Quaternionic Geometry and Einstein Manifolds

We consider a Riemannian manifold (M, g). On any manifold a linear connection ∇ gives a way of comparing tangent vectors at different points through the notion of parallel transport.

The First Fundamental Theorem of Riemannian geometry says that there exists a unique torsionfree connection that leaves the metric tensor g invariant, i.e. $\nabla g = 0$.

The Riemann curvature tensor R measures the "curvature" of (M, g). It is given by

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$$

where X, Y, Z are vector fields on M.

Two important irreducible components of the Riemann curvature tensor R are: the Ricci curvature

$$\mathrm{Ric}(X,Y) = \mathrm{Tr}\ (Z \mapsto R(X,Z)Y)$$

which is a symmetric tensor field and the scalar curvature

$$s_g = \mathrm{Tr}\ \mathrm{Ric}$$

which is a function on M.

Important Question (R. Thom): Is there a distinquished class or best class of Riemannian manifolds?

Answer: Perhaps the answer is the class of:

Einstein manifolds.

DEFINITION: *A Riemannian manifold (M, g) is called an Einstein manifold if there exists a smooth function λ on M such that*

$$Ric = \lambda g.$$

The Bianchi identities imply that λ is actually a constant, called the *Einstein constant*. So there are 3 cases

1. $\lambda > 0$.
2. $\lambda = 0$.
3. $\lambda < 0$.

There are two reasons why one might believe that the above answer is correct.

1. Einstein manifolds are the critical points of the "Hilbert Action" given by the total scalar curvature

$$S(g) = \int s_g \mu_g$$

   where the measure μ_g is normalized to volume one, i.e. $\int \mu_g = 1$.

2. The class of Einstein manifolds is a natural generalization of the class of Riemannian manifolds of constant curvature. For dimension 2 the well known Uniformization Theorem says that every orientable surface admits a metric with curvature either $1, 0$, or -1. These are automatically Einstein. In higher dimensions the condition of constant curvature is too strong, and the Einstein condition is a natural generalization.

Two natural questions arise:

1. Is there any relation between the geometry of an Einstein manifold and its topology?
2. How numerous are Einstein metrics and are they easy or difficult to find?

The answer to question 1 is generally **yes** in dimensions 3 and 4, but **no** in dimensions greater than 4. The following is well known:

1. In dimension 3, g is Einstein $\iff$ g has constant curvature. Not all 3-manifolds admit such a metric.
2. In dimension 4, the Hitchin-Thorpe inequality implies that if (M, g) is compact orientable Einstein manifold, then

$$\chi(M) \geq \frac{3}{2}|\tau(M)|$$

where $\chi(M)$ is the Euler characteristic and $\tau(M)$ is the Hirzebruch signature. For example the connected sum $\#_k\mathbb{CP}^2$ does not admit an Einstein metric if $k > 4$.

In higher dimensions there are no known general obstructions to admitting an Einstein metric. Of course, there are obstructions to specific cases. For example, if (M, g) is Einstein with Einstein constant $\lambda > 0$ then Myers' Theorem implies that if (M, g) is complete it is compact with finite fundamental group.

As for question 2 very little is known as to how numerous Einstein metrics really are. There appear to be two basic methods for finding Einstein metrics and both require the addition of more structure.

1. The analytic approach which proves existence of Einstein metrics by working directly with the PDE's. An

example of this is the celebrated proof by Yau and Aubin of Calabi's conjecture, and more recently the work of G. Tian on the case of positive first Chern class.

2 The constructive approach using the method of symmetry reduction. It is this approach that the remainder of this talk is dedicated to.

A typical way of introducing more structure is through reduced holonomy. Recall that the *Riemannian holonomy group* $\mathrm{Hol}(g)$ of a metric g is the subgroup of the orthogonal group generated by the parallel transport of tangent vectors around closed loops. If we restrict ourselves to loops that are homotopic to the constant loop we obtain a subgroup $\mathrm{Hol}^0(g)$ of $\mathrm{Hol}(g)$ called the *restricted holonomy group*. One can show that both $\mathrm{Hol}(g)$ and $\mathrm{Hol}^0(g)$ only depend on the metric g.

If (M, g) is reducible in the sense of de Rham, that is a product $(M, g) = (M_1 \times M_2, g_1 \times g_2)$ then the holonomy representation is reducible, that is,

$$\mathrm{Hol}(g) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2).$$

Conversely, if (M, g) is complete and simply connected, and its holonomy representation is reducible then (M, g) is a product.

The main theorem on Riemannian holonomy is:

Berger's Classification Theorem: *Let (M, g) be a Riemannian manifold of dimension n which is not locally symmetric and with $Hol^0(g)$ irreducible. Then $Hol^0(g)$ is one of the following:*

(1) $SO(n)$.

(2) $U(m)$ *where* $n = 2m$.

(3) $SU(m)$ *where* $n = 2m$.

