3-Sasakian Manifolds

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An Introduction

We begin this review with a brief history of the subject for our exposition shall have little to do with the chronology. In 1960 Sasaki [Sas 1] introduced a geometric structure related to an almost contact structure. This geometry became known as Sasakian geometry and has been studied extensively ever since. In 1970 Kuo [Kuo] refined this notion and introduced manifolds with Sasakian 3-structures (see also [Kuo-Tach, Tach-Yu]). Independently, the same concept was invented by Udrişte [Ud]. Between 1970 and 1975 this new kind of geometry was investigated almost exclusively by a group of Japanese geometers, including Ishihara, Kashiwada, Konishi, Kuo, Tachibana, Tanno, and Yu. Already in [Kuo] we learn that the 3-Sasakian geometry has some interesting topological implications. Using earlier results of Tachibana about the harmonic forms on compact Sasakian spaces [Tach], Kuo showed that odd Betti numbers up to the middle dimension must be divisible by 4. In 1971 Kashiwada observed that every 3-Sasakian manifold is Einstein with a positive Einstein constant [Kas]. In the same year Tanno proved an interesting theorem about the structure of the isometry group of every 3-Sasakian space [Tan 1]. In a related paper he studied a natural 3-dimensional foliation on such spaces showing that, if the foliation is regular, then the space of leaves is an Einstein manifold of positive scalar curvature [Tan 2]. Tanno clearly points to the importance of the analogy with the quaternionic Hopf fibration $S^3 \to S^7 \to S^4$, but does not go any further. In fact, Kashiwada's paper mentions a conjecture speculating that every 3-Sasakian manifold is of constant curvature [Kas]. She attributed this conjecture to Tanno and, at the time, these were the only known examples.

Very soon after, however, it became clear that such a conjecture could not possibly be true. This is due to a couple of papers by Ishihara and Konishi [I-Kon, Ish 1]. They made a fundamental observation that the space of leaves of the natural 3-dimensional foliations mentioned above has a "quaternionic structure", part of which is the Einstein metric discovered by Tanno. This led Ishihara to an independent study of this "sister geometry": quaternionic Kähler manifolds [Ish 2]. His paper is very well-known and is almost always cited as the source of the explicit coordinate description of quaternionic Kähler geometry. Among other results Ishihara showed that his definition implies that the holonomy group of the metric is a subgroup of $Sp(n)\cdot Sp(1)$, thus providing an important connection with the earlier studies of such manifolds by Alekseevsky [Al 1], Bonan [Bon], Gray [Gra 1], Kraines [Kra], and Wolf [Wol]. In 1975 Konishi [Kon] proved the existence of a Sasakian 3-structure on a natural principal $SO(3)$-bundle over any quaternionic Kähler manifold of positive scalar curvature. This, with the symmetric examples of Wolf, gives precisely all of the homogeneous 3-Sasakian spaces. Yet, at the time they did not appear explicitly and escaped any systematic study until much later.

In fact, 1975 seems to be the year when 3-Sasakian manifolds are relegated to an almost complete obscurity which lasted for about 15 years. From that point on the two "sisters" fair very differently. The extent of this can be best illustrated by the famous book on Einstein manifolds by Besse [Bes]. The book appeared in 1987 and provided the reader with an excellent, up-to-date, and very complete account of what was known about

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During the preparation of this work the authors were supported by an NSF grant.

Einstein manifolds 10 years ago. But one is left in the dark when trying to find references to any of the papers on 3-Sasakian manifolds we have cited; 3-Sasakian manifolds are never mentioned in Besse. The other “sister”, on the contrary, received a lot of space in a separate chapter. Actually Einstein metrics on Konishi's bundle do appear in Besse (see [Bes] 14.85, 14.86) precisely in the context of the $SO(3)$-bundles over positive quaternionic Kähler manifolds as a consequence of a theorem of Bérard-Bergery ([Bes], 9.73). Obviously, the absence of 3-Sasakian spaces in Besse's book was the result rather than the cause of this obscurity. One could even say it was justified by the lack of any interesting examples. The authors have puzzled over this phenomenon without any sound explanation. One can only speculate that it is the holonomy reduction that made quaternionic Kähler manifolds so much more attractive an object. Significantly, the holonomy group of a 3-Sasakian manifold never reduces to a proper subgroup of the special orthogonal group. And when in 1981 Salamon [Sal 1,2], independently with Bérard-Bergery [BeBèr], generalized Penrose's twistor construction for self-dual 4-manifolds introducing the twistor space over an arbitrary quaternionic Kähler manifold, the research on quaternionic Kähler geometry flourished, fueled by powerful tools from complex algebraic geometry.

Finally, in the early nineties, 3-Sasakian manifolds start a comeback. They begin to appear in two completely different contexts. First, in the study of manifolds with real Killing spinors, Friedrich and Kath notice that the existence of one such spinor leads naturally to a Sasakian-Einstein structure while three of them give the manifold a 3-Sasakian structure [B-G-F-K, Fr-Kat 1]. Assuming regularity they are able to combine the result of Hitchin [Hit 1] and Friedrich and Kurke [Fr-Kur] and obtain a classification of all regular complete 7-manifolds with 3-Sasakian structure [Fr-Kat 2]. This appears to be the first classification result about 3-Sasakian manifolds. In 1993 the classification problem for manifolds admitting Killing spinors found an elegant formulation in terms of holonomy groups [Bär]. Bär observes that if (M,g) is a simply connected spin manifold with a non-trivial real Killing spinor then the metric cone $(C(M),\bar{g})$ must admit a parallel spinor. In particular $(C(M),\bar{g})$ is Ricci-flat and $\mathrm{Hol}(\bar{g})$ is quite restricted so that only very few groups can occur. One such possibility is $\mathrm{Hol}(\bar{g})=Sp(m+1)$ which gives the cone a hyperkähler structure. It easily follows that M must be 3-Sasakian.

