The Kähler Geometry of Bott Manifolds

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Gauge Theories, Monopoles, Moduli Spaces and Integrable Systems
A Conference honouring Jacques Hurtubise on his 60th birthday
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BON ANNIVERSAIRE JACQUES
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5 They are best approached through the notion of a Bott Tower which we now describe.
Consider **Closed Complex Manifolds** $M_k$ for $k = 0, 1, \ldots, n$ with $M_0 = \{pt\}$ and $M_k$ the total space of the $\mathbb{C}P^1$-bundle $\pi_k : \mathbb{P}(1 \oplus L_k) \to M_{k-1}$ giving the sequence

$$M_n \xrightarrow{\pi_n} M_{n-1} \xrightarrow{\pi_{n-1}} \cdots M_2 \xrightarrow{\pi_2} M_1 = \mathbb{C}P^1 \to \{pt\}$$

where $L_k$ is a holomorphic line bundle on $M_{k-1}$.
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3. Stage 2 Bott manifolds are nothing but **Hirzebruch surfaces**.

4. A **Bott tower** is a collection $(M_k, \pi_k, \sigma_0^k, \sigma_\infty^k)_{k=1}^n$ where $\sigma_0^k$ and $\sigma_\infty^k$ are the zero and infinity sections of $\mathcal{L}_k$, respectively.

5. The **Quotient Construction**: Any Bott tower is obtained from the complex torus action $(t_i)_{n_i=1}^n \in (\mathbb{C}^*)^n$ on $(z_j, w_j)_{n_j=1}^n \in (\mathbb{C}^2 \{0\})^n$ by

$$(t^i)_{n_i=1}^n : (z_j, w_j)_{n_j=1}^n \mapsto (t^j z_j, (\prod_{i=1}^n t^A_{ij}) w_j)_{n_j=1}^n$$

where $A_{ij}$ are the entries of a lower triangular unipotent integer-valued matrix $A$.

6. The **Cohomology Ring**: $H^\ast(M_n, \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \ldots, x_n] / I$ where $I$ is generated by $x_k y_k$ with $y_k = \sum_{n_j=1}^{n} A_{jk} x_j$.

7. **Problem**: When does the cohomology ring determine the diffeomorphism type? (Choi, Masuda, Panov, Suh)
Consider **Closed Complex Manifolds** $M_k$ for $k = 0, 1, \ldots, n$ with $M_0 = \{pt\}$ and $M_k$ the total space of the $\mathbb{C}P^1$-bundle $\pi_k: \mathbb{P}(\mathbb{1} \oplus \mathcal{L}_k) \rightarrow M_{k-1}$ giving the sequence

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Consider **Closed Complex Manifolds** $M_k$ for $k = 0, 1, \ldots, n$ with $M_0 = \{pt\}$ and $M_k$ the total space of the $\mathbb{CP}^1$-bundle $\pi_k: \mathbb{P}(1 \oplus L_k) \to M_{k-1}$ giving the sequence

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$$(t_i)_{i=1}^n : (z_j, w_j)_{j=1}^n \mapsto (t_jz_j, \left(\prod_{i=1}^n t_i^{A_j^i}\right)w_j)_{j=1}^n$$

where $A_j^i$ are the entries of a **lower triangular unipotent integer-valued matrix** $A$. 
Consider **Closed Complex Manifolds** $M_k$ for $k = 0, 1, \ldots, n$ with $M_0 = \{pt\}$ and $M_k$ the total space of the $\mathbb{CP}^1$-bundle $\pi_k : \mathbb{P}(L \oplus L_k) \to M_{k-1}$ giving the sequence

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where $A^i_j$ are the entries of a lower triangular unipotent integer-valued matrix $A$.
5. The **Cohomology Ring**: $H^*(M_n, \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \ldots, x_n]/J$ where $J$ is generated by $x_k y_k$ with $y_k = \sum_{j=1}^n A^i_j x_j$. 
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**Problem**: When does the cohomology ring determine the **diffeomorphism type**? (Choi, Masuda, Panov, Suh)
Bott towers form the objects $g^0_{BT}$ of a groupoid $g^{BT}$ (Bott tower groupoid) whose morphisms $g^1_{BT}$ are $T^n$ equivariant biholomorphisms.
Bott towers form the objects $g^{BT}_0$ of a groupoid $g^{BT}$ (Bott tower groupoid) whose morphisms $g^{BT}_1$ are $T^n$ equivariant biholomorphisms.

