Extremal Sasakian Geometry

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Problems:

Given a contact structure or isotopy class of contact structures:

1. Determine the space of compatible Sasakian structures.
2. Determine the moduli space of extremal Sasakian structures; those of constant scalar curvature (cscS).

• **Contact Manifold** M (compact). *A* **contact 1-form** η *such that*

$$\eta \wedge (d\eta)^n \neq 0.$$

defines a **contact structure**

$$\eta' \sim \eta \iff \eta' = f\eta$$

for some $f \neq 0$, *take* $f > 0$. *or equivalently a codimension 1 sub-bundle* $\mathcal{D} = \mathrm{Ker}\ \eta$ *of* TM *with a conformal symplectic structure.*

A **contact invariant***:* $c_1(\mathcal{D})$

Unique vector field ξ, *called the* **Reeb vector field**, *satisfying*

$$\xi \rfloor \eta = 1, \qquad \xi \rfloor d\eta = 0.$$

The **characteristic foliation** $\mathcal{F}_\xi$ *each leaf of* $\mathcal{F}_\xi$ *passes through any nbd* U *at most* k *times* $\Longleftrightarrow$ **quasi-regular**, $k = 1 \leftrightarrow$ *regular, otherwise* **irregular**

Quasi-regularity is strong, most contact 1-forms are irregular.

Contact bundle $\mathcal{D} \longrightarrow$ *choose* **almost complex structure** J *extend to* Φ *with* $\Phi\xi = 0$
with a compatible metric

$$g = d\eta \circ (\Phi \otimes 1\!\!1) + \eta \otimes \eta$$

Quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ *called* **contact metric structure**

The pair $(\mathcal{D}, J)$ *is a strictly pseudo-convex almost CR structure*.

Definition: *The structure* $\mathcal{S} = (\xi, \eta, \Phi, g)$ *is* **K-contact** *if* $\pounds_\xi g = 0$ *(or* $\pounds_\xi \Phi = 0$*). It is* **Sasakian** *if in addition* $(\mathcal{D}, J)$ *is integrable.*

Transverse Metric $g_{\mathcal{D}}$ *is Kähler*

Cone (Symplectization)

$C(M) = M \times \mathbb{R}^+$

symplectic form $d(r^2\eta), \; r \in \mathbb{R}^+$.

Cone Metric $g_C = dr^2 + r^2 g$

- g_C *is Kähler* $\iff$ g *is Sasaki* $\iff$ $g_{\mathcal{D}}$ *is Kähler.*

Sasaki-Kähler Sandwich

 $\Longleftrightarrow$  $\Longleftrightarrow$ 

Symmetries

- **Contactomorphism Group**

$\mathfrak{Con}(M,\mathcal{D}) =$

$\{\phi \in \mathfrak{Diff}(M) \mid \phi_* \mathcal{D} \subset \mathcal{D}\}.$

CR transformation group: $\mathfrak{CR}(\mathcal{D},J)$

$= \{\phi \in \mathfrak{Con}(M,\mathcal{D}) \mid \phi_* J = J \phi_*\}$

Have: $T^k \subset \mathfrak{CR}(\mathcal{D},J) \subset \mathfrak{Con}(M,\mathcal{D})$

T^k *a* **max'l torus** $0 \leq k \leq n+1$.

$\mathcal{J}(\mathcal{D})$ *space of compatible almost CR structures, then a map*

$\mathfrak{Q} : \mathcal{J}(\mathcal{D}) \to \{$*conjugacy classes of maximal tori*$\}$ *in* $\mathfrak{Con}(M,\mathcal{D})$

Sasaki cones

$\mathfrak{t}_k^+(\mathcal{D},J)=\{\xi\in\mathfrak{t}_k \mid \eta'(\xi)>0,\}$ *s.t.* $\mathcal{S}=(\xi,\eta,\Phi,g)\in(\mathcal{D},J)$ *is Sasakian*

• *finite dim'l moduli of Sasakian structures within CR structure*

$\kappa(\mathcal{D},J)=\mathfrak{t}_k^+(\mathcal{D},J)/\mathcal{W}(\mathcal{D},J)$

A given $\mathcal{D}$ can have many Sasaki cones $\mathfrak{t}_k^+(\mathcal{D},J_\alpha)$ labelled by complex structures, and $k=k(\alpha)$.

