive. We then observe that (32) and (33) are equivalent to

$$(34) \quad \begin{aligned} \frac{m_\infty}{m_0} &= \frac{3q_1q_2 + p_1p_2 - p_1q_2 - p_2q_1}{3q_1q_2 + p_1p_2 + p_1q_2 + p_2q_1} \\ \frac{n_1}{m_0} &= \frac{2p_1(3q_1q_2 + p_1p_2 - p_1q_2 - p_2q_1)}{3q_1^2q_2 + 2p_1p_2q_1 + p_1^2q_2} \\ \frac{n_2}{m_0} &= \frac{2p_2(3q_1q_2 + p_1p_2 - p_1q_2 - p_2q_1)}{3q_1q_2^2 + 2p_1p_2q_2 + p_2^2q_1}. \end{aligned}$$

Note that, due to  $0 < |p_i| < q_i$ , the denominator of each the right hand fractions in (34) are all positive. Further, the numerator of the first two right hand fractions are positive while the numerator of the last right hand fraction has the same sign as  $p_2$ . Thus if we set

$$k = \gcd \left( \begin{array}{l} (3q_1q_2 + p_1p_2 + p_1q_2 + p_2q_1)(3q_1^2q_2 + 2p_1p_2q_1 + p_1^2q_2)(3q_1q_2^2 + 2p_1p_2q_2 + p_2^2q_1), \\ (3q_1q_2 + p_1p_2 - p_1q_2 - p_2q_1)(3q_1^2q_2 + 2p_1p_2q_1 + p_1^2q_2)(3q_1q_2^2 + 2p_1p_2q_2 + p_2^2q_1), \\ 2p_1(3q_1q_2 + p_1p_2 - p_1q_2 - p_2q_1)(3q_1q_2 + p_1p_2 + p_1q_2 + p_2q_1)(3q_1q_2^2 + 2p_1p_2q_2 + p_2^2q_1), \\ 2|p_2|(3q_1q_2 + p_1p_2 - p_1q_2 - p_2q_1)(3q_1q_2 + p_1p_2 + p_1q_2 + p_2q_1)(3q_1^2q_2 + 2p_1p_2q_1 + p_1^2q_2) \end{array} \right)$$

we have an appropriate solution  $(n_1, n_2, m_0, m_\infty)$  to (34) given by

$$(35) \quad \begin{aligned} n_1 &= \frac{2p_1(3q_1q_2 + p_1p_2 - p_1q_2 - p_2q_1)(3q_1q_2 + p_1p_2 + p_1q_2 + p_2q_1)(3q_1q_2^2 + 2p_1p_2q_2 + p_2^2q_1)}{k} \\ n_2 &= \frac{2p_2(3q_1q_2 + p_1p_2 - p_1q_2 - p_2q_1)(3q_1q_2 + p_1p_2 + p_1q_2 + p_2q_1)(3q_1^2q_2 + 2p_1p_2q_1 + p_1^2q_2)}{k} \\ m_0 &= \frac{(3q_1q_2 + p_1p_2 + p_1q_2 + p_2q_1)(3q_1^2q_2 + 2p_1p_2q_1 + p_1^2q_2)(3q_1q_2^2 + 2p_1p_2q_2 + p_2^2q_1)}{k} \\ m_\infty &= \frac{(3q_1q_2 + p_1p_2 - p_1q_2 - p_2q_1)(3q_1^2q_2 + 2p_1p_2q_1 + p_1^2q_2)(3q_1q_2^2 + 2p_1p_2q_2 + p_2^2q_1)}{k}. \end{aligned}$$

Each of these solutions then yields a KE example on the KS orbifold  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}}) = (\mathbb{P}(\mathbf{1} \oplus \mathcal{O}(n_1, n_2)), \Delta_{\mathbf{m}})$  with fibers  $\mathbb{C}\mathbb{P}^1[m_0/\gcd(m_0, m_\infty), m_\infty/\gcd(m_0, m_\infty)]/\mathbb{Z}_{\gcd(m_0, m_\infty)}$ .