(4) $Sp(m) \cdot Sp(1)$ *where* $n = 4m$ *and* $m \geq 2$.

(5) $Sp(m)$ *where* $n = 4m$ *and* $m \geq 2$.

(6) $Spin(9)$ *where* $n = 16$.

(7) $Spin(7)$ *where* $n = 8$.

(8) G_2 *where* $n = 7$.

These geometries go by other names as follows:

(1) Generic Riemannian geometry.

(2) Kähler geometry.

(3) Special Kähler geometry.

(4) Quaternionic Kähler geometry.

(5) Hyperkähler geometry.

(6-8) Exotic holonomy.

Remarks:

1. Geometries (3-8) are automatically Einstein.
2. Geometries (2) and (3) are related to the complex numbers.
3. Geometries (4) and (5) are related to the quaternions.
4. Geometries (6-8) are related to the octonions.

Our work concerns geometries of generic holonomy, i.e type (1). Nevertheless, the Einstein metrics that we are interested in are quite special and, as the title of my talk suggests, related to the quaternions. The key is to consider the Riemannian cone of certain Riemannian manifolds.

In recent work in progress with K. Galicki, we are studying Sasakian-Einstein manifolds using the cone technique. An important special case is what is called a *3-Sasakian structure.*

3-Sasakian Manifolds

DEFINITION: *Let $(\mathcal{S}, g)$ be a Riemannian manifold of real dimension m. We say that $(\mathcal{S}, g)$ is 3-Sasakian if the holonomy group of the Riemannian cone on $\mathcal{S}$,*

$$(C(\mathcal{S}), \bar{g}) = (\mathbb{R}_+ \times \mathcal{S},\ dr^2 + r^2 g)$$

reduces to a subgroup of $Sp(\frac{m+1}{4})$. In particular, $m = 4n + 3, n \geq 1$ and $(C(\mathcal{S}), \bar{g})$ is hyperkähler.

3-Sasakian manifolds are manifolds with a certain metric quaternionic contact structure. They were first defined by Kuo in 1970 using tensorial methods, and the definition is quite technical. Shortly thereafter, Kashiwada proved that every 3-Sasakian manifold is Einstein with positive scalar curvature. Other important properties of 3-Sasakian manifolds were obtained early to mid 1970's by Ishihara, Konishi, Sasaki, Tachibana, Tanno, and Yu. Nevertheless, the only examples known until fairly recently were spheres and their quotients.

PROPOSITION: *Let $(\mathcal{S}, g)$ be a Riemannian manifold Then $\mathcal{S}$ is 3-Sasakian if and only if it admits three characteristic vector fields $\{\xi^1, \xi^2, \xi^3\}$ of 3 different Sasakian structures such that $g(\xi^a, \xi^b) = \delta_{ab}$ and $[\xi^a, \xi^b] = 2\epsilon_{abc}\xi^c$.*

The triple $\{\xi^1, \xi^2, \xi^3\}$ defines $\eta^a(Y) = g(\xi^a, Y)$ and $\Phi^a(Y) = \nabla_Y \xi^a$ for each $a = 1, 2, 3$. We call $\{\xi^a, \eta^a, \Phi^a\}$ the *3-Sasakian structure* on $(\mathcal{S}, g)$. The hyperkähler geometry of the cone $C(\mathcal{S})$ gives $\mathcal{S}$ a "quaternionic contact structure" reflected by the composition laws of the (1,1) tensors Φ^a.

$$\begin{aligned}
\eta^a(\xi^b) &= \delta^{ab}, \\
\Phi^a \xi^b &= -\epsilon^{abc}\xi^c, \\
\Phi^a \circ \Phi^b - \xi^a \otimes \eta^b &= -\epsilon^{abc}\Phi^c - \delta^{ab}\mathrm{id}.
\end{aligned}$$

¿From our point of view, without refering to any of the tensorial properties, we notice that since a hyperkähler manifold is Ricci-flat, the Definition and Lemma immediately imply:

COROLLARY: *Every 3-Sasakian manifold $(\mathcal{S}, g)$ of dimension $4n+3$ is Einstein with Einstein constant $\lambda = 2(2n+1)$. If $\mathcal{S}$ is complete it is compact with finite fundamental group and diam $\leq \pi$. Moreover, every 3-Sasakian manifold admits two distinct homothety classes of positive Einstein metrics.*

COROLLARY(B-Galicki) : *Every complete 3-Sasakian manifold $(\mathcal{S}, g)$ is irreducible with holonomy $SO(4n+3)$.*

Before discussing the fundamental structure theorems of a 3-Sasakian manifold we mention an important vanishing theorem of Galicki and Salamon:

THEOREM: *Let $(\mathcal{S}, g)$ be a compact 3-Sasakian manifold of dimension $4n+3$. Then the odd Betti numbers b_{2k+1} of $\mathcal{S}$ are all zero for $0 \leq k \leq n$.*

The Fundamental Foliations

The work of the author on 3-Sasakian manifolds has been done in a series of papers in collaboration with K. Galicki and B. Mann, including one with E. Rees. In what follows $\mathcal{S}$ will denote a complete hence compact 3-Sasakian manifold.