Get **bouquet** $\bigcup_\alpha \mathfrak{t}_{k(\alpha)}^+(\mathcal{D},J_\alpha)$ of Sasaki cones

• **Extremal Sasakian metrics**

(B-,Galicki,Simanca)

$E(g) = \int_M s_g^2 d\mu_g,$

• Deform contact structure

Vary $\eta \mapsto \eta + td^c\varphi,\ \varphi$ basic, gives critical point of $E(g) \iff \partial_g^{\#} s_g$ is transversely holomorphic. $s_g =$ scalar curvature.

Special case: constant scalar curvature Sasakian (cscS) . If $c_1(\mathcal{D}) = 0 \Rightarrow$ Sasaki-η-Einstein (SηE)

$\mathrm{Ric}_g = ag + b\eta \otimes \eta,\ a, b$ constants.

Sasaki-Einstein (SE) $b = 0$

Extremal Set $\mathfrak{e}(\mathcal{D},J)$

$\mathfrak{e}(\mathcal{D},J)\subset\mathfrak{t}_k^+(\mathcal{D},J)$ is open in Sasaki cone B-,Galicki,Simanca

If $\mathcal{S}=\mathcal{S}_1\in\mathfrak{e}(\mathcal{D},J)$ then entire ray $\mathcal{S}_a=(a^{-1}\xi,a\eta,\Phi,g_a)\in\mathfrak{e}(\mathcal{D},J)$

When is $\mathfrak{e}(\mathcal{D},J)=\mathfrak{t}_k^+(\mathcal{D},J)$?

Many ex's if $\dim\mathfrak{t}_k^+(\mathcal{D},J)=1$

- If $\dim\ \kappa(\mathcal{D},J)>1$, there are only two known cases when $\mathfrak{e}(\mathcal{D},J)=\mathfrak{t}_k^+(\mathcal{D},J)>1$.

(1) standard CR structure on S^{2n+1}

Toric ($\dim \kappa(\mathcal{D}, J) = n + 1.$)

$\kappa(\mathcal{D}, J) = \{\mathbf{w} = (w_0, \cdots, w_n) \in \mathbb{R}^{n+1} \mid w_0 \leq w_1 \leq \cdots \leq w_n\}$

All $\mathcal{S}_{\mathbf{w}}$ have extremal representatives, but only Φ-sect. curv. $c > -3$ has (cscS), and only the round sphere ($c = 1$) is SE.

(B,Galicki,Simanca)

(2) The Heisenberg group $\mathfrak{H}^{2n+1}$ with standard CR structure (non-compact), $\dim \kappa(\mathcal{D}, J) = n$. (B-)

All $\mathcal{S} \in \kappa(\mathcal{D}, J)$ have extremal representatives, but there is only one with constant scalar curvature, SηE with Φ-holomorphic curvature $= -3$. Here transverse homothety is induced by diffeomorphism.

Probably $\mathfrak{e}(\mathcal{D}, J) = \mathfrak{t}_k^+(\mathcal{D}, J)$ also holds for standard CR structure on the hyperbolic ball $B^n_{\mathbb{C}} \times \mathbb{R}$. Here Φ-sect. curv. $c < -3$ is (cscS).

Sometimes: extremal bouquets

Toric Contact Manifold

$(M^{2n+1}, \mathcal{D})$, effective action of torus T^{n+1} leaving $\mathcal{D}$ invariant.

(1): **Reeb Type** Reeb field ξ lies in $\mathfrak{t}_{n+1}$, Lie algebra of T^{n+1}.
(2): $\xi \not\in \mathfrak{t}_{n+1}$. (less interesting)
Reeb type are Sasakian. B-/Galicki.
Other References: Banyaga/Molino, Lerman, Falcao de Moraes/Tomei.
Complete classification: Lerman.