Elements of $g^{BT}_1$ give equivalences of Bott towers.
1. Bott towers form the objects $G^B_0$ of a groupoid $G^B_T$ (Bott tower groupoid) whose morphisms $G^B_1$ are $\mathbb{T}^n$ equivariant biholomorphisms.

2. Elements of $G^B_1$ give equivalences of Bott towers.

3. The set of $n$ dimensional Bott towers $G^B_0$ can be identified with the set of $n \times n$ lower triangular unipotent matrices $A$ over the integers $\mathbb{Z}$, hence with $\mathbb{Z}^{n(n-1)/2}$. 
Bott towers form the objects $\mathcal{G}_{\text{BT}}^0$ of a groupoid $\mathcal{G}_{\text{BT}}$ (Bott tower groupoid) whose morphisms $\mathcal{G}_{\text{BT}}^1$ are $\mathbb{T}^n$ equivariant biholomorphisms.

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The isotropy subgroup $\text{Iso}(M_n(A)) \subset \mathcal{G}_{\text{BT}}^1$ at $M_n(A) \in \mathcal{G}_{\text{BT}}^0$ is $\text{Aut}(M_n(A))$. 

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The quotient stack $\mathcal{G}_0^{BT} / \mathcal{G}_1^{BT}$ is the set of Bott manifolds.
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6 A Bott manifold is a smooth projective toric variety whose polytope $P$ is combinatorially equivalent to an $n$-cube.
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A Bott manifold is a smooth projective toric variety whose polytope $P$ is combinatorially equivalent to an $n$-cube.

A Bott manifold has a $\mathbb{T}^n$ invariant compatible Kähler form $\omega$. In fact its Kähler cone $\mathcal{K}$ is $n$ dimensional.
Bott towers form the objects $G_{0}^{BT}$ of a groupoid $G^{BT}$ (Bott tower groupoid) whose morphisms $G_{1}^{BT}$ are $\mathbb{T}^{n}$ equivariant biholomorphisms.

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$\mathcal{K}$ is isomorphic to the ample cone $\mathcal{A}$ of $\mathbb{T}^{n}$ invariant ample divisors.
Given a **Bott tower** $M_n(A)$ choose a $\mathbb{T}^n$ invariant compatible symplectic form $\omega$. Then say that the symplectic manifold $(M^{2n}, \omega)$ is of **Bott type**.
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$N_B(M^{2n}, \omega)$ denotes the number of $\mathbb{T}^n$ invariant complex structures that are compatible with $(M^{2n}, \omega)$ which is isomorphic to the number of compatible Bott manifolds.
Symplectic Structures

- Given a **Bott tower** $M_n(A)$ choose a $\mathbb{T}^n$ invariant compatible symplectic form $\omega$. Then say that the **symplectic manifold** $(M^{2n}, \omega)$ is of **Bott type**.
- $N_B(M^{2n}, \omega)$ denotes the number of $\mathbb{T}^n$ invariant **complex structures** that are compatible with $(M^{2n}, \omega)$ which is isomorphic to the number of compatible **Bott manifolds**.
- The number $N_B(M^{2n}, \omega)$ is **finite** (McDuff). 

**Theorem (1)**

Let $(M^{2n}, \omega)$ be a symplectic manifold of Bott type. Then the number of conjugacy classes of maximal tori of dimension $n$ in the symplectomorphism group $\text{Symp}(M^{2n}, \omega)$ equals $N_B(M^{2n}, \omega)$. 