Independently, the hyperkähler geometry of the cone $C(\mathcal{S})$ was the starting point of our research on 3-Sasakian manifold. In 1991 the authors, together with Ben Mann, discovered that 3-Sasakian manifolds appear naturally as levels sets of a certain moment map on a hyperkähler manifold with an isometric $SU(2)$-action rotating the triple of complex structures [B-G-M 1]. In fact, if some obstructions for the $SU(2)$-action vanish, then the hyperkähler manifold is precisely a cone on a 3-Sasakian space and, at the same time, it is the Swann's bundle over the associated quaternionic Kähler orbifold of positive scalar curvature [Sw]. We quickly realized that $\mathcal{S}$ is ultimately related to three other Einstein geometries: its hyperkähler cone $C(\mathcal{S})$, the associated twistor space $\mathcal{Z}$, and the associated quaternionic Kähler orbifold $\mathcal{O}$. In this review we call the collection of these four geometries together with all the relevant maps $\diamondsuit(\mathcal{S})$. Thus, every $\mathcal{S}$ comes together with a fundamental diagram

$$\begin{tikzcd} & C(\mathcal{S}) \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathcal{Z} \arrow[dr] & & \mathcal{S}. \arrow[ll] \arrow[dl] \\ & \mathcal{O} & \end{tikzcd}$$

More importantly we also realized that, even when $\mathcal{O}$ and $\mathcal{Z}$ are compact Riemannian orbifolds, $\mathcal{S}$ can be a smooth manifold. This moment marks the beginning of our efforts to understand the geometry and topology of 3-Sasakian manifolds.

They have led us through the classification of all 3-Sasakian homogeneous spaces and a discovery of a new quotient construction of infinitely many homotopy types of non-regular compact 3-Sasakian manifolds [B-G-M-2]. In dimension 7 these examples turned out to be certain Eschenburg bi-quotients of $U(3)$ by a 2-torus [Esch 1-2]. We gave a complete analysis of the geometry and topology of such spaces [B-G-M 2]. The next important step was the second author's work with Simon Salamon [G-Sal]. There we noticed that Kuo's theorem about odd Betti numbers of 3-Sasakian manifolds being divisible by 4 missed a crucial point. Because of the isometric $SU(2)$-action, all odd Betti numbers up to the middle dimension must actually vanish. In the regular case we were able to show that 3-Sasakian cohomology is just the primitive cohomology of both $\mathcal{Z}$ and $\mathcal{O}$. These results were then extended to the orbifold case in [B-G 1], where we also made a systematic study of the orbifold twistor spaces $\mathcal{Z}$ and gave an orbifold extension of the LeBrun's inversion theorem [Le 3]. Finally, the Vanishing Theorem for Betti numbers provided us with the tools to study the geometry and topology of more complicated examples. This study [B-G-M-R 1,2] used a rational spectral sequence and culminated in discovering that, in dimension 7, all rational homology types not excluded by the Vanishing Theorem do occur and can be constructed explicitly. These examples illustrate the richness of 3-Sasakian geometry in dimension 7. For example, there is an infinite family of 3-Sasakian 7-manifolds that admit metrics of positive sectional curvature, while there is another infinite family that can admit no metrics whose sectional curvature is bounded below by an arbitrary fixed negative number! Later in [B-G-M 8] we discovered how to handle the integral spectral sequence giving integral results for our 7-dimensional examples up through the second homology group. We also studied [B-G-M 7] the higher dimensional analogue showing that these meet with an entirely different fate.

This review chapter is intended to give the reader a self-contained account of everything we have learned about such spaces to date. We have tried to gather all the known results. In a chapter like this it would be impossible to present every proof so we do quote some theorems just referring to the literature. But we have tried to include as many proofs as possible so that the review is not simply a long dry list of theorems, propositions, and corollaries. When it comes to references we make no claim of completeness, though we have tried to do our best. We apologize for any omissions. At the end we hope to be able to convince our reader that the 3-Sasakian geometry is at least as fascinating as any other "sister" geometry of the fundamental diagram $\diamondsuit(\mathcal{S})$.

Our review is organized as follows: We begin by setting up definitions, notation, and describing elementary properties of Sasakian, Sasakian-Einstein, and 3-Sasakian manifolds in Section 1. Next we discuss fundamentals about the geometry of the associated foliations (arrows in the diagram $\diamondsuit(\mathcal{S})$). We then give a classification of homogeneous geometries in Section 3. Section 4 is all about Betti numbers of Sasakian and 3-Sasakian manifolds while Section 5 is a very brief look at the Killing spinors and G_2 structures. The following section describes the geometry of the 3-Sasakian quotient construction. After this we give a detailed study of "toric" 3-Sasakian manifolds. We conclude with a handful of open problems, questions, and some conjectures followed by an appendix on fundamental properties of orbifolds.

Acknowledgments: The authors would like thank Ben Mann who is a friend and has been a collaborator on much of our work. We also thank our other collaborators Simon Salamon and Elmer Rees. We thank Roger Bielawski, Alex Buium, Claude LeBrun, Liviu Ornea, and Uwe Semmelmann for discussions and valuable comments. Last, but not least, the second named author would like to than Max-Planck-Institute für Mathematik in Bonn for support and hospitality. This review was written during his stay in Bonn.

1. Definitions and Basic Properties

In this section we introduce notation, definitions, and discuss some elementary properties of Sasakian, Sasakian-Einstein, and 3-Sasakian manifolds. Traditionally Sasakian structures were defined via contact structures by adding a Riemannian metric with some additional conditions. We take a simpler and more geometric approach that uses the holonomy reduction of the associated metric cone.

1.1 Sasakian Manifolds

Definition 1.1.1: *Let $(\mathcal{S}, g)$ be a Riemannian manifold of real dimension m. We say that $(\mathcal{S}, g)$ is Sasakian if the holonomy group of the metric cone on $\mathcal{S}$ $(C(\mathcal{S}), \bar{g}) = (\mathbb{R}_+ \times \mathcal{S},\ dr^2 + r^2 g)$ reduces to a subgroup of $U(\frac{m+1}{2})$. In particular, $m = 2n+1, n \geq 1$ and $(C(\mathcal{S}), \bar{g})$ is Kähler.*

The following proposition provides three alternative characterizations of the Sasakian property, the first one, perhaps, most in the spirit of the the original definition of Sasaki [Sas 1]:

Proposition 1.1.2: *Let $(\mathcal{S}, g)$ be a Riemannian manifold, ∇ the Levi-Civita connection of g, and let $R(X,Y) : \Gamma(T\mathcal{S}) \rightarrow \Gamma(T\mathcal{S})$ denote the Riemann curvature tensor of ∇. Then the following conditions are equivalent:*

(i) There exists a Killing vector field ξ of unit length on $\mathcal{S}$ so that the tensor field Φ of type $(1,1)$, defined by $\Phi(X) = \nabla_X \xi$, satisfies the condition

1.1.3
$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X,Y)\xi$$

for any pair of vector fields X and Y on $\mathcal{S}$.