**Proposition 4.11.** *There exists a four-parameter family of KS orbifolds with KE orbifold metrics. The parameters  $(p_1, p_2, q_1, q_2)$  are integers such that  $0 < p_1 < q_1$ ,  $0 < |p_2| < q_2$ , and  $\gcd(|p_i|, q_i) = 1$ .*

**Remark 4.12.** One might ask the following question. Suppose we have a co-prime quadruple  $(n_1, n_2, m_0, m_\infty)$  satisfying the conditions in Proposition 4.9, that is,  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  admits a KE metric with  $\mathbf{n} = (n_1, n_2)$  and  $\mathbf{m} = (m_0, m_\infty)$ . Fixing  $\mathbf{n} = (n_1, n_2)$ , does there exist another pair  $\tilde{\mathbf{m}} = (\tilde{m}^0, \tilde{m}^\infty)$  with  $\tilde{\mathbf{m}} \neq \mathbf{m}$ , such that  $(n_1, n_2, \tilde{m}^0, \tilde{m}^\infty)$  is coprime and also satisfies the conditions in Proposition 4.9. In other words, if  $\mathbf{n}$  has an appropriate choice of  $\mathbf{m}$  such that  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  admits a KE metric, is this  $\mathbf{m}$  then unique?

**Example 4.3.** If we pick  $p_1 = p_2 = 1$ ,  $q_1 = 2$ , and  $q_2 = q > 2$ , then (35) becomes

$$\begin{aligned} n_1 &= \frac{4(5q-1)(7q+3)(3q^2+q+1)}{k} = \frac{2(5q-1)(7q+3)(3q^2+q+1)}{\hat{k}} \\ n_2 &= \frac{2(5q-1)(7q+3)(13q+4)}{k} = \frac{(5q-1)(7q+3)(13q+4)}{\hat{k}} \\ m_0 &= \frac{2(7q+3)(13q+4)(3q^2+q+1)}{k} = \frac{(7q+3)(13q+4)(3q^2+q+1)}{\hat{k}} \\ m_\infty &= \frac{2(5q-1)(13q+4)(3q^2+q+1)}{k} = \frac{(5q-1)(13q+4)(3q^2+q+1)}{\hat{k}}, \end{aligned}$$

where

$$\hat{k} = \gcd \begin{pmatrix} 2(5q-1)(7q+3)(3q^2+q+1), \\ (5q-1)(7q+3)(13q+4), \\ (7q+3)(13q+4)(3q^2+q+1), \\ (5q-1)(13q+4)(3q^2+q+1) \end{pmatrix}.$$

Likewise, if we pick  $p_1 = 1$ ,  $p_2 = -1$ ,  $q_1 = 2$ , and  $q_2 = q > 2$ , then (35) becomes

$$\begin{aligned} n_1 &= \frac{4(5q+1)(7q-3)(3q^2-q+1)}{k} = \frac{2(5q+1)(7q-3)(3q^2-q+1)}{\hat{k}} \\ n_2 &= \frac{-2(5q+1)(7q-3)(13q-4)}{k} = \frac{-(5q+1)(7q-3)(13q-4)}{\hat{k}} \\ m_0 &= \frac{2(7q-3)(13q-4)(3q^2-q+1)}{k} = \frac{(7q-3)(13q-4)(3q^2-q+1)}{\hat{k}} \\ m_\infty &= \frac{2(5q+1)(13q-4)(3q^2-q+1)}{k} = \frac{(5q+1)(13q-4)(3q^2-q+1)}{\hat{k}}, \end{aligned}$$

where

$$\hat{k} = \gcd \begin{pmatrix} 2(5q+1)(7q-3)(3q^2-q+1), \\ (5q+1)(7q-3)(13q-4), \\ (7q-3)(13q-4)(3q^2-q+1), \\ (5q+1)(13q-4)(3q^2-q+1) \end{pmatrix}.$$

The appendix contains a table representing a sample family of these solutions as well as the classic smooth Koiso-Sakane example from Example 4.2. Using (4), we calculated the (orbifold) index  $\mathcal{I}_{\mathbf{n},\mathbf{m}}$  of each  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ .

**4.5. Extremal and CSC Kähler metrics.** More generally, the admissible metrics are *extremal*, as defined by Calabi [Cal82], if and only if the scalar curvature of  $g$ , which is given as a function of  $\mathfrak{z}$  by

$$(36) \quad \text{Scal}(g) = \frac{2s_1 r_1}{1+r_1 \mathfrak{z}} + \frac{2s_2 r_2}{1+r_2 \mathfrak{z}} - \frac{F''(\mathfrak{z})}{p_c(\mathfrak{z})},$$

is a holomorphic potential, i.e., a linear affine function of  $\mathfrak{z}$ . Following the arguments in [ACGTF08a] and considering the orbifold case at hand, it is easy to see that the proof of Proposition 11 together with Section 2.2 in [ACGTF08a] adapts to give us the following result.