**INGREDIENTS OF PROOF**

- To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional torus in $\text{Symp}(M^{2n}, \omega)$ and hence its conjugacy class.
- There is a bijection between the set of Bott manifolds compatible with $(M^{2n}, \omega)$ and the set of $G_{BT1}$ orbits in $G_{BT0}$ with an element compatible with $\omega$.
- Elements of $G_{BT0}/G_{BT1}$ have distinct complex structures.
- Then a cohomological rigidity result of Choi-Suh and Masuda-Panov gives a bijection between the set of Bott manifolds compatible with $(M^{2n}, \omega)$ and the set of $\mathbb{T}^n$ invariant integrable complex structures $J$ compatible with $(M^{2n}, \omega)$.
- Delzant’s Theorem $\Rightarrow$ $(M^{2n}, \omega, J)$ is a smooth projective toric variety.
- The corresponding Delzant polytope $P$ is combinatorially equivalent to $n$ cube.
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- Given a Bott tower \( M_n(A) \) choose a \( \mathbb{T}^n \) invariant compatible symplectic form \( \omega \). Then say that the symplectic manifold \((M^{2n}, \omega)\) is of Bott type
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Ingredients of Proof:

- Delzant's Theorem \(\Rightarrow\) \((M^{2n}, \omega, J)\) is a smooth projective toric variety.
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- To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional **torus** in $\text{Symp}(M^{2n}, \omega)$ and hence its **conjugacy class**.
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### Theorem (1)

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- To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional **torus** in $\text{Symp}(M^{2n}, \omega)$ and hence its **conjugacy class**.
- There is a **bijection** between the set of **Bott manifolds** compatible with $(M^{2n}, \omega)$ and the set of $G^B_T$ **orbits** in $G^{B_T}_0$ with an element compatible with $\omega$. 

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### Theorem (1)

**Let** $(M^{2n}, \omega)$ **be a symplectic manifold of Bott type.** Then the number of conjugacy classes of **maximal tori** of dimension $n$ in the symplectomorphism group $\text{Symp}(M^{2n}, \omega)$ equals $N_B(M^{2n}, \omega)$.

**INGREDIENTS OF PROOF:**

- To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional **torus** in $\text{Symp}(M^{2n}, \omega)$ and hence its **conjugacy class**.
- There is a **bijection** between the set of **Bott manifolds** compatible with $(M^{2n}, \omega)$ and the set of $g_{BT}^1$ **orbits** in $g_{BT}^0$ with an element compatible with $\omega$.
- Elements of $g_{BT}^0 / g_{BT}^1$ have **distinct complex structures**.
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- To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional **torus** in $\text{Symp}(M^{2n}, \omega)$ and hence its **conjugacy class**.
- There is a **bijection** between the set of **Bott manifolds** compatible with $(M^{2n}, \omega)$ and the set of $G_1^{BT}$ **orbits** in $G_0^{BT}$ with an element compatible with $\omega$.
- Elements of $G_0^{BT} / G_1^{BT}$ have **distinct complex structures**.
- Then a **cohomological rigidity** result of Choi-Suh and Masuda-Panov gives a **bijection** between the set of **Bott manifolds compatible** with $(M^{2n}, \omega)$ and the set of $T^n$ invariant integrable **complex structures** $J$ compatible with $(M^{2n}, \omega)$.
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- Given a Bott tower $M_n(A)$ choose a $\mathbb{T}^n$ invariant compatible symplectic form $\omega$. Then say that the symplectic manifold $(M^{2n}, \omega)$ is of Bott type.
- $N_B(M^{2n}, \omega)$ denotes the number of $\mathbb{T}^n$ invariant complex structures that are compatible with $(M^{2n}, \omega)$ which is isomorphic to the number of compatible Bott manifolds.
- The number $N_B(M^{2n}, \omega)$ is finite (McDuff).