(ii) There exists a Killing vector field ξ of unit length on $\mathcal{S}$ so that the Riemann curvature satisfies the condition

1.1.4
$$R(X,\xi)Y = g(\xi, Y)X - g(X,Y)\xi,$$

for any pair of vector fields X and Y on $\mathcal{S}$.

(iii) $(\mathcal{S}, g)$ is Sasakian.

Proof: We outline the proof of the equivalence of (i) and (iii). The equivalence of (i) and (ii) is a simple calculation relating $(\nabla_X \Phi)(Y)$ to $R(X,\xi)Y$ and is left to the reader (see [Y-K]).

We first show how (iii) implies (i). Let X, Y be any two vector fields on $\mathcal{S}$ viewed as vector fields on $C(\mathcal{S})$ and $\bar{\nabla}$ be the Levi-Civita connection of $\bar{g}$. Then we have the following warped product formulas for the cone metric connection [O'N, p. 206]:

1.1.5
$$\bar{\nabla}_{\partial_r}\partial_r = 0, \qquad \bar{\nabla}_{\partial_r} X = \bar{\nabla}_X \partial_r = \frac{1}{r}X, \qquad \bar{\nabla}_X Y = \nabla_X Y - r g(X,Y)\partial_r.$$

Since the holonomy group of the cone $(C(\mathcal{S}), \bar{g})$ reduces to a subgroup of $U(\frac{m+1}{2})$ there is a parallel complex structure I on $C(\mathcal{S})$, i.e., I commutes with $\bar{\nabla}$. We can identify $\mathcal{S}$ with $\mathcal{S} \times \{1\} \subset C(\mathcal{S})$ and define

1.1.6
$$\xi = I(\partial_r), \qquad \eta(Y) = g(\xi, Y), \qquad \Phi(Y) = \nabla_Y \xi$$

for any vector field $Y \in \Gamma(T\mathcal{S})$. It is then a simple calculation to show that ξ is actually a unit Killing vector field on $\mathcal{S}$ and it satisfies the curvature condition 1.1.3. Clearly, ξ is unit by definition and we have

$$g(\nabla_Y \xi, X) = \bar{g}(\bar{\nabla}_Y \xi + g(\xi, Y)\partial_r, X) = \bar{g}(\bar{\nabla}_Y I(\partial_r), X) =$$

$$= \bar{g}(I(\bar{\nabla}_Y \partial_r), X) = \bar{g}(I(Y), X)$$

which is skew-symmetric in X and Y. The second condition follows from $\bar{\nabla} I = 0$, definition of $\Phi(Y) = \nabla_Y(I\partial_r)$, and the formulas 1.1.5. Conversely, we can construct a Kähler structure on $C(\mathcal{S})$ as follows: Let $\Psi = r\partial_r$ denote the Euler field on $C(\mathcal{S})$ and define smooth section of End $TC(\mathcal{S})$ by the formula

1.1.7 $$IY \;=\; \Phi(Y) - \eta(Y)\Psi, \qquad I\Psi \;=\; \xi,$$

where $\eta(Y) = g(\xi, Y)$ is the dual 1-form of ξ. It is easy to see that I is an almost complex structure on $C(\mathcal{S})$ and the metric $\bar{g}$ is Hermitian. To show that $C(\mathcal{S})$ is Kähler it is enough to show that $\bar{\nabla} I = 0$. This is done by a direct calculation using the definition of I and equations 1.1.6. ∎

The above discussion shows that there is a natural splitting of the tangent bundle $TC(\mathcal{S})$ as $TC(\mathcal{S}) = \mathcal{L}_\Psi \oplus \mathcal{L}_\xi \oplus \mathcal{H}$ where $\mathcal{L}_X$ denotes the trivial line bundle generated by the nowhere vanishing vector field X, and $\mathcal{H}$ is a complement with respect to the metric $\bar{g}$. It follows immediately that the frame bundle of any Sasakian manifold of dimension $2n+1$ reduces to the group $1 \times U(n)$ [Sas 1]. It follows that every Sasakian manifold has a canonical Spin^c structure [Mor].

In view of the above proposition the triple $\{\xi, \eta, \Phi\}$ is called a *Sasakian structure* on $(\mathcal{S}, g)$, the Killing vector field ξ and the 1-form η are called the *characteristic vector field* and the *characteristic 1-form* of the Sasakian structure, respectively. We next give some elementary properties of Sasakian structures. All of them follow as an immediate consequence of the definition and Proposition 1.1.2.

PROPOSITION 1.1.8: *Let $(\mathcal{S}, g)$ be a Sasakian manifold, $\{\xi, \eta, \Phi\}$ its Sasakian structure, and X and Y any pair of vector fields on $\mathcal{S}$. Furthermore, let $N_\Phi(Y, X) \;=\; [\Phi Y, \Phi X] + \Phi^2[Y, X] - \Phi[Y, \Phi X] - \Phi[\Phi Y, X]$ be the Nijenhuis torsion tensor of Φ. Then*

(i) $$\Phi \circ \Phi(Y) \;=\; -Y + \eta(Y)\xi,$$

(ii) $$\Phi\xi \;=\; 0, \qquad \eta(\Phi Y) \;=\; 0,$$

(iii) $$g(X, \Phi Y) + g(\Phi X, Y) \;=\; 0, \qquad g(\Phi Y, \Phi X) \;=\; g(Y, X) - \eta(Y)\eta(X),$$

(iv) $$d\eta(Y, X) \;=\; 2g(\Phi Y, X), \qquad N_\Phi(Y, X) \;=\; d\eta(Y, X) \otimes \xi.$$