The 3 characteristic vector fields ξ^a for $a = 1, 2, 3$ generate a locally free action of the group $Sp(1) \simeq SU(2)$ acting as isometries on $\mathcal{S}$ and hence, define a 3-dimensional Riemannian foliation $\mathcal{F}_3$ on $\mathcal{S}$ all leaves of which are compact.

Theorem: *Let $(\mathcal{S}, g, \xi^a)$ be a compact 3-Sasakian manifold of dimension $4n + 3$. Then*

(i) The metric g is bundle-like with respect to the foliation $\mathcal{F}_3$.

(ii) All leaves are totally geodesic.

(iii) Each leaf $\mathcal{L}$ is a 3-dimensional homogeneous spherical space form of constant sectional curvature one, and the generic leaves are either $Sp(1) = S^3$ or $SO(3)$.

(iv) The space of leaves $\mathcal{S}/\mathcal{F}_3$ is a quaternionic Kähler orbifold of dimension $4n$ with scalar curvature $16n(n+2) > 0$.

Recall that an *orbifold* or *V-manifold* is a generalization of a manifold where the local Euclidean condition is replaced by the condition that every point has a neighborhood that is homeomorphic to $\mathbb{R}^n/\Gamma$ where Γ is a finite group, called a *local uniformizing group*. So locally we have finite branched covers. By a quaternionic Kähler orbifold we mean an orbifold whose local uniformizing groups preserve the quaternionic Kähler structure.

The regular version of this theorem, that is, the case when the space of leaves is a smooth manifold is essentially due to Ishihara and Konishi. In the regular case, Konishi also proved an important inversion theorem. More generally we have

THEOREM: *Let $(\mathcal{O}, g_{\mathcal{O}})$ be a quaternionic Kähler orbifold of dimension $4n$ with positive scalar curvature $16n(n+2)$. Then there is a principal $SO(3)$ V-bundle over $\mathcal{O}$ whose total space admits a 3-Sasakian structure with scalar curvature* $2(2n+1)(4n+3)$.

REMARK: The local geometry of a 3-Sasakian manifold $\mathcal{S}$ is determined completely by the local geometry of the quaternionic Kähler orbifold $\mathcal{O} = \mathcal{S}/\mathcal{F}_3$. Moreover, the local uniformizing groups of $\mathcal{O}$ are just the leaf holonomy groups of $\mathcal{F}_3$, and they are invariants of the 3-Sasakian structure.

There is a 1-dimensional foliation $\mathcal{F}_1$ on $\mathcal{S}$ obtained by choosing a 1-dimensional subalgebra of the Lie algebra $\mathfrak{sp}(1)$. (Actually there is a 2-sphere's worth of equivalent 1-dimensional foliations.) This gives the analogue of the Salamon twistor space of a quaternionic Kähler orbifold, and its geometry is important in the study of 3-Sasakian geometry. We shall refer to the space of leaves $\mathcal{Z} = \mathcal{S}/\mathcal{F}$ as the *twistor space* of the 3-Sasakian manifold $\mathcal{S}$.

Theorem: *Let $(\mathcal{S}, g)$ be a compact 3-Sasakian manifold. Then*

(i) *The leaves of $\mathcal{F}$ are all diffeomorphic to circles with cyclic leaf holonomy groups.*

(ii) *The space of leaves $\mathcal{Z} = \mathcal{S}/\mathcal{F}$ is a compact Kähler-Einstein orbifold $(\mathcal{Z}, h)$ with scalar curvature $4n(n+1)$ in such a way that $\pi : (\mathcal{S}, g) \longrightarrow (\mathcal{Z}, h)$ is an orbifold Riemannian submersion.*

(iii) *$\mathcal{Z}$ has a complex contact structure that is compatible with the Kähler-Einstein structure.*

(iv) *$\mathcal{Z}$ is a simply connected normal projective algebraic variety of Kodaira dimension $\kappa(\mathcal{Z}) = -\infty$.*

(v) $\mathcal{Z}$ has the structure of a $\mathbb{Q}$-factorial Fano variety.

Recall that a *Fano variety* means that the anti-canonical sheaf K^{-1} is ample, $\mathbb{Q}$-factorial means that for every Weil divisor D on $\mathcal{Z}$ there is an integer m such that mD is a Cartier divisor.

Since an orbifold fibration is not a fibration in the usual sense, the usual techniques in topology for fibrations do not apply directly. However, Haefliger [Hae] has defined orbifold homology, cohomology, and homotopy groups which do have an analogue in the standard theory. Let X be an orbifold of dimension n, and let P denote the bundle of orthonormal frames on X. It is a smooth manifold on which the orthogonal group $O(n)$ acts locally freely with the quotient X. Let $EO(n) \longrightarrow BO(n)$ denote the universal $O(n)$ bundle. Consider the diagonal action of $O(n)$ on $EO(n) \times P$ and denote the quotient by BX. Now there is a natural projection $p : BX \longrightarrow X$ with generic fiber the contractible space $EO(n)$.