**Theorem (1)**

Let $(M^{2n}, \omega)$ be a symplectic manifold of Bott type. Then the number of conjugacy classes of maximal tori of dimension $n$ in the symplectomorphism group $\text{Symp}(M^{2n}, \omega)$ equals $N_B(M^{2n}, \omega)$.

**INGREDIENTS OF PROOF:**

- To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional torus in $\text{Symp}(M^{2n}, \omega)$ and hence its conjugacy class.
- There is a bijection between the set of Bott manifolds compatible with $(M^{2n}, \omega)$ and the set of $g_1^{BT}$ orbits in $g_0^{BT}$ with an element compatible with $\omega$.
- Elements of $g_0^{BT}/g_1^{BT}$ have distinct complex structures.
- Then a cohomological rigidity result of Choi-Suh and Masuda-Panov gives a bijection between the set of Bott manifolds compatible with $(M^{2n}, \omega)$ and the set of $T^n$ invariant integrable complex structures $J$ compatible with $(M^{2n}, \omega)$.
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Symplectic Structures

Given a **Bott tower** $M_n(A)$ choose a $\mathbb{T}^n$ invariant compatible symplectic form $\omega$. Then say that the **symplectic manifold** $(M^{2n}, \omega)$ is of **Bott type**.

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**The Kähler Geometry of Bott Manifolds**

August 25, 2017Gauge Theories, Monopoles, Moduli SpacesA Conference honouring Jacques Hurtubise on his 60th birthdayMontreal, Quebec, Canada
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**INGREDIENTS OF PROOF:**

- To each Bott tower $M_n(A)$ compatible with $(M^{2n}, \omega)$ we can assign an $n$ dimensional torus in $\text{Symp}(M^{2n}, \omega)$ and hence its **conjugacy class**.
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- The corresponding Delzant polytope $P$ is combinatorially equivalent to $n$ cube.
- **smooth projective toric varieties** $\{M_\mathcal{F}\} \approx \{\mathcal{F}_M\}$ smooth normal fans $\mathcal{F}$ over $\{P\}$. 
An Example

Karshon proved the Theorem for Hirzebruch surface (stage 2 Bott manifolds) and gave a formula for $N_B(M^4, \omega)$.
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Example: **Stage 3 Bott manifolds** diffeomorphic to $(S^2)^3 = S^2 \times S^2 \times S^2$ with symplectic form $\omega_{k_1,k_2,k_3}$ with $k_i \in \mathbb{R}^+$ and ordered $0 < k_3 \leq k_2 \leq k_1$. 
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$((S^2)^3, \omega_{k_1,k_2,k_3})$ is Kähler with respect to the Bott manifold $M_3(2a, 2b, 2c)$ if and only if one of the following two cases hold:
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Then $N_B(M^6, \omega_{k_1,k_2,k_3}) = \sum_{j=0}^{b_{\text{max}}} \left\lfloor \frac{k_1-jk_3}{k_2} \right\rfloor + \sum_{j=1}^{c_{\text{max}}} \left\lfloor \frac{k_1}{k_2-jk_3} \right\rfloor$ where $\left\lfloor \frac{a}{b} \right\rfloor$ is least integer greater than or equal to $\frac{a}{b}$.
Calabi’s Extremal Kähler Metrics

- **Calabi Energy functional**  \( E(g) = \int_M s_g^2 d\mu_g \), where \( s_g \) is the scalar curvature of a Kähler metric \( g \) with Kähler form \( \omega \) on a compact complex manifold \( M \).
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- $\text{Aut}(X_F)_0$ is reductive if and only if and only if $R(\mathcal{F}) = -R(\mathcal{F})$. 
The Generalized Calabi Construction

**Ingredients**

- Given a **principal** $T^\ell$ bundle $\mathcal{G} \rightarrow \mathbb{C}P^1$ construct the associated fiber bundle $M = \mathcal{G} \times_{T^\ell} V$ with **fiber** $V$ where $V$ is a compact **toric Kähler manifold** of complex dimension $\ell$.