**Proposition 4.13.** *Any Kähler class  $\Omega_{\mathbf{r}}$  on  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  admits an admissible extremal metric with scalar curvature equal to an affine linear function of  $\mathfrak{z}$ . Moreover, this metric is (positive) CSC (i.e., the function is constant) if and only if  $\alpha_0 \beta_1 - \alpha_1 \beta_0 = 0$ ,*

where

$$(37) \quad \begin{aligned} \alpha_r &= \int_{-1}^1 t^r p_c(t) dt \\ \beta_r &= \int_{-1}^1 \left( r_1 s_1 (1 + r_2 t) + r_2 s_2 (1 + r_1 t) \right) t^r dt \\ &\quad + (-1)^r p_c(-1)/m_\infty + p_c(1)/m_0. \end{aligned}$$

In the smooth case, this has been thoroughly explored in [Gua95], [Hwa94], and [HS02], as well as Section 3.4 of [ACGTF08a]. Here we will just mention that with  $s_1 = 2/n_1$  and  $s_2 = 2/n_2$ , the equation  $\alpha_0\beta_1 - \alpha_1\beta_0 = 0$  is equivalent to  $f(r_1, r_2) = 0$  where

$$(38) \quad \begin{aligned} f(r_1, r_2) &= 9(m_0 - m_\infty)n_1n_2 - 6(m_0 + m_\infty)n_1n_2(r_1 + r_2) + 6(m_0 - m_\infty)n_1n_2r_1r_2 \\ &\quad + 3n_2(4m_0m_\infty - n_1(m_0 - m_\infty))r_1^2 + 3n_1(4m_0m_\infty - n_2(m_0 - m_\infty))r_2^2 \\ &\quad - (4m_0m_\infty(n_1 + n_2) - 3(m_0 - m_\infty)n_1n_2)r_1^2r_2^2. \end{aligned}$$

**Proposition 4.14.** *For any value of  $n_1, n_2 \neq 0$ , there exist a choice of  $m_0$  and  $m_\infty$  such that  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  admits Kähler classes  $\Omega_{\mathbf{r}}$  with admissible constant scalar curvature Kähler metrics.*

*Proof.* In the case of  $n_1n_2 < 0$  we can choose  $m_0 = m_\infty$  and the existence of CSC Kähler classes follows from [Hwa94] (see also [ACGTF08a] for details using the present notation). In the case of  $n_1n_2 > 0$ , notice that  $f(0, 0) = 9(m_0 - m_\infty)n_1n_2$  while  $f(1, 1) = 8m_\infty(m_0(n_1 + n_2) - 3n_1n_2)$ . So, if we choose  $m_0$  and  $m_\infty$  such that  $m_\infty > m_0 > \frac{3n_1n_2}{n_1+n_2}$ , we have  $f(0, 0) < 0$  and  $f(1, 1) > 0$ . This means that any continuous curve going from  $(0, 0)$  to  $(1, 1)$  in the square  $0 < r_1, r_2 < 1$  must contain at least one point  $(r_1, r_2)$  where  $f$  vanishes. This ensures the existence of solutions  $0 < r_1, r_2 < 1$  to the equation  $f(r_1, r_2) = 0$ . In turn, this implies the existence of classes  $\Omega_{\mathbf{r}}$  with admissible constant scalar curvature Kähler metrics.  $\square$

**Example 4.4.** When  $m_0 = m_\infty = 1$ ,  $n_1 = 1$ , and  $n_2 = -1$ , we have  $f(r_1, r_2) = -12(-1 + r_1 - r_2)(r_1 + r_2)$ , reconfirming the smooth CSC Kähler metrics on  $\mathbb{P}(\mathbb{1} \oplus \mathcal{O}(1, -1)) \rightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  as stated in Theorem 9 of [ACGTF08a].

**Remark 4.15.** Note that in general there is no guarantee the CSC Kähler classes  $\Omega_{\mathbf{r}}$  from Proposition 4.14 are rational (i.e. both  $r_1$  and  $r_2$  are rational) and thus useful from the Sasakian geometry point of view. With the constraint that  $r_1, r_2$  are rational, the equation  $f(r_1, r_2) = 0$  is diophantine in nature. Suppose  $f(r_1, r_2) \neq 0$  for some rational class  $\Omega_{\mathbf{r}}$  and assume we have a Boothby-Wang constructed Sasaki manifold defined by an integer orbifold Kähler class obtained by an appropriate rescale of  $\Omega_{\mathbf{r}}$ . We shall see in the next subsection that then we will have a CSC Sasaki metric somewhere else in the Sasaki cone of this Sasaki manifold. The key to seeing this is to use the connection discovered by Apostolov and Calderbank in [AC21] between the so-called *weighted extremal metrics* and extremal metrics appearing as transverse structures for different rays in the same Sasaki cone.