Toric contact manifolds of Reeb type are classified by certain convex polyhedral cones in $\mathfrak{t}^*_{n+1}$ up to T^{n+1}-equivariant equivalence. (Lerman).

Theorem: Every toric contact structure of Reeb type with $c_1(\mathcal{D}) = 0$ admits a unique Sasaki-Einstein metric (Futaki,Ono,Wang,Cho)

There is a ray of cscS metrics in an open set of extremal rays. How big is $\mathfrak{e}(\mathcal{D}, J)$?

Simply connected 5-manifolds

Barden-Smale classification

$H_2(M^5,\mathbb{Z})$ torsionfree

$S^5, S^2\times S^3, X_\infty, k\#(S^2\times S^3),$
$X_\infty\#k\#(S^2\times S^3)$.

All admit toric contact structures of Reeb type. (B-/Galicki,Ornea)

All but S^5 admit infinitely many.

All can be obtained by **Symmetry Reduction**.

- S^3-bundles over S^2:

$M^5 = S^2 \times S^3$ or X_∞. Which?

$w_2(M^5) \equiv c_1(\mathcal{D}_{\mathbf{p}})\ mod\ 2$. $\Rightarrow$

$M^5 = S^2 \times S^3 (X_\infty)$ if $c_1(\mathcal{D}_{\mathbf{p}})$ is even (odd).

S^1 reduction of S^7 (B-/Pati).

$S^1_{\mathbf{p}}$-action $(z_1, z_2, z_3, z_4) \mapsto (e^{ip_1\theta}z_1, e^{ip_2\theta}z_2, e^{-ip_3\theta}z_3, e^{-ip_4\theta}z_4)$

where $p_i \in \mathbb{Z}^+$, $\gcd(p_i, p_j) = 1$ for $i = 1, 2, j = 2, 3$.

moment map $\mu : S^7 \to \mathbb{R}$ is $\mu(\mathbf{z}) = p_1|z_1|^2 + p_2|z_2|^2 - p_3|z_3|^2 - p_4|z_4|^2$.

$\mu^{-1}(0)/S^1_{\mathbf{p}} = M^5$ compact simply connected, $H_2(M^5, \mathbb{Z}) = \mathbb{Z}$, induced contact structure $\mathcal{D}_{\mathbf{p}}$ with $c_1(\mathcal{D}_{\mathbf{p}}) = (p_1 + p_2 - p_3 - p_4)\gamma$.

Two Special Cases:

1. $\mathcal{D}_{\mathbf{p}} = \mathcal{D}_{j,2k-j,l,l}$

$c_1(\mathcal{D}_{\mathbf{p}}) = 2(k-l)\gamma \Rightarrow S^2 \times S^3$.

2. $\mathcal{D}_{\mathbf{p}} = \mathcal{D}_{j,2k-j+1,l,l}$

$c_1(\mathcal{D}_{\mathbf{p}}) = (2(k-l)+1)\gamma \Rightarrow X_\infty$.

Case 1: regular $\iff$ l divides $2(k-j)$

Case 2: regular $\iff$ l divides $2(k-j)+1$

Calabi extremal Kähler metrics on Hirzebruch surfaces give a bouquet of extremal Sasakian structures on $S^2 \times S^3$ and X_∞. Moduli space is non-Hausdorff.

General Procedure: Choose a Reeb vector field get S^1_ϕ-action

$$(z_1, z_2, z_3, z_4) \mapsto$$
$$(e^{i(2k-j)\phi} z_1, e^{ij\phi} z_2, e^{il\phi} z_3, e^{il\phi} z_4)$$

The quotient M^5/S^1_ϕ is an orbifold Hirzebruch surface.

Special case of our special case: $Y^{p,q} \approx S^2 \times S^3$. Physicists:

Gauntlett, Martelli, Sparks, Waldram.
Infinitely many toric contact structures. Each $Y^{p,q}$ admits unique Sasaki-Einstein metrics.