- The **Moment map** $\mathcal{Z} : V \rightarrow t^{\ast} \ell$ with image the Delzant polytope $P$ in the dual of the Lie algebra $t^{\ast} \ell$.

- A principal connection on $G$ with curvature $\omega_{FS} \otimes p \in C^\infty(\Sigma, \bigwedge^1, 1 \otimes t^{\ell})$ where $\omega_{FS}$ is the Fubini-Study form on $\mathbb{C}P^1$ and $p \in t^{\ell}$.

- A constant $\hat{c} \in \mathbb{R}$ such that the $(1,1)$-form $\hat{c} \omega_{\Sigma} + \langle v, \omega_{\Sigma} \otimes p \rangle$ is positive for $v \in P$.

- The **generalized Calabi data** on $\mathring{M} = \mathcal{G} \times_{T^\ell} \mathring{P}$ is $g = (\hat{c} + \langle p, \mathcal{Z} \rangle) g_{\mathbb{C}P^1} + \langle d\mathcal{Z}, G, d\mathcal{Z} \rangle + \langle \theta, H, \theta \rangle \omega = (\hat{c} + \langle p, \mathcal{Z} \rangle) \omega_{\mathbb{C}P^1} + \langle d\mathcal{Z} \wedge \theta \rangle d\theta = \omega_{\mathbb{C}P^1} \otimes p$, where $G = \text{Hess}(U) = H - 1$, $U$ is the symplectic potential of the chosen toric Kähler structure $g_V$ on $V$, and $\langle \cdot, \cdot, \cdot \rangle$ denotes the pointwise contraction $g^{\ast} \times S^2 t^{\ell} \times t^{\ast} \ell \rightarrow \mathbb{R}$ or the dual contraction.

- Get compatible Kähler metrics on $M$ and

**Lemma (Apostolov, Calderbank, Gauduchon, Tønnesen-Friedmann)**

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A constant $\hat{c} \in \mathbb{R}$ such that the $(1,1)$-form $\hat{c} \omega_{\Sigma} + \langle v, \omega_{\Sigma} \otimes p \rangle$ is positive for $v \in P$. 

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where $G = \text{Hess}(U) = H^{-1}$, $U$ is the **symplectic potential** of the chosen *toric Kähler* structure $g_V$ on $V$, and $\langle \cdot, \cdot, \cdot \rangle$ denotes the pointwise **contraction** $g^* \times S^2 t_\ell \times t_\ell^* \rightarrow \mathbb{R}$ or the dual contraction.
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- Given a principal $T^\ell$ bundle $\mathcal{G} \rightarrow \mathbb{C}P^1$ construct the associated fiber bundle $M = \mathcal{G} \times_{T^\ell} V$ with fiber $V$ where $V$ is a compact toric Kähler manifold of complex dimension $\ell$.

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Any Bott manifold $M_n$ admits a toric extremal Kähler metric. Alternatively, the extremal Kähler cone $\mathcal{E}(M_n)$ is a non-empty open cone in the Kähler cone $\mathcal{K}(M_n)$. 
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**Problem**

*Describe the extremal Kähler cone $\mathcal{E}(M_n)$. In particular, when is $\mathcal{E}(M_n) = \mathcal{K}(M_n)$?*
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Any Bott manifold $M_n$ admits a toric extremal Kähler metric. Alternatively, the extremal Kähler cone $\mathcal{E}(M_n)$ is a non-empty open cone in the Kähler cone $\mathcal{K}(M_n)$.