**4.6. Weighted Extremal Metrics.** Let  $(M, g, \omega)$  be a Kähler orbifold of complex dimension  $m$ ,  $f$  a positive Killing potential on  $M$ , and (weight)  $p \in \mathbb{R}$ . Then the

$(f, p)$ -Scalar curvature of  $g$  is given by

$$(39) \quad Scal_{f,p}(g) = f^2 Scal(g) - 2(p-1)f\Delta_g f - p(p-1)|df|_g^2,$$

If  $Scal_{f,p}(g)$  is a Killing potential,  $g$  is said to be a  $(f, p)$ -extremal Kähler metric. The case  $p = 2m$  has been studied by several people and is interesting due to the fact that  $Scal_{f,2m}(g)$  computes the scalar curvature of the Hermitian metric  $h = f^{-2}g$ .

However, the case of interest to us here is when  $p = m + 2$ . This case is related to the study of extremal Sasaki metrics [AC21, ACL21, AJL21]. Indeed, if we assume that the Kähler class  $[\omega/2\pi]$  is an integer orbifold class giving a Boothby-Wang constructed (smooth) Sasaki manifold over  $(M, g, \omega)$ , then  $\frac{Scal_{f,m+2}(g)}{f}$  is equal to the transverse scalar curvature of a certain Sasaki structure (determined by  $f$ ) in the Sasaki cone. More precisely, if  $\chi$  is the Reeb vector field of the Sasaki structure coming directly from the Boothby-Wang construction over  $(M, g, \omega)$  and  $f$  is viewed as a pull-back to the Sasaki manifold, then (mod  $\mathcal{D}$ )  $\xi := f\chi$  is a Reeb vector field in the Sasaki cone giving a new Sasaki structure. While the pull-back from  $M$  of  $Scal(g)$  is the Tanaka-Webster scalar curvature of the Tanaka-Webster connection induced by  $\chi$ , the expression  $\frac{Scal_{f,m+2}(g)}{f}$  pulls back from  $M$  to be the Tanaka-Webster scalar curvature of the Tanaka-Webster connection induced by  $\xi$ . The latter is then also identified with the transverse scalar curvature of the Sasaki structure defined by  $\xi$ . This fact is seen from the details of the proof of Lemma 3 in [AC21]. As also follows from [AC21] (see their Theorem 1), the Sasaki structure determined by  $f$  is extremal if and only if  $g$  is  $(f, m+2)$ -extremal.

Now, returning to our orbifolds at hand we have that  $m = 3$  and so  $m + 2 = 5$ . Note that  $|\mathfrak{z} + b|$  for  $b \in \mathbb{R}$  such that  $|b| > 1$  defines a Killing potential on  $(S_{\mathfrak{n}}, \Delta_{\mathfrak{m}})$ .

It is not hard to check that Section 2 and the existence result in the second half of Theorem 3.1. of [AMTF] adapts to get us the next proposition. Note in particular, that whether we use  $f = \mathfrak{z} + b$  (when  $b > 1$ ) or  $f = -(\mathfrak{z} + b)$  (when  $b < -1$ ), the formula for  $Scal_{f,p}$  in (39) will give us the same result, i.e., the right hand side of (19) in [AMTF]. In general the assumption of  $b > 1$  in [AMTF] is merely practical and all the arguments are easily adapted to include the  $b < -1$  case as well. Further, the root counting argument in the proof of Theorem 3.1. in [AMTF] is not affected by our mild orbifold conditions.