A Sasakian manifold is not necessarily Einstein. As a simple consequence of the relation between Ricci curvature of $\mathcal{S}$ and its metric cone $C(\mathcal{S})$, the Einstein condition can be expressed in terms of Ricci-flatness of the cone metric $\bar{g}$ and we get

PROPOSITION 1.1.9: *Let $(\mathcal{S}, g)$ be be a Sasakian manifold of dimension $2n+1$. Then the metric g is Einstein if and only if the cone metric $\bar{g}$ is Ricci-flat, i.e., $(C(\mathcal{S}), \bar{g})$ is Kähler Ricci-flat (Calabi-Yau). In particular, it follows that the restricted holonomy group $\mathrm{Hol}^0(\bar{g}) \subset SU(n+1)$ and that the Einstein constant of g is positive and equals $2n$.*

An immediate consequence of the this proposition and Myers' Theorem is:

COROLLARY 1.1.10: *A complete Sasakian-Einstein manifold is compact with diameter less than or equal to π and with finite fundamental group.*

Now $\mathrm{Hol}^0(\bar{g})$ is the normal subgroup of the full holonomy group $\mathrm{Hol}(\bar{g})$ that is the component connected to the identity. There is a canonical epimorphism

$$\pi_1(\mathcal{S}) = \pi_1(C(\mathcal{S})) \longrightarrow \mathrm{Hol}(\bar{g})/\mathrm{Hol}^0(\bar{g}),$$

so if $\mathcal{S}$ is simply-connected its structure group reduces to $1 \times SU(n)$ and it will admit a spin structure. We have

COROLLARY 1.1.11: *Let $\mathcal{S}$ be a Sasakian-Einstein manifold such that the full holonomy group of the cone metric $\mathrm{Hol}(\bar{g})$ is contained in $SU(m+1)$. Then $\mathcal{S}$ admits a spin structure. In particular, every simply-connected Sasakian-Einstein manifold admits a spin structure.*

We give some examples that illustrate the complications in the presence of fundamental group. The hypothesis of this corollary is not necessary as the second example shows.

EXAMPLES 1.1.12: The real projective space $\mathcal{S} = \mathbb{RP}^{2n+1}$ with its canonical metric is Sasakian-Einstein, and the cone $C(\mathcal{S}) = (\mathbb{C}^{n+1} - \{0\})/\mathbb{Z}_2$ with the usual antipodal identification. We have $\mathrm{Hol}(\bar{g}) \simeq \pi_1(\mathcal{S}) \simeq \mathbb{Z}_2$. When n is odd the antipodal map τ is in $SU(n+1)$, so $\mathcal{S} = \mathbb{RP}^{2n+1}$ admits a spin structure. But when n is even the antipodal map τ does not lie in $SU(n+1)$, which obstructs a further reduction of the structure group. In this case it is well-known that $\mathcal{S} = \mathbb{RP}^{2n+1}$ does not admit a spin structure. In fact the generator of $\mathrm{Hol}(\bar{g}) \simeq \mathbb{Z}_2$ is the obstruction. There are many other similar examples. An example that shows that the hypothesis in Corollary 1.1.11 is not necessary is the following: Consider the lens space $L(p;, q_1, \cdots, q_n) \simeq S^{2n+1}/\mathbb{Z}_p$ where the q_i's are relatively prime to p. The action on $\mathbb{C}^{n+1} - \{0\}$ is generated by $(z_0, z_1, \cdots, z_n) \mapsto (\eta z_0, \eta^{q_1} z_1, \cdots, \eta^{q_n} z_n)$ where η is a primitive pth root of unity. It is known [Fra] that if p is odd, $L(p; q_1, \cdots, q_n)$ admits a spin structure. However, if $\sum_i q_i + 1$ is not divisible by p, the holonomy group $\mathrm{Hol}(\bar{g}) \simeq \mathbb{Z}_p$ does not lie in $SU(n+1)$.

Let $\mathcal{S}$ be a Sasakian manifold, suppose that the characteristic vector field ξ is complete. Since ξ has unit norm, it defines a 1-dimensional foliation $\mathcal{F}$ on $\mathcal{S}$. We shall be interested in the case when all the leaves of $\mathcal{F}$ are compact.

DEFINITION 1.1.13: *Let $(\mathcal{S}, g)$ be be a compact Sasakian manifold and let $\mathcal{F}$ be the 1-dimensional foliation defined by ξ. We say that $\mathcal{S}$ is quasi-regular if the foliation $\mathcal{F}$ is quasi-regular, i.e., each point $p \in \mathcal{S}$ has a cubical neighborhood U such that any leaf $\mathcal{L}$ of $\mathcal{F}$ intersects a transversal through p at most a finite number of times $N(p)$. Furthermore, $\mathcal{S}$ is called regular if $N(p) = 1$ for all $p \in \mathcal{S}$.*

It is known that the quasi-regular property is equivalent to the condition that all the leaves of the foliation are compact. In the regular case, the foliation $\mathcal{F}$ is simple, and defines a global submersion. In fact it defines a principal S^1 bundle over its space of leaves. In the quasi-regular case it is well-known [Tho, Mol] that ξ generates a locally free circle action on $\mathcal{S}$, and that the space of leaves is a compact orbifold (See the appendix for a brief review of orbifolds and their relation to foliations, in particular see A.3). We shall denote the space of leaves of the foliation $\mathcal{F}$ on $\mathcal{S}$ by $\mathcal{Z}$. Then the natural projection $\pi : \mathcal{S} \longrightarrow \mathcal{Z}$ is a Siefert fibration. It is an example of what we call a principal V-bundle over $\mathcal{Z}$. In Section 2 we shall study this foliation in detail.

1.2 3-Sasakian Spaces

Using all the definitions of the previous section we now describe a more specialized situation. Again, this can be done by an additional holonomy reduction requirement.