Haefliger defines the orbifold cohomology, homology, and homotopy groups by

$$H^i_{orb}(X,\mathbb{Z}) = H^i(BX,\mathbb{Z}), \quad H_i^{orb}(X,\mathbb{Z}) = H_i(BX,\mathbb{Z}),$$

$$\pi_i^{orb}(X) = \pi_i(BX).$$

This definition of π_1^{orb} is equivalent to Thurston's better known definition in terms of orbifold deck transformations, and when X is a smooth manifold these orbifold groups coincide with the usual groups. The map $p : BX \longrightarrow X$ induces an isomorphism $H^i_{orb}(\mathcal{S},\mathbb{Z}) \otimes \mathbb{Q} \simeq H^i(\mathcal{S},\mathbb{Z}) \otimes \mathbb{Q}$.

An important invariant of $\mathcal{Z}$ is the group $\mathrm{Pic}^{orb}(\mathcal{Z})$ of line V-bundles on $\mathcal{Z}$, and this is related to the orbifold cohomology and homology groups of Haefliger.

PROPOSITION: *Let $\mathcal{Z}$ be the twistor space of the 3-Sasakian manifold $\mathcal{S}$. Then Pic($\mathcal{Z}$) is free, $H_1^{orb}(\mathcal{Z})$ is torsion, and*

$$Pic^{orb}(\mathcal{Z}) \simeq Pic(\mathcal{Z}) \oplus H_1^{orb}(\mathcal{Z})$$

Hence,

$$Pic^{orb}(\mathcal{Z}) \otimes \mathbb{Q} \simeq Pic(\mathcal{Z}) \otimes \mathbb{Q}.$$

We have:

THEOREM: (B-Galicki) *If $Pic^{orb}(\mathcal{Z})$ is torsionfree, then there is a 1-1 correspondence between a 3-Sasakian manifold and its twistor space, with the exception of the case that $\mathcal{Z} = \mathbb{CP}^{2n+1}$ in which case there are precisely two 3-Sasakian manifolds corresponding to this twistor space, namely S^{4n+3} and $\mathbb{RP}^{4n+3}$.*

The more general case when $\mathrm{Pic}^{orb}(\mathcal{Z})$ has torsion is not fully understood yet. In this case the 1-1 nature clearly fails in the orbifold category, that is when $\mathcal{S}$ is a 3-Sasakian orbifold. We suspect that the only case when $\mathcal{S}$ is smooth where there is not a 1-1, but a 2-1 correspondence is when $\mathcal{S}$ has constant curvature.

The geometry of the fundamental foliations fits nicely into a diagram of orbifold fibrations:

$$
\begin{tikzcd}
 & C(\mathcal{S}) \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathcal{Z} \arrow[dr] & & \mathcal{S}. \arrow[ll] \arrow[dl] \\
 & \mathcal{O} &
\end{tikzcd}
$$

All the spaces in this diagram are Einstein. $\mathcal{S}$ is 3-Sasakian, $C(\mathcal{S})$ is hyperkähler, $\mathcal{Z}$ is Kähler-Einstein, and $\mathcal{O}$ is quaternionic Kähler.

The leaf holonomy groups for both foliations are important invariants of the 3-Sasakian structure on $\mathcal{S}$. In fact, the leaf holonomy groups of the foliation $\mathcal{F}_3$ give a rough classification of 3-Sasakian manifolds. First, there is whether the generic leaves are an S^3 or an $SO(3)$. Then the leaf holonomy groups must be one of the finite subgroup Γ of $SU(2)$ or a $\mathbb{Z}_2$ quotient of it, namely: (1) $\Gamma = \mathrm{id}$, (2) $\Gamma = \mathbb{Z}_m$ the cyclic group of order m, (3) $\Gamma = \mathbb{D}^*_m$ a binary dihedral group with m an integer greater than 2, (4) $\Gamma = \mathbb{T}^*$ the binary tetrahedral group, (5) $\Gamma = \mathbb{O}^*$ the binary octahedral group, (6) $\Gamma = \mathbb{I}^*$ the binary ocosahedral group.

A 3-Sasakian manifold of type (1) is a regular 3-Sasakian manifold for which we have a conjecture equivalent to the LeBrun-Salamon Conjecture that any quaternionic Kähler manifold of positive scalar curvature is symmetric.