**Proposition 4.16.** *Let  $b \in \mathbb{R}$  such that  $|b| > 1$ . Any Kähler class  $\Omega_{\mathfrak{r}}$  on  $(S_{\mathfrak{n}}, \Delta_{\mathfrak{m}})$  admits a  $(|\mathfrak{z} + b|, 5)$ -extremal Kähler metric with weighted scalar curvature  $Scal_{|\mathfrak{z}+b|,5} = A_1\mathfrak{z} + A_2$  for constants  $A_1, A_2$  given as the unique solutions of the following linear system*

$$(40) \quad \begin{aligned} \alpha_{1,-6}A_1 + \alpha_{0,-6}A_2 &= 2\beta_{0,-4} \\ \alpha_{2,-6}A_1 + \alpha_{1,-6}A_2 &= 2\beta_{1,-4}, \end{aligned}$$

where

$$(41) \quad \begin{aligned} \alpha_{r,-6} &= \int_{-1}^1 (t+b)^{-6} t^r p_c(t) dt \\ \beta_{r,-4} &= \int_{-1}^1 \left( \frac{2r_1}{n_1}(1+r_2t) + \frac{2r_2}{n_2}(1+r_1t) \right) t^r (t+b)^{-4} dt \\ &\quad + (-1)^r (b-1)^{-4} p_c(-1)/m_{\infty} + (1+b)^{-4} p_c(1)/m_0. \end{aligned}$$

Moreover, assuming that  $\Omega_{\mathbf{r}}$  is rational and we can form a Boothby-Wang constructed Sasaki manifold with respect to an appropriate rescale of  $\Omega_{\mathbf{r}}$ , the extremal Sasaki structure determined by  $f = |\mathfrak{z} + b|$  has constant scalar curvature if and only if  $\text{Scal}_{|\mathfrak{z}+b|,5} = A_1\mathfrak{z} + A_2$  is a constant multiple of  $|\mathfrak{z} + b|$ , i.e., if and only if  $A_1b - A_2 = 0$ .

Suppose a rational  $\Omega_{\mathbf{r}}$  is given on a specific  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ . Solving the linear system for  $A_1$  and  $A_2$  in Proposition 4.16 (note that  $\alpha_{1,-6}^2 - \alpha_{0,-6}\alpha_{2,-6} \neq 0$ ) and simplifying a bit, the equation  $A_1b - A_2 = 0$  may be re-written as  $h(b) = 0$  where

$$(42) \quad h(b) = (b^2 - 1)^7(b(\alpha_{1,-6}\beta_{0,-4} - \alpha_{0,-6}\beta_{1,-4}) - (\alpha_{1,-6}\beta_{1,-4} - \alpha_{2,-6}\beta_{0,-4})).$$

Using (41) we see that  $h(b)$  simplifies as a polynomial of degree 5 in the variable  $b$  with leading coefficient equal to  $\frac{2f(r_1, r_2)}{9m_0m_\infty n_1 n_2}$ , where  $f(r_1, r_2)$  is given by (38). Since further  $\lim_{b \rightarrow \pm 1^\mp} h(b) > 0$ , we conclude that unless  $f(r_1, r_2) = 0$  (in which case  $\Omega_{\mathbf{r}}$  itself has an admissible CSC representative), there must exist at least one value  $b \in (-\infty, -1) \cup (1, +\infty)$  such that  $h(b) = 0$ , hence  $A_1b - A_2 = 0$ , and therefore, for this  $b$  value, the extremal Sasaki structure determined by  $f = |\mathfrak{z} + b|$  has constant scalar curvature. If  $b$  is an irrational number, then the corresponding Sasaki structure is irregular.

**Theorem 4.17.** *Suppose  $\Omega_{\mathbf{r}}$  is a rational admissible Kähler class on a KS orbifold of the form  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$ . Let  $\mathcal{S}$  be a Boothby-Wang constructed Sasaki manifold given by an appropriate rescale of  $\Omega_{\mathbf{r}}$ . Then the corresponding Sasaki cone will always have a (possibly irregular) CSC-ray (up to isotopy).*

**Remark 4.18.** It is not too difficult (using (36) and (21)) to produce some admissible Kähler metric of positive scalar curvature in  $\Omega_{\mathbf{r}}$ . Using the corresponding Kähler form of this metric for the Boothby-Wang construction, we can use Lemma 5.2 in [BHL18]) to conclude that the CSC-ray alluded to in theorem above has positive transverse scalar curvature. This together with Theorem 4.17 proves the main theorem in the introduction.

**Remark 4.19.** The observations in this section - in particular, Proposition 4.16 and Theorem 4.17 - can be generalized to no-blown-down admissible manifolds in general [BHLTF]. Indeed, in [BHLTF] we use a more general version of Proposition 4.16 to answer the open question in Problem 6.1.2 of [BHLTF21] by providing a counter example where the Sasaki cone contains some extremal Sasaki structures, but no CSC Sasaki metric at all.