DEFINITION 1.2.1: *Let $(\mathcal{S}, g)$ be a Riemannian manifold of real dimension m. We say that $(\mathcal{S}, g)$ is 3-Sasakian if the holonomy group of the metric cone on $\mathcal{S}$ $(C(\mathcal{S}), \bar{g}) = (\mathbb{R}_+ \times \mathcal{S},\ dr^2 + r^2 g)$ reduces to a subgroup of $Sp(\frac{m+1}{4})$. In particular, $m = 4n+3, n \geq 1$ and $(C(\mathcal{S}), \bar{g})$ is hyperkähler.*

Since $C(\mathcal{S})$ is hyperkähler it has a hypercomplex structure $\{I^1, I^2, I^3\}$. We can define $\xi^a = I^a(\partial_r)$ for each $a = 1,2,3$. Then using the well-known properties of a hypercomplex structure together with Proposition 1.1.2 gives:

PROPOSITION 1.2.2: *Let $(\mathcal{S}, g)$ be a Riemannian manifold and let ∇ denote the Levi-Civita connection of g. Then $\mathcal{S}$ is 3-Sasakian if and only if it admits three characteristic vector fields $\{\xi^1, \xi^2, \xi^3\}$ (that is, satisfying any of the corresponding conditions in Proposition 1.1.2) such that $g(\xi^a, \xi^b) = \delta_{ab}$ and $[\xi^a, \xi^b] = 2\epsilon_{abc}\xi^c$.*

REMARK 1.2.3: By using Proposition 1.2.2 we can easily generalize the definition of a 3-Sasakian structure to orbifolds. A Riemannian orbifold $\mathcal{S}$ is a *3-Sasakian orbifold* if it admits three characteristic vector fields satisfying the conditions of Proposition 1.2.2, and if the action of the local uniformizing groups leaves the characteristic vector fields $\{\xi^1, \xi^2, \xi^3\}$ invariant.

The triple $\{\xi^1, \xi^2, \xi^3\}$ defines $\eta^a(Y) = g(\xi^a, Y)$ and $\Phi^a(Y) = \nabla_Y \xi^a$ for each $a = 1,2,3$. We call $\{\xi^a, \eta^a, \Phi^a\}_{a=1,2,3}$ the *3-Sasakian structure* on $(\mathcal{S}, g)$. The hyperkähler geometry of the cone $C(\mathcal{S})$ gives $\mathcal{S}$ a "quaternionic structure" reflected by the composition laws of the (1,1) tensors Φ^a. The following proposition describes additional properties of $\{\xi^a, \eta^a, \Phi^a\}$ not listed in Proposition 1.1.8(i-iv).

PROPOSITION 1.2.4: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold and let $\{\xi^a, \eta^a, \Phi^a\}_{a=1,2,3}$ be its 3-Sasakian structure. Then*

$$\begin{aligned} \eta^a(\xi^b) &= \delta^{ab}, \\ \Phi^a \xi^b &= -\epsilon^{abc}\xi^c, \\ \Phi^a \circ \Phi^b - \xi^a \otimes \eta^b &= -\epsilon^{abc}\Phi^c - \delta^{ab} id. \end{aligned}$$

REMARK 1.2.5: For any $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ such that $\tau_1^2 + \tau_3^2 + \tau_3^2 = 1$ the vector field $\xi(\tau) = \tau_1\xi^1 + \tau_2\xi^2 + \tau_3\xi^3$ has the Sasakian property. Therefore a 3-Sasakian manifold has not just 3 but an S^2 worth of Sasakian structures. This is in complete analogy with the hyperkähler case, and perhaps the name *hypersasakian* would have been more consistent. However, most of the existing literature uses the name *Sasakian 3-structure* or, as we do, *3-Sasakian structure*. Thus we have decided to stay with the latter.

Since a hyperkähler manifold is Ricci-flat, Proposition 1.1.9 and its corollary immediately imply:

COROLLARY 1.2.6: *Every 3-Sasakian manifold $(\mathcal{S}, g)$ of dimension $4n+3$ is Einstein with Einstein constant $\lambda = 2(2n+1)$. Moreover, if $\mathcal{S}$ is complete it is compact with finite fundamental group.*

The important result that every 3-Sasakian manifold is Einstein was first obtained by Kashiwada [Kas] using tensorial methods. One can also easily verify the structure group of any 3-Sasakian manifold is reducible to $Sp(n)\times\mathbb{I}_3$, where $\mathbb{I}_3$ denotes the three by three identity matrix [Kuo]. It follows [B-G-M 2] that

COROLLARY 1.2.7: *Every 3-Sasakian manifold* $(\mathcal{S},g)$ *is spin.*

If $(\mathcal{S},g)$ is compact the characteristic vector fields $\{\xi^1,\xi^2,\xi^3\}$ are complete and define a 3-dimensional foliation $\mathcal{F}_3$ on $\mathcal{S}$. The leaves of this foliation are necessarily compact as $\{\xi^1,\xi^2,\xi^3\}$ defines a locally free $Sp(1)$ action on $\mathcal{S}$. Hence, the foliation $\mathcal{F}_3$ is automatically quasi-regular and the space of leaves is a compact orbifold. We shall denote it by $\mathcal{O}$.

DEFINITION 1.2.8: *Let* $(\mathcal{S},g)$ *be be a compact 3-Sasakian manifold of dimension* $4n+3$*,* $n\geq 1$*, and let* $\mathcal{F}_3$ *be the 3-dimensional foliation defined by* $\{\xi^1,\xi^2,\xi^3\}$*. We say that* $\mathcal{S}$ *is regular if* $\mathcal{F}_3$ *is regular.*

REMARK 1.2.9: When $\dim(\mathcal{S})=3$ the leaf space of the foliation $\mathcal{F}_3$ is a single point so it makes no sense to talk about the regularity of $\mathcal{F}_3$. In this case we will say that $\mathcal{S}$ is *regular* if the foliation $\mathcal{F}_1$ defined by the characteristic vector field ξ^1 is regular.