CONJECTURE(B-Galicki): *Let $(\mathcal{S}, g)$ be a compact regular 3-Sasakian manifold. Then $\mathcal{S}$ is homogeneous.*

3-Sasakian Homogeneous Manifolds

DEFINITION: *Let $I_0(\mathcal{S}, g) \subset I(\mathcal{S}, g)$ be the subgroup of the isometry group which preserves the 3-Sasakian structure, that is if $\phi^k \in \mathit{Diff}\,\mathcal{S}$ corresponds to $k \in I_0(\mathcal{S}, g)$ then $\phi^k_* \xi^a = \xi^a$, for all $a = 1, 2, 3$. Then $I_0(\mathcal{S}, g)$ is called the group of 3-Sasakian isometries and when it acts transitively on $(\mathcal{S}, g)$ the space $\mathcal{S}$ is said to be a 3-Sasakian homogeneous space.*

The following describes 3-Sasakian homogeneous spaces, and is analogous to Boothby's classification of all homogeneous complex contact manifolds and Wolf's classification of homogeneous quaternionic Kähler manifolds.

THEOREM(B-Galicki-Mann): *Any* 3-*Sasakian homogeneous space $\mathcal{S} = G/H$ is one of the following:*

$$\frac{Sp(n+1)}{Sp(n)}, \quad \frac{Sp(n+1)}{Sp(n)\times\mathbb{Z}_2},$$

$$\frac{SU(m)}{S\big(U(m-2)\times U(1)\big)}, \quad \frac{SO(k)}{SO(k-4)\times Sp(1)},$$

$$\frac{G_2}{Sp(1)}, \quad \frac{F_4}{Sp(3)}, \quad \frac{E_6}{SU(6)}, \quad \frac{E_7}{\mathrm{Spin}(12)}, \quad \frac{E_8}{E_7}.$$

Here $n \geq 0$, $Sp(0)$ denotes the trivial group, $m \geq 3$, and $k \geq 7$.

The Quotient Construction

Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold with a nontrivial group $I_0(\mathcal{S}, g)$ of automorphisms of the 3-Sasakian structure. Now $I_0(\mathcal{S}, g)$ extends to a group of hyperkähler automorphisms $I_0(\mathcal{S}, \bar{g})$ on the cone $C(\mathcal{S})$ by defining each element to act trivially on $\mathbb{R}^+$. Recall that the hyperkähler quotient of Hitchin, Karlhede, Linström, and Roček shows that any subgroup $G \subset I_0(M, \bar{g})$ gives rise to a hyperkähler moment map $\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$, where $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}^*$ is its dual. Thus, we can define a 3-Sasakian moment map

$$\mu_{\mathcal{S}} : \mathcal{S} \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

simply by restriction $\mu_{\mathcal{S}} = \mu \,|\, \mathcal{S}$. Such a moment map takes the simple form

$$< \mu^a_{\mathcal{S}}, \tau > = \frac{1}{2}\eta^a(X^\tau).$$

Then we have

Theorem (B-G-M): *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ smoothly and properly by 3-Sasakian isometries. Let $\mu_{\mathcal{S}}$ be the corresponding 3-Sasakian moment map and assume both that 0 is a regular value of $\mu_{\mathcal{S}}$ and that G acts freely on the submanifold $\mu_{\mathcal{S}}^{-1}(0)$. Furthermore, let $\iota : \mu_{\mathcal{S}}^{-1}(0) \longrightarrow \mathcal{S}$ and $\pi : \mu_{\mathcal{S}}^{-1}(0) \longrightarrow \mu_{\mathcal{S}}^{-1}(0)/G$ denote the corresponding embedding and submersion. Then $(\mathcal{S} /\!\!/\!\!/ G = \mu_{\mathcal{S}}^{-1}(0)/G, \check{g})$ is a smooth 3-Sasakian manifold of dimension $4(n - dim\ \mathfrak{g}) + 3$ with metric $\check{g}$ and Sasakian vector fields $\check{\xi}^a$ determined uniquely by the two conditions $\iota^* g = \pi^* \check{g}$ and $\pi_*(\xi^a \,|\, \mu_{\mathcal{S}}^{-1}(0)) = \check{\xi}^a$.*

3-Sasakian Toric Manifolds

We can use the 3-Sasakian reduction theorem to obtain new 3-Sasakian manifolds by toral reductions from the 3-Sasakian sphere S^{4n+3}. On S^{4n+3} the group $I_0(\mathcal{S}, g)$ of 3-Sasakian automorphisms is $Sp(n+1)$. We choose a k-dimensional subtorus T^k in the maximal torus T^n. Every quaternionic representation of a k-torus T^k on $\mathbb{H}^{n+1}$ can be described via a homomorphism $f_\Omega : T^k \longrightarrow T^{n+1}$

$$f_\Omega(\tau_1, \ldots, \tau_k) = \begin{pmatrix} \prod_{i=1}^{k} \tau_i^{a_1^i} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \prod_{i=1}^{k} \tau_i^{a_{n+1}^i} \end{pmatrix},$$

where $(\tau_1, .., \tau_k) \in S^1 \times \cdots \times S^1 = T^k$ are the complex coordinates on T^k, and $a_j^i \in \mathbb{Z}$ are the coefficients of a $k \times (n+1)$ integral *weight* matrix $\Omega = (a_\alpha^l)_{\alpha=1,\ldots,n+1}^{l=1,\ldots,k} \in \mathcal{M}_{k,n+1}(\mathbb{Z})$.