**Example 4.5.** Suppose  $m_1 = m_2 = 1$  and suppose  $n_1 > 4$ . Now let  $r_1 = \frac{5n_1^2 - 4}{n_1(n_1^2 + 4)}$ ,  $r_2 = 2/n_1$ , and  $n_2$  be equal to any positive integer. Note that since  $n_1 > 4$  we have ensured that  $r_i \in (0, 1)$  for  $i = 1, 2$  and  $r_1 \neq r_2$ . Then  $\Omega_{\mathbf{r}}$  is a rational (non-diagonally) admissible Kähler class on the smooth KS manifold  $(S_{\mathbf{n}})$ . Note that since  $\Omega_{\mathbf{r}}$  is non-diagonally admissible, we know from Theorem 4.3 that  $S_{\mathbf{n}}$  (with any appropriate rescale of  $\Omega_{\mathbf{r}}$ ) is not the quotient of a regular ray in the  $\mathbf{w}$ -cone of the type of  $S_{\mathbf{w}}^3$ -join described in Theorem 4.3. Further, it can be shown (as we will do near the end of this example) that  $S_{\mathbf{n}}$  (with any appropriate rescale of  $\Omega_{\mathbf{r}}$ ) is also not the quotient of the regular ray in the  $\mathfrak{t}_{spher}^+$  cone of a Yamazaki fiber join over  $\mathbb{C}P^1 \times \mathbb{C}P^1$ . The Boothby-Wang constructed

Sasaki manifold over  $S_{\mathbf{n}}$ , given by an appropriate rescale of  $\Omega_{\mathbf{r}}$  is thus in a certain sense a truly new example.

By Theorem 8 and Proposition 11 of [ACGTF08a] we know that the Kähler class  $\Omega_{\mathbf{r}}$ , while having an extremal admissible Kähler metric representative, does not admit a CSC Kähler metric. [This fact can also easily be checked by seeing that  $f(r_1, r_2)$  from (38) cannot be zero in this case.] We now calculate that with the given choices of  $(\mathbf{m}, \mathbf{n}, \mathbf{r})$ , we have that  $h(b)$  from (42) is equal to

$$h(b) = \frac{4(n_1 - 2b)p(b)}{9n_1^5 (n_1^2 + 4)^2 n_2}$$

where  $p(b)$  is given by

$$\begin{aligned} p(b) &= 3n_1^8 n_2 - 4n_1^7 - 13n_1^6 n_2 + 268n_1^5 + 1252n_1^4 n_2 - 544n_1^3 - 1456n_1^2 n_2 + 192n_1 + 192n_2 \\ &\quad - 4n_1 (9n_1^6 n_2 + 20n_1^5 + 482n_1^4 n_2 + 64n_1^3 - 464n_1^2 n_2 - 64n_1 + 224n_2) b \\ &\quad + 16 (n_1^7 + 79n_1^6 n_2 - 17n_1^5 - 35n_1^4 n_2 + 56n_1^3 + 32n_1^2 n_2 - 16n_1 - 16n_2) b^2 \\ &\quad - 4n_1 (63n_1^6 n_2 - 20n_1^5 + 106n_1^4 n_2 - 64n_1^3 - 16n_1^2 n_2 + 64n_1 - 32n_2) b^3 \\ &\quad + (21n_1^8 n_2 - 12n_1^7 + 21n_1^6 n_2 + 4n_1^5 + 268n_1^4 n_2 - 352n_1^3 - 208n_1^2 n_2 + 64n_1 + 64n_2) b^4. \end{aligned}$$

Clearly,  $b = n_1/2$  is a rational root of  $h(b)$ . Note that since  $n_1 > 4$  and  $n_2 > 0$ , we also have that

- The  $b^4$  coefficient of  $p(b)$  is positive
- $p(\pm 1) > 0$
- $p'(-1) < 0$  and  $p'(1) > 0$
- $f(b) := p''(b)$  is a concave up 2nd order polynomial with  $f(\pm 1) > 0$ ,  $f'(-1) < 0$ , and  $f'(1) > 0$ . Thus any roots of  $f(b)$  would be inside the interval  $(-1, 1)$ .

Putting this together we have that  $p(b)$  is positive and decreasing at  $b = -1$ , positive and increasing at  $b = 1$ , and concave up for  $|b| \geq 1$ . Hence  $p(b)$  has no roots in  $(-\infty, -1) \cup (1, +\infty)$  and hence  $b = n_1/2$  is the ONLY root of  $h(b)$ .