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Let $M_n(A)$ be a Bott tower. If the elements below the diagonal of any row of the lower triangular unipotent matrix $A$ all have the same sign, then $M_n(A)$ does not admit a Kähler metric with constant scalar curvature. In particular, if $A^1_2 \neq 0$ then $M_n(A)$ does not admit a Kähler metric with constant scalar curvature.
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- The proof essentially follows from Demazure’s Theorem by computing possible root vectors.
Following Choi-Suh we let \( t \) denote the number of non-trivial topological fibrations in the defining sequence of a Bott tower \( M_n(A) \). It is well defined and \( t = 0, 1, \ldots, n - 1 \).
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**Theorem**

Let $M_n(A)$ be a Bott tower with twist $t$ and matrix $A$ of the form

$$A = \begin{pmatrix}
\tilde{A} & 0 & \cdots & 0 \\
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\end{pmatrix}, \quad A^i_j \in \mathbb{Z},$$

where $\tilde{A} \neq \mathbb{1}_n$ has 0-twist. Then $M_n(A)$ does not admit a compatible Kähler metric with constant scalar curvature. In particular, if $t = 0$ and the Bott manifold $M_n(A)$ has a compatible Kähler metric with constant scalar curvature, then it is the product $(\mathbb{C}P^1)^n$. 

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- The only 0 twist Fano Bott manifold is the product $(\mathbb{CP}^1)^n$.

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Charles Boyer (University of New Mexico)
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- A 1 twist Bott manifold is diffeomorphic to a non-trivial $\mathbb{CP}^1$ bundle over $(S^2)^{n-1}$.
- The **diffeomorphism type** of a 1 twist Bott manifold is determined by its **cohomology ring** (Choi-Suh).

Consider Bott manifolds $M_n(k)$ with $k = (k_1, \ldots, k_{n-1})$ satisfying $k_1 k_2 \cdots k_{n-1} \neq 0$ with a matrix $A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ k_1 & k_2 & \cdots & k_{n-1} & 1 \end{pmatrix}$.

Then $M_n(k)$ admits an extremal Kähler metric in every Kähler class. If not all $k_i$ have the same sign, then some of these metrics will have constant scalar curvature.

$M_n(k)$ is Fano if and only if $k_i = \pm 1$ for all $i$.

The monotone Kähler class admits a Kähler-Ricci soliton which is Kähler-Einstein if and only if the number of $+1$ in $k$ equals the number of $-1$ in $k$. Much of this case recovers previous work of Koiso, Sakane, Guan, Hwang, and Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman.
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There are 5 stage 3 Fano Bott manifolds \( M^3(A_1^2, A_1^3, A_2^3) \), up to equivalence, with representatives \( M^3(0, 0, 0) \), \( M^3(0, 1, -1) \), \( M^3(0, 1, 1) \), \( M^3(1, 0, 0) \), \( M^3(-1, 0, 1) \). The first 2 admit constant scalar curvature Kähler metrics, the remaining 3 do not.

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There are many extremal orbifold Kähler metrics on stage 3 Bott manifolds. There is an uncountable number of extremal almost Kähler metrics (non integrable) on stage 3 Bott manifolds. These are not constant scalar curvature.
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- A 0 twist **stage 3 Bott manifold** is $M_3(2A_3^1, 2A_3^1, 0)$ or $M_3(2A_2^1, 2A_2^1A_3^2, 2A_3^2)$. The former is type 3 whereas generically the later is type 1.

- There are 5 stage 3 **Fano Bott manifolds** $M_3(A_1^1, A_3^1, A_3^2)$, up to equivalence, with representatives $M_3(0, 0, 0), M_3(0, 1, -1), M_3(0, 1, 1), M_3(1, 0, 0), M_3(-1, 0, 1)$. The first 2 admit **constant scalar curvature** Kähler metrics, the remaining 3 do not.

- There is an infinite number of pairs of **c-projectively equivalent** (Calderbank-Eastwood-Mateev-Neusser) **constant scalar curvature** Kähler metrics that are not affinely equivalent.

- There are many **extremal orbifold** Kähler metrics on **stage 3 Bott manifolds**.

- There is an **uncountable** number of **extremal almost Kähler metrics** (non integrable) on **stage 3 Bott manifolds**. These are not **constant scalar curvature**
THANK YOU FOR YOUR ATTENTION

and

HAPPY BIRTHDAY JACQUES