For any $\tau\in S^2$ we can consider again the characteristic vector field $\xi(\tau)$ associated with the direction τ. This vector field defines a 1-dimensional foliation $\mathcal{F}_\tau\subset\mathcal{F}_3\subset\mathcal{S}$. This foliation has compact leaves and defines a locally free circle action $U(1)_\tau\subset Sp(1)$ on $\mathcal{S}$. In the next section we will describe the geometry of these foliations. Here, we simply conclude by the following observation concerning regularity properties of the foliations $\mathcal{F}_\tau\subset\mathcal{F}_3$ [Tan 2]:

PROPOSITION 1.2.10: *Let* $(\mathcal{S},g)$ *be a compact 3-Sasakian manifold. If* $\mathcal{F}_3$ *is regular then* $\mathcal{F}_\tau$ *is regular for all* $\tau\in S^2$*. Conversely, if* $\mathcal{F}_\tau$ *is regular for some* $\tau=\tau_0\in S^2$ *then it is regular for all* τ *and, hence,* $\mathcal{F}_3$ *is regular. Furthermore, if* $\mathcal{F}_3$ *is regular then either all the leaves are diffeomorphic to* $SO(3)$ *or all the leaves are diffeomorphic to* S^3*.*

Actually in the regular case it follows from a deeper result of Simon Salamon [Sal 1] that all leaves are diffeomorphic to S^3 in precisely one case, namely when $\mathcal{S}=S^{4n+3}$. (See the next section for further discussion.)

REMARK 1.2.11: Note that every Sasakian-Einstein 3-manifold must also have a 3-Sasakian structure. This is because in dimension four Ricci-flat and Kähler is equivalent to hyperkähler. Every compact 3-Sasakian 3-manifold, by Proposition 1.1.2(iii), must be a space of constant curvature 1. Hence, $\mathcal{S}$ is covered by a unit round 3-sphere and , in fact, it is always the homogeneous spherical space form S^3/Γ, where Γ is a discrete subgroup of $Sp(1)$ [Sas 2]. The homogeneous spherical space forms in dimension 3 are well-known. They are $Sp(1)/\Gamma$ where Γ is one of the finite subgroups of $Sp(1)$, namely: $\Gamma=\mathbb{Z}_m$ the cyclic group of order m, $\Gamma=\mathbb{D}^*_m$ a binary dihedral group with m is an integer greater than 2, $\Gamma=\mathbb{T}^*$ the binary tetrahedral group, $\Gamma=\mathbb{O}^*$ the binary octahedral group, $\Gamma=\mathbb{I}^*$ the binary icosahedral group. The only regular 3-Sasakian manifolds in dimension 3 are S^3 and $SO(3)$. More generally, the diffeomorphism classification of compact Sasakian 3-manifolds was recently completed by Geiges [Gei]. In addition to S^3/Γ one gets compact quotients of the double cover of $PSL_2(\mathbb{R})$ and the 3-dimensional Heisenberg group.

REMARK 1.2.12: A Sasakian-Einstein structure on a 3-Sasakian manifold does not have to be a part of the 3-Sasakian structure. The simplest example when this is the case is the lens space $\mathbb{Z}_k\backslash S^3$. Consider the unit 3-sphere $S^3\simeq Sp(1)$ as the unit quaternion $\sigma\in\mathbb{H}$. Such a sphere has two 3-Sasakian structures generated by the left and the right multiplication. Consider the homogeneous space $\mathbb{Z}_k\backslash S^3$, where the $\mathbb{Z}_k$-action is given by the multiplication from the left by $\rho \in Sp(1)$, $\rho^k = 1$. The quotient still has the "right" 3-Sasakian structure. But it also has a "left" Sasakian structure (the centralizer of $\mathbb{Z}_k$ in $Sp(1)$ is an S^1 and it acts on the coset from the left). This left Sasakian structure is actually regular while none of the Sasakian structures of the right 3-Sasakian structure can be regular unless $k = 1, 2$ [Tan 3].

2. The Fundamental Foliations

In this section we discuss the foliations associated with Sasakian and 3-Sasakian manifolds and describe their consequences.

2.1 The Sasakian Foliation

As mentioned in Section 1.1 a Sasakian manifold defines a Riemannian foliation of dimension 1. Using the basic properties described in Propositions 1.1.2 and 1.1.8. we have

PROPOSITION 2.1.1: *Let $(\mathcal{S}, g)$ be a Sasakian manifold, and let $\mathcal{F}$ denote the foliation defined by the characteristic vector field ξ. Then*

(i) *The metric g is bundle-like.*

(ii) *The leaves of $\mathcal{F}$ are totally geodesic.*

(iii) *The complementary vector bundle $\mathcal{H}$ to the trivial line subbundle of $T\mathcal{S}$ generated by ξ defines a strictly pseudoconvex CR structure on $\mathcal{S}$ with vanishing Webster torsion.*

In order to have a well behaved space of leaves we need a further assumption on the foliation. We have a generalization of the well-known Boothby-Wang fibration Theorem:

THEOREM 2.1.2: *Let $\mathcal{S}$ be a complete quasi-regular Sasakian manifold. Then*

(i) *The leaves of $\mathcal{F}$ are all diffeomorphic to circles with cyclic leaf holonomy groups.*

(ii) *The space of leaves $\mathcal{Z} = \mathcal{S}/\mathcal{F}$ has the structure of a Kähler orbifold.*

Suppose further that $(\mathcal{S}, g)$ is Sasakian-Einstein. Then

(iii) *The leaf space $\mathcal{Z}$ is a simply-connected normal projective algebraic variety with a Kähler-Einstein metric h of positive scalar curvature $4n(n+1)$ in such a way that $\pi : (\mathcal{S}, g) \longrightarrow (\mathcal{Z}, h)$ is an orbifold Riemannian submersion.*

(iv) $\mathcal{Z}$ has the structure of a $\mathbb{Q}$-factorial Fano variety. Hence, it is uniruled with Kodaira dimension $\kappa(\mathcal{Z}) = -\infty$.