All in all we conclude that while the original regular ray coming from the Boothby-Wang constructed Sasaki manifold over  $S_{\mathbf{n}}$ , given by an appropriate rescale of  $\Omega_{\mathbf{r}}$  is not CSC, the CSC ray alluded to in Theorem 4.17 is quasi-regular. It would be interesting to explore what the transverse Kähler orbifold is for this ray. Conjecturely, it may be a KS orbifold.

Finally, let us finish this example by confirming that  $S_{\mathbf{n}}$  (with any appropriate rescale of  $\Omega_{\mathbf{r}}$ ) is also not the quotient of the regular ray in the  $\mathfrak{t}_{spher}^+$  cone of a Yamazaki fiber join over  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ :

For  $S_{\mathbf{n}}$  (with an appropriate rescale of  $\Omega_{\mathbf{r}}$ ) to be such a quotient, it follows from Section 4.3 and (23) that there should exist  $k_1^1, k_1^2, k_2^1, k_2^2 \in \mathbb{Z}^+$  such that

$$k_1^1 - k_2^1 = n_1, \quad k_1^2 - k_2^2 = n_2, \quad \frac{k_1^1 - k_2^1}{k_1^1 + k_2^1} = \frac{5n_1^2 - 4}{n_1 (n_1^2 + 4)}, \quad \frac{k_1^2 - k_2^2}{k_1^2 + k_2^2} = 2/n_1.$$

This would place further restrictions on  $n_1$  and  $n_2$  and, in particular, we would need  $n_1$  to satisfy that  $\frac{n_1^2(n_1^2+4)}{5n_1^2-4}$  is a positive integer (that positive integer is  $k_1^1 + k_2^1$ ). By the following elementary lemma this is not possible and so  $S_{\mathbf{n}}$  (with any appropriate rescale of  $\Omega_{\mathbf{r}}$ ) is not the quotient of the regular ray in the  $\mathfrak{t}_{spher}^+$  cone of a Yamazaki fiber join.

**Lemma 4.20.** *For all integers  $x \geq 3$ ,  $\frac{x^2(x^2+4)}{5x^2-4}$  is NOT an integer.*

*Proof.* Assume for contradiction that there exists an integer  $x \geq 3$  such that  $\frac{x^2(x^2+4)}{5x^2-4}$  is an integer. Let  $d > 1$  be a prime divisor of  $5x^2 - 4$ . Then, due to our assumption,  $d \mid x^2$  or  $d \mid (x^2 + 4)$ . If  $d \mid x^2$ , then, since also  $d \mid (5x^2 - 4)$ , we have that  $d \mid 4$  and so  $d = 2$ . If  $d \mid (x^2 + 4)$ , then, since again  $d \mid (5x^2 - 4)$ , we have that  $d \mid (5(x^2 + 4) - (5x^2 - 4))$ , i.e.,  $d \mid 24$ , and so  $d = 2$  or  $d = 3$ .

By considering the three possible cases  $x = 3k$ ,  $x = 3k + 1$ , and  $x = 3k + 2$  with  $k \in \mathbb{Z}$ , we realize that  $3 \nmid (5x^2 - 4)$ . Therefore,  $d = 2$  and  $5x^2 - 4$  is even. This implies that  $x^2$  is even and hence  $x$  is even. In particular,  $x \neq 3$  and moving forward we may assume  $x \geq 4$ .

Since  $5x^2 - 4$  is even, we have that  $(5x^2 - 4) = 2^i$  for some non-negative integer  $i$ . We also know that  $5x^2 - 4 \geq 76$  (since  $x \geq 4$ ), so  $i > 6$ .

Now, writing  $x = 2k$  for some  $k \in \mathbb{Z}^+$  we see that  $(5x^2 - 4) = 2^i$  implies that  $20k^2 - 4 = 2^i$  and hence  $5k^2 = 2^{i-2} + 1$ . Thus  $5k^2$  is odd, which gives us that  $k^2$  is odd and hence  $k$  is odd. Let us write  $k = 2l + 1$  for some  $l \in \mathbb{Z}^+$ . So  $x = 2(2l + 1)$ . Then

$$\begin{aligned} x^2(x^2 + 4) &= 2^2(4l^2 + 4l + 1)(2^2(4l^2 + 4l + 1) + 4) \\ &= 2^2(4l^2 + 4l + 1)(2^2(4l^2 + 4l + 2)) \\ &= 2^2(4l^2 + 4l + 1)(2^3(2l^2 + 2l + 1)) \\ &= 2^5(2(2l^2 + 2l) + 1)(2(l^2 + l) + 1). \end{aligned}$$

Since  $(2(2l^2 + 2l) + 1)(2(l^2 + l) + 1)$  is odd, we realize that  $\frac{x^2(x^2+4)}{5x^2-4} = \frac{(2(2l^2+2l)+1)(2(l^2+l)+1)}{2^{i-5}}$ . Since  $i > 6$ , this cannot possibly be an integer and hence we have arrived at a contradiction. This completes the proof of the lemma.  $\square$

As an immediate consequence of Theorem 4.17 we get the following special case of the existence result for toric Sasaki-Einstein metrics due to Futaki, Ono, and Wang [FOW09].