Let $\{e_l\}_{l=1}^k$ denote the standard basis for $\mathfrak{t}_k^* \simeq \mathbb{R}^k$. Then the 3-Sasakian moment map $\mu_\Omega : S^{4n+3} \longrightarrow \mathfrak{t}_k^* \otimes \mathbb{R}^3$ of the k-torus action defined by $\varphi_{(\tau_1,\ldots,\tau_k)}(\mathbf{u}) = f_\Omega(\tau_1, \ldots, \tau_k)\mathbf{u}$, is given by $\mu_\Omega = \sum_l \mu_\Omega^l e_l$ with

$$\mu_\Omega^l(\mathbf{u}) = \sum_\alpha \bar{u}_\alpha i a_\alpha^l u_\alpha.$$

Let us denote further denote the triple $(T^k, f_\Omega, \varphi_{(\tau_1,\ldots,\tau_k)})$ by $T^k(\Omega)$. We want to study the reduced space

$$\mathcal{S}(\Omega) = S^{4n+3} /\!\!/\!\!/ T^k(\Omega) = \mu_\Omega^{-1}(0)/T^k(\Omega)$$

.

Consider the $\binom{n}{k}$ minor determinants

$$\Delta_{\alpha_1\ldots\alpha_k} = \det \begin{pmatrix} a^1_{\alpha_1} & \ldots & a^1_{\alpha_k} \\ \vdots & & \vdots \\ a^k_{\alpha_1} & \ldots & a^k_{\alpha_k} \end{pmatrix}$$

obtained by deleting $n+1-k$ columns of Ω.

Definition: *Let $\Omega \in \mathcal{M}_{k,n+1}(\mathbb{Z})$ be the weight matrix.*

(i) If $\Delta_{\alpha_1\cdots\cdots\alpha_k} \neq 0, \forall \quad 1 \leq \alpha_1 < \cdots < \alpha_k \leq n+1$, then we say that Ω is non-degenerate.

Suppose Ω is non-degenerate and let g be the kth determinantal divisor, i.e., the gcd of all the k by k minor determinants $\Delta_{\alpha_1\cdots\cdots\alpha_k}$. Then Ω is said to be admissible *if in addition we have*

(ii) $\gcd(\Delta_{\alpha_2\cdots\alpha_{k+1}}, ..., \Delta_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, ..., \Delta_{\alpha_1\cdots\alpha_k}) = g$ *for all sequences of length (k+1) such that $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n+1$.*

Theorem (B-G-M): *Let $\mathcal{S}(\Omega)$ be a 3-Sasakian toral quotient space. Then*

(i) if Ω is non-degenerate, $\mathcal{S}(\Omega)$ is an orbifold.

(ii) If Ω is degenerate, then either $\mathcal{S}(\Omega)$ is a singular stratified manifold which is not *an orbifold or it is an orbifold obtained by reduction of a lower dimensional sphere $S^{4n-4r-1}$ by a torus $T^{k-r}(\Omega')$ or a finite quotient of such, where $1 \leq r \leq k$ and Ω' is non-degenerate. (When $r = k$ the quotient is the sphere $S^{4n-4k-1}$).*

(iii) Assuming that Ω is non-degenerate $\mathcal{S}(\Omega)$ is a smooth manifold if and only if Ω is admissible.

When can we find admissible Ω?

Generally this question is non-trivial. Admissible Ω clearly exist for $k = 1$ for any n, i.e. any dimension by simply choosing the elements of Ω to be pairwise relatively prime. They also exist in dimension 7 for any k as the following Ω shows:

$$\Omega = \begin{pmatrix} 1 & 0 & \dots & 0 & a_1 & b_1 \\ 0 & 1 & \dots & 0 & a_2 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_k & b_k \end{pmatrix}.$$

We simply choose (a_i, b_i) to be relatively prime, and if for some pair i, j $a_i = \pm a_j$ or $b_i = \pm b_j$ then we must have $b_i \neq \pm b_j$ or $a_i \neq \pm a_j$, respectively.

However, there are mod 2 obstructions to smoothness which show somewhat surprisingly:

THEOREM (B-G-M): *Let Ω be admissible and $k > 1$. Then dim $\mathcal{S}(\Omega) \leq 15$. Furthermore, if $k > 4$, then dim $\mathcal{S}(\Omega) = 7$.*

This says that there are **smooth** toral quotients $\mathcal{S}(\Omega)$ only in the cases indicated, and we shall see shortly that k is precisely the second Betti number $b_2(\mathcal{S}(\Omega))$. Moreover, a theorem of R. Bielawski says that any 3-Sasakian manifold $(\mathcal{S}, g)$ whose automorphism group $I_0(\mathcal{S}, g)$ contains a torus T^{n+1-k} is homeomorphic to a $\mathcal{S}(\Omega)$.