PROOF: Parts (i) and (ii) are straightforward generalizations of the Boothby-Wang fibration in the Sasakian setting [Bl, K-Y] to the quasi-regular case. The point is that the CR structure on $\mathcal{S}$ pushes down to give a complex structure on $\mathcal{Z}$ and the Sasakian nature of $\mathcal{S}$ guarantees that the complex structure will be Kähler. That $\mathcal{Z}$ is projective algebraic is a consequence of Baily's version [Bai 2] of the Kodaira Embedding Theorem. Simple connectivity follows essentially from Kobayashi's argument in the smooth case by using the singular version of the Riemann-Roch Theorem due to Baum, Fulton, and Macpherson. The uniruledness is a result of Miyaoka and Mori [Mi-Mo]. For details we refer the reader to [B-G 1, B-G 2]. ∎

Let us recall that a complex variety X is $\mathbb{Q}$-factorial variety if for every Weil divisor D there exists a positive integer m such that mD is a Cartier divisor. The smallest such integer $m(D)$ is called the order of D. If X is compact the least common multiple taken over all Weil divisors on X is the order of X. Now on a compact complex orbifold Weil divisors coincide with Baily divisors [B-G 1] and Baily divisors correspond to line V-bundles. On X we have the group $\mathrm{Pic}^{orb}(X)$ of holomorphic line V-bundles on X and its subgroup

Pic(X) of holomorphic line bundles or absolute line V-bundles in Baily's terminology [Bai 1, Bai 2]. It is not difficult to prove [B-G 2]

PROPOSITION 2.1.3: *Let $\mathcal{S}$ be a complete Sasakian-Einstein manifold, and let $\mathcal{Z}$ be the space of leaves of the foliation $\mathcal{F}$ on $\mathcal{S}$. Then Pic$(\mathcal{Z})$ is free, and a subgroup of Pic$^{orb}(\mathcal{Z})$ which satisfies*

(i) Pic$^{orb}(\mathcal{Z}) \otimes \mathbb{Q} \simeq$ Pic$(\mathcal{Z}) \otimes \mathbb{Q}$.

(ii) If $\pi_1^{orb}(\mathcal{Z}) \simeq 0$, then Pic$^{orb}(\mathcal{Z}) \simeq$ Pic$(\mathcal{Z})$.

For an inversion theorem to Theorem 2.1.2 in the Sasakian-Einstein case and the construction of many nontrivial examples the reader is referred to [B-G 2] and [B-F-G-K] in the regular case. In particular, in dimension 5 we have

THEOREM 2.1.4[B-F-G-K]: *Let $\mathcal{S}$ be a simply-connected regular Sasakian-Einstein manifold of dimension 5. Then $\mathcal{S}$ is one of the following: S^5, the Stiefel manifold $V_2(\mathbb{R}^4)$ of 2-frames in $\mathbb{R}^4$, or the total space S_k of the S^1 bundles $S_k \to P_k$ for $3 \leq k \leq 8$ where P_k is a Del Pezzo surface with a Kähler-Einstein metric [T-Y]. It is known that S_k is diffeomorphic to the k-fold connected sum $S^2 \times S^3 \# \cdots \# S^2 \times S^3$.*

2.2 The One Dimensional 3-Sasakian Foliation

Fixing a Sasakian structure, say (ξ^1, Φ^1, η^1) in the 3-Sasakian structure, we notice that subbundle $\mathcal{H} = \ker\, \eta^1$ of $T\mathcal{S}$ together with $I = -\Phi^1|\mathcal{H}$ define the CR structure on $\mathcal{S}$. Actually a 3-Sasakian structure gives a special kind of CR structure, namely, a CR structure with a compatible holomorphic contact structure. Notice that the complex valued one form on $\mathcal{S}$ defined by $\eta^+ = \eta^2 + i\eta^3$ is type $(1,0)$ on $\mathcal{S}$. Moreover, one checks that η^+ is holomorphic with respect to the CR structure I. Although the 1-form η^+ is not invariant under the circle action generated by ξ^1, the trivial complex line bundle L^+ generated by η^+ is invariant. Thus, the complex line bundle L^+ pushes down to a nontrivial complex V-line bundle $\mathcal{L}$ on $\mathcal{Z}$. Let V denote the one dimensional complex vector space generated by L^+. Writing the circle action as exp $(i\phi\xi^1)$ shows that V is the representation with character $e^{-2i\phi}$, and since $\mathcal{S}$ is a principal S^1 V-bundle over $\mathcal{Z}$, the twisted product $\mathcal{L} \simeq \mathcal{S} \times_{S^1} V$ is a complex line V-bundle on $\mathcal{Z}$. Now we can define a map of V-bundles $\theta : T^{(1,0)}\mathcal{Z} \longrightarrow \mathcal{L}$ by

$$\theta(X) = \eta^+(\hat{X}), \tag{2.2.1}$$

where $\hat{X}$ is the horizontal lift of the vector field X on $\mathcal{Z}$. Notice that $\theta(X)$ is not a function on $\mathcal{Z}$ but a section of $\mathcal{L}$. Now a straightforward computation shows that $\eta^+ \wedge (d\eta^+)^n$ is a nowhere vanishing section of $\Lambda^{(2n+1,0)}\mathcal{H}$ on $\mathcal{S}$, and thus $\theta \wedge (d\theta)^n$ is a nowhere vanishing section of $K \otimes \mathcal{L}^{n+1}$, where K is the canonical V-line bundle (see the Appendix) on $\mathcal{Z}$. Hence, in Pic$^{orb}(\mathcal{Z})$ we have the relation $\mathcal{L}^{n+1} \otimes K = 1$. So the contact line V-bundle is $\mathcal{L} \simeq K^{-\frac{1}{n+1}}$ in Pic$^{orb}(\mathcal{Z})$. Alternatively, the subbundle ker θ is a holomorphic subbundle of $T^{(1,0)}\mathcal{Z}$ which is maximally non-integrable. This defines the complex contact structure on $\mathcal{Z}$. Of course, this construction depends on a choice of direction $\tau \in S^2$ in the 2-sphere of complex structures. However, the transitive action of $Sp(1)$ on S^2 guarantees that this structure is unique up to isomorphism as complex contact manifolds. We have [see B-G-M 1, B-G 1]:

THEOREM 2.2.2 *Let $\mathcal{S}$ be a complete 3-Sasakian manifold, choose a direction $\tau \in S^2$, and let $\mathcal{Z}_\tau$ denote the space of leaves of the corresponding foliation $\mathcal{F}_\tau$. Then $\mathcal{Z}_\tau$ is a compact*

$\mathbb{Q}$-factorial contact Fano variety with a Kähler-Einstein metric h of scalar curvature $8(2n+1)(n+1)$ such that the natural projection $\pi : \mathcal{S} \longrightarrow \mathcal{Z}_\tau$ is an orbifold Riemannian submersion with respect to the Riemannian metrics g on $\mathcal{S}$ and h on $\mathcal{Z}_\tau$.