**Corollary 4.21.** *Suppose a KS orbifold of the form  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  is log Fano and consider a Boothby-Wang constructed Sasaki manifold over  $(S_{\mathbf{n}}, \Delta_{\mathbf{m}})$  given by some Kähler form representing  $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})/\mathcal{I}_{\mathbf{n}, \mathbf{m}}$ . Then the corresponding Sasaki cone will always have a (possibly irregular) Sasaki-Einstein structure (up to isotopy).*

*Proof.* First, we note that  $c_1(\mathcal{D}) = 0$  and so the CSC Sasaki metrics from Theorem 4.17 above are now  $\eta$ -Einstein. Second, we see from Proposition 5.3 in [BHL18]) that since the basic first Chern class of the initial Sasaki metric is positive (as a pullback of the positive class  $c_1^{orb}(S_{\mathbf{n}}, \Delta_{\mathbf{m}})/\mathcal{I}_{\mathbf{n}, \mathbf{m}}$ ), the average transverse scalar curvature of any Sasaki

structure in the Sasaki cone must be positive. In particular, the transverse (constant) scalar curvature of any  $\eta$ -Einstein structure in the cone must be positive. This means that any  $\eta$ -Einstein ray admits a Sasaki-Einstein structure.  $\square$

**Remark 4.22.** Examples similar in spirit to Corollary 4.21, but also including non-toric cases have been given by Mabuchi and Nakagawa [MN13]. See also [AJL21] and references therein.

APPENDIX

$p_1$	$q_1$	$p_2$	$q_2$	$n_1$	$n_2$	$m_0$	$m_\infty$	$m$	$v_0$	$v_\infty$	$\mathcal{I}_{n,m}$
1	2	-1	15	5124072	-740316	6438801	4797538	126251	51	38	89
1	2	-1	14	775675	-120061	972325	726685	10235	95	71	83
1	2	-1	13	48	-8	60	45	15	4	3	7
1	2	-1	12	2080161	-375516	2591676	1951756	31996	81	61	71
1	2	-1	11	1462832	-288008	1815479	1373876	49067	37	28	65
1	2	-1	10	110483	-23919	136479	103887	2037	67	51	59
1	2	-1	9	129720	-31188	159330	122153	5311	30	23	53
1	2	-1	8	80401	-21730	98050	75850	1850	53	41	47
1	2	-1	7	25944	-8004	31349	24534	1363	23	18	41
1	2	-1	6	124527	-44733	148629	118141	3811	39	31	35
1	2	-1	5	59072	-25376	69296	56303	4331	16	13	29
1	2	-1	4	525	-280	600	504	24	25	21	23
1	2	-1	3	1440	-1008	1575	1400	175	9	8	17
1	2	-1	2	1	-1	1	1	1	1	1	1
1	2	1	2	51	51	85	45	5	17	9	13
1	2	1	3	10416	7224	15996	9331	1333	12	7	19
1	2	1	4	31217	16492	46004	28196	1484	31	19	25
1	2	1	5	8208	3496	11799	7452	621	19	12	31
1	2	1	6	30015	10701	42435	27347	943	45	29	37
1	2	1	7	54808	16796	76570	50065	2945	26	17	43
1	2	1	8	51389	13806	71154	47034	1206	59	39	49
1	2	1	9	552	132	759	506	253	3	2	5
1	2	1	10	1112447	239659	1521101	1021013	20837	73	49	61
1	2	1	11	36000	7056	49000	33075	1225	40	27	67
1	2	1	12	456837	82128	619440	420080	7120	87	59	73
1	2	1	13	3134336	520384	4236251	2884256	90133	47	32	79
1	2	1	14	466923	72013	629331	429939	6231	101	69	85
1	2	1	15	5522472	795204	7425486	5087833	137509	54	37	91

TABLE: This gives a sample of Kähler-Einstein orbifold solutions. See Example 4.3.

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