Furthermore, we have an existence theorem:

THEOREM (B-G-M-Rees,B-G-M): *There exist smooth toric quotients* $\mathcal{S}(\Omega)$ *in dimension 7 with any second Betti number* $b_2(\mathcal{S}(\Omega)) = k$, *and in dimensions* 11 *and* 15 *with second Betti numbers* $b_2(\mathcal{S}(\Omega)) = 2, 3,$ *and* 4.

The Topology of $\mathcal{S}(\Omega)$

The full isometry group of $\mathcal{S}(\Omega)$ is $Tn + 1 - k \times Sp(1)$ and in the case dim $\mathcal{S}(\Omega) = 7$ one can obtain a stratification of $\mathcal{S}$ by analysing the quotient $Q(\Omega) = \mathcal{S}(\Omega)/(T^2 \times Sp(1)$. For general dimension Bielawski has obtained the rational homology of $\mathcal{S}(\Omega)$ by viewing $\mathcal{S}(\Omega)$ as a boundary at infinity of a hyperkähler orbifold that is the limit of certain toric hyperkähler orbifolds studied by Bielawski and Dancer.

The orbit space $Q(\Omega)$ can be stratified as follows:

$$Q(\Omega) = Q_0(\Omega) \sqcup Q_1(\Omega) \sqcup Q_2(\Omega)$$

and:

LEMMA (B-G-M-R): *Let dim* $\mathcal{S}(\Omega) = 7$. *Then*

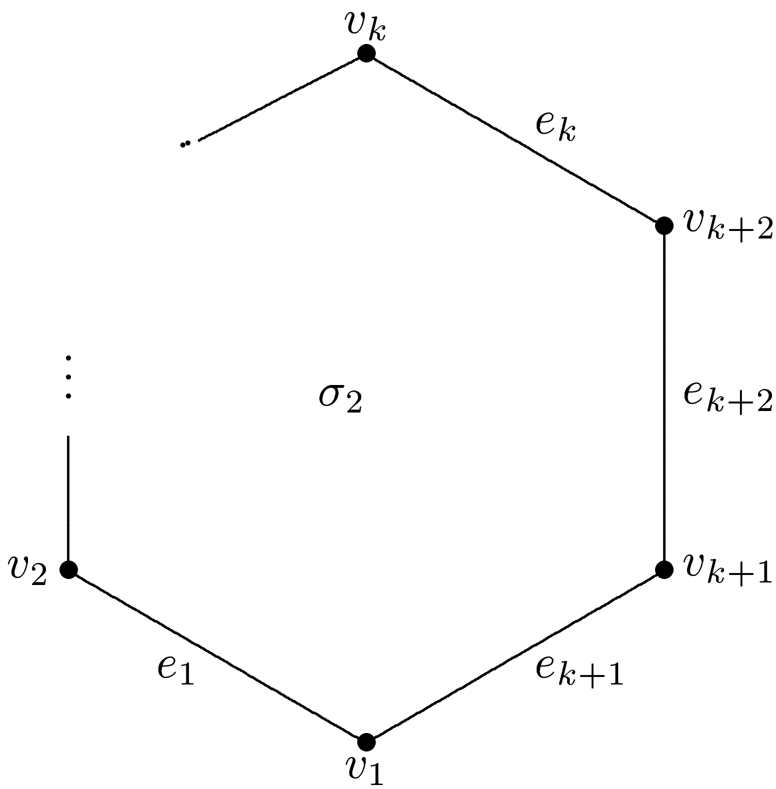
(i) *The orbit space* $Q(\Omega)$ *is homeomorphic to the closed disc* $\bar{D}^2$, *and the subset of singular orbits* $Q_1(\Omega) \sqcup Q_0(\Omega)$ *is homeomorphic to the boundary* $\partial\bar{D}^2 \simeq S^1$.

(ii) $Q_2(\Omega)$ *is homeomorphic to the open disc* D^2.

(iii) $Q_1(\Omega)$ *is homeomorphic to the disjoint union of* $k + 2$ *copies of the open unit interval.*

(iv) $Q_0(\Omega)$ *is a set of* $k + 2$ *points.*

Hence, the quotient $Q(\Omega)$ can be depicted by the polygon:



The orbit space $Q(\Omega)$

The topology of $\mathcal{S}(\Omega)$ can then be analysed by using a Leray spectral sequence on the induced stratification of $\mathcal{S}(\Omega)$. This gives