In our notation $\mathcal{D}_{p-q,p+q,p,p}$ with $\gcd(p,q) = 1$ and $1 \leq q < p$.

$c_1(\mathcal{D}_{p-q,p+q,p,p}) = 0$.

$Y^{p,q}$ and $Y^{p',q'}$ are toric contact equivalent $\iff (p',q') = (p,q)$.

Theorem: $Y^{p,q}$ and $Y^{p',q'}$ are contact equivalent $\iff p' = p$.

Outline of Proof: Equivalence when $p' = p$: Amounts to proving that the 3-tori corresponding to $Y^{p,q}$ and $Y^{p,q'}$ belong to distinct conjugacy classes of maximal tori in $\mathfrak{Con}(\mathcal{D}_{p-q,p+q,p,p})$.

Simplicity take p odd. The quotient $Y^{p,q}/S^1_\phi$ is orbifold Hirzebruch surface (S_{2q}, Δ_p)

$S_{2q} \approx S^2 \times S^2$, but with a twisted complex structure. Represent as $S_{2q} = \mathbb{P}(H^{2q} \oplus \mathbb{C})$, H is the hyperplane bundle over $\mathbb{P}^1$, $\mathbb{C}$ is the trivial line bundle.

$\Delta_p = (1 - \frac{1}{p})(E + F)$ is branch divisor with ramification index p, divisors $E \cdot E = 2q$ and $F \cdot F = -2q$.

$\mathbb{1} : (S_{2q}, \Delta_p) \to (S_{2q}, \emptyset)$ identity map is a Galois cover. $\emptyset$ means trivial orbifold structure. Karshon: symplectomorphism $S_{2q} \overset{K}{\to} S_{2q-2}$ with same symplectic form ω.

Get commutative diagram

$$\begin{array}{ccc} (S_{2q}, \Delta_p) & \xrightarrow{K_p} & (S_{2q'}, \Delta_p) \\ \downarrow \mathbb{1} & & \uparrow \mathbb{1}^{-1} \\ (S_{2q}, \emptyset) & \xrightarrow{K} & (S_{2q'}, \emptyset) \end{array}$$

K_p is an orbifold symplectomorphism. Neither K nor K_p are T^2-equivariant symplectomorphisms. Their toric structures belong to distinct conjugacy classes of maximal tori in $\mathfrak{Ham}(S_{2q}, \omega)$.

These toric symplectic structures on $S^2 \times S^2$ lift to toric contact structures

$\mathcal{D}_{p-q,p+q,p,p}$ and $\mathcal{D}_{p-q',p+q',p,p}$ on $S^2 \times S^3$ that are contactomorphic, but not T^3-equivariantly contactomorphic.

Non-equivalence if $p' \neq p$:

Contact homology

(Eliashberg,Givental,Hofer)

(Abreu,Macarini): $Y^{p,1} \not\cong Y^{p',1}$ when $p' \neq p$. $\square$

Remark: The orbifold Hirzebruch surfaces (S_{2q}, Δ_p) are log del Pezzo, i.e. K_{orb}^{-1} is positive. So this gives Sasakian metrics with positive Ricci curv.

Extremal set for contact structure $\mathcal{D}_p$ of $Y^{p,q}$s:
A bouquet $\bigcup_k \mathfrak{t}_3^+(\mathcal{D}_p, J_k)$ of Sasaki cones intersect in 2-dim subspace. For $\mathcal{D}_p$ each cone contains extremal rays with SE metric. Number of cones in bouquet is Euler phi function $\phi(p)$.

There are other extremal Sasakian structures with positive Ricci curvature and not cscS in the Sasaki cones of $\mathcal{D}_p$ and the extremal set may not be connected.

- Work of E. Legendre on cscS metrics:
M^5 5-manifold with $b_2(M^5) = 1$ with toric contact structure of Reeb type. There is at least 1 and at most 7 rays of cscS metrics.

In particular, there are 2 rays of cscS non-isometric metrics on the Wang-Ziller manifolds $M_{k,l}^{1,1} \approx S^2 \times S^3$ if $k > 5l$.

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