We call the space $\mathcal{Z}_\tau$, usually just written $\mathcal{Z}$, the *twistor space* associated to $\mathcal{S}$. Actually there is another object that could merit the name the twistor space of $\mathcal{S}$, namely the trivial 2-sphere bundle $S^2 \times \mathcal{S}$ with the structure induced from the twistor space $S^2 \times C(\mathcal{S})$ of the hyperkähler cone.

An important property of the twistor space in the case of quaternionic Kähler manifolds is that it is ruled by rational curves. The same is true in our case as long as one allows for singularities. We have

Proposition 2.2.3: *$\mathcal{Z}$ is ruled by a real family of rational curves C with possible singularities on the singular locus of $\mathcal{Z}$. All the curves C are simply-connected, but $\pi_1^{orb}(C)$ can be a non-trivial cyclic group.*

For any line V-bundle $\mathcal{L}$ we let $\hat{\mathcal{L}}$ denote $\mathcal{L}$ minus its zero section.

Proposition 2.2.4: *Let $\mathcal{Z}$ be the twistor space of a 3-Sasakian manifold $\mathcal{S}$ of dimension $4n+3$, and assume that $\pi_1^{orb}(\mathcal{Z}) = 0$. If the contact line V-bundle $\mathcal{L}$ (or equivalently its dual $\mathcal{L}^{-1}$) has a root in $Pic^{orb}(\mathcal{Z})$, then it must be a square root, namely $\mathcal{L}^{\frac{1}{2}}$. Moreover, in this case if both $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}^{\frac{1}{2}}$ are proper in the sense of Kawasaki, then we must have $\mathcal{Z} = \mathbb{P}^{2n+1}$. In particular, this holds if the total space of $\hat{\mathcal{L}}$ is smooth.*

Proof: By Proposition 2.1.3 $\mathrm{Pic}^{orb}(\mathcal{Z})$ is torsion free. So the proof in [B-G 1] now goes through. By Proposition 2.2.3 $\mathcal{Z}$ is ruled by rational curves C which on the singular locus take the form $\Gamma\backslash\mathbb{P}^1$. Now the restriction $\mathcal{L}^{-1}|C$ is $\mathcal{O}(-2)$ which is a V-bundle if C is singular. In either case it has only a square root namely the tautological V-bundle $\mathcal{O}(-1)$. Since these curves C cover $\mathcal{Z}$ this proves the first statement. The second statement follows from a modification of an argument due to Kobayashi and Ochiai [K-O] and used by Salamon [Sal 1]. The main point is that since $\hat{\mathcal{L}}, \hat{\mathcal{L}}^{\frac{1}{2}}$ are proper and it follows that we can apply Kawasaki's Riemann-Roch Theorem [Kaw 1] together with the Kodaira-Baily Vanishing Theorem [Bai 2] to arbitrary powers of the line V-bundle $\mathcal{L}^{\frac{1}{2}}$ to give $(n+1)(2n+3)$ infinitesimal automorphisms of the complex contact structure on $\mathcal{Z}$. Since $\pi_1^{orb}(\mathcal{Z}) = 0$, these integrate to global automorphisms on $\mathcal{Z}$ and the result follows. See the Appendix and [B-G 1] for details. ∎

Remark 2.2.5: There is an error in the statement of Proposition 4.3 of [B-G 1]. The error is in leaving out the assumptions that $\pi_1^{orb}(\mathcal{Z})$ is trivial and that the contact line bundle is proper. Example 2.2.6 below shows that the conclusion in Proposition 2.2.4 does not necessarily hold if the hypothesis $\pi_1^{orb}(\mathcal{Z}) = 0$ is omitted. Likewise, Example 2.2.7 below gives a counterexample when the condition that $\mathcal{L}$ be proper is omitted.

Example 2.2.6: Consider the 3-Sasakian lens space $L(p;q) = \mathbb{Z}_p\backslash S^7$ constructed as follows: S^7 is the unit sphere in the quaternionic vector space $\mathbb{H}^2$ with quaternionic coordinates u_1, u_2. The action of $\mathbb{Z}_p$ is the left action defined by $(u_1, u_2) \mapsto (\tau u_1, \tau^q u_2)$, where $\tau^p = 1$ and p and q are relatively prime positive integers. If $p = 2m$ for some integer m then $-\mathrm{id}$ is an element of $\mathbb{Z}_{2m}$, so the 3-Sasakian manifolds $L(2m;q)$ and $L(m;q)$ both have the same twistor space, namely $\mathcal{Z} = \mathbb{Z}_m\backslash\mathbb{P}^3$, and $\pi_1^{orb}(\mathcal{Z}) \simeq \mathbb{Z}_m$. There are clearly many similar examples in all dimensions equal to 3 mod 4.

Example 2.2.7: Consider the 3-Sasakian 7 manifolds $\mathcal{S}(p_1, p_2, p_3)$ described in Section 7.4 below, where the p_i's are pairwise relatively prime, and precisely one of the p_i's is even, say p_1. $\mathcal{S}(p_1,p_2,p_3)$ is simply-connected and its twistor space $\mathcal{Z}(p_1,p_2,p_3)$ has $\pi_1^{orb}(\mathcal{Z}(p_1,p_2,p_3))=0$. Now there is a $\mathbb{Z}_2$ acting on $\mathcal{S}(p_1,p_2,p_3)$, but not freely, which acts as the identity on $\mathcal{Z}(p_1,p_2,p_3)$. Thus, $\mathbb{Z}_2\backslash\mathcal{S}(p_1,p_2,p_3)$ has $\mathcal{Z}(p_1,p_2,p_3)$ as its twistor space, and as a V-bundle $\mathbb{Z}_2\backslash\mathcal{S}(p_1,p_2,p_3)\longrightarrow\mathcal{Z}(p_1,p_2,p_3)$ is not proper in the sense of Kawasaki [Kaw 2]. Thus, the V-bundle $\hat{\mathcal{L}}$ is not proper, and Kawasaki's Riemann-Roch theorem [Kaw 1] cannot be applied.

We now wish to formulate a converse to Theorem 2.2.2.

DEFINITION 2.2.8: *A complete $\mathbb{Q}$-factorial Fano contact variety $\mathcal{Z}$ is said to be good if the total space of the principal circle bundle $\mathcal{S