Theorem (B-G-M-R,B-G-M): *Let $\mathcal{S}(\Omega)$ be a smooth 3-Sasakian toric quotient of dimension 7. Then $\mathcal{S}(\Omega)$ is simply connected and $H_2(\mathcal{S}(\Omega),\mathbb{Z}) \simeq \mathbb{Z}^k$ where $k = b_2(\mathcal{S}(\Omega))$. Furthermore, the rational homology is given by*

$$b_i(\mathcal{S}(\Omega)) = \begin{cases} 1 & \text{if } i = 0,7; \\ 0 & \text{if } i = 1,3,4,6; \\ k & \text{if } i = 2,5. \end{cases}$$

There is certainly torsion in $H_3(\mathcal{S}(\Omega), \mathbb{Z})$, but we have only been able to determine this in the case $b_2(\mathcal{S}(\Omega)) = k = 1$. In this case we have determined the complete cohomology ring in all dimensions. Here, we have $\Omega = \mathbf{p} = (p_1, \cdots, p_n)$ with the p_i's relatively prime. Then

THEOREM (B-G-M): *Let $\mathcal{S}(\mathbf{p})$ be a 3-Sasakian manifold with $b_2 = k = 1$ obtained by a circle reduction from S^{4n+3}. Then, as rings,*

$$H^*\big(\mathcal{S}(\mathbf{p}), \mathbb{Z}\big) \;\cong\; \left(\frac{\mathbb{Z}[e_2]}{[e_2^{n+1} = 0]} \otimes E[f_{2n+1}]\right) / \mathcal{R}(\mathbf{p}).$$

Here e_2 and f_{2n+1} are generators with the subscripts indicating the cohomological dimension of each generator. Furthermore, the relations $\mathcal{R}(\mathbf{p})$ are generated by $\sigma_n(\mathbf{p})e_2^n = 0$ and $f_{2n+1}e_2^n = 0$, where $\sigma_n(\mathbf{p}) = \sum_{j=1}^{n+1} p_1 \cdots \hat{p_j} \cdots p_{n+1}$ is the $(n)^{th}$ elementary symmetric polynomial in the entries of $\mathbf{p}$.

Notice this theorem shows that $H^{2n}(\mathcal{S}(\mathbf{p}); \mathbb{Z}) \;=\; \mathbb{Z}_{\sigma_n(\mathbf{p})}$.

In more general dimension there is:

THEOREM (Bielawski): *Let $\mathcal{S}(\Omega)$ is a toric 3-Sasakian orbifold of dimension $4(n-k)+3$. Then we have*

$$b_{2i}(\mathcal{S}(\Omega)) = \binom{k+i-1}{k}, \qquad i < n+1-k.$$

Consequences

We give a list of some of the consequences that follow from our work together with other known results.

Corollary 1: *The quotients* $(\mathcal{S}(\mathbf{p}), g(\mathbf{p}))$ *give infinitely many non-homotopy equivalent simply-connected compact inhomogeneous 3-Sasakian manifolds, hence positive Einstein manifolds, in dimension* $4n-1$ *for every* $n \geq 2$.

A remarkable theorem of Gromov says that if there is a negative number κ and a Riemannian metric with bounded diameter such that all sectional curvatures are $\geq \kappa$, then the total Betti number of the manifold must be bounded. This together with our results give

Corollary 2: *For any negative real number* κ *there are infinitely many 3-Sasakian 7-manifolds, hence positive Einstein 7-manifolds, which do not admit metrics whose sectional curvatures are all greater than or equal to* κ.

Corollary 3: *There are infinitely many 3-Sasakian 7-manifolds with arbitrarily small injectivity radius.*

Corollary 4: *In dimension seven there exist 3-Sasakian manifolds with every rational homology type not excluded by the Galicki-Salamon Vanishing Theorem.*

Corollary 5: *There exist 7-manifolds with arbitrary second Betti number having two distinct positive Einstein metrics of weak* G_2 *holonomy.*

Corollary 6: *There exist* $\mathbb{Q}$*-factorial contact Fano 3-folds* X *with* $b_2(X) = l$ *for any positive integer* l.

A theorem Mori and Mukai says that for smooth Fano 3-folds, $b_2 \leq 10$.

COROLLARY 7: *If the second Betti number* $b_2(\mathcal{S}(\Omega)) = k > 3$, *the 3-Sasakian manifolds* $\mathcal{S}(\Omega)$ *are not homotopy equivalent to any homogeneous space.*

COROLLARY 8: *There exist compact, self-dual,* T^2 *symmetric Einstein orbifolds of positive scalar curvature with arbitrary second Betti number.*

So the rigidity that is known to hold in the smooth quaternionic Kähler category fails drastically in the orbifold category.

In the case of a regular 3-Sasakian manifold $\mathcal{S}$ Galicki and Salamon used an index calculation to derive the following remarkable relation among the Betti numbers of $\mathcal{S}$:

$$\sum_{k=1}^{n} k(n+1-k)(n+1-2k)b_{2k} = 0.$$

Bielawski's formula for the Betti numbers of toric quotients shows that in the non-regular case this formula holds if and only if $k = 1$. There are no relations in dimension 7 and our construction gives explicit examples where these relations fail in dimensions 11 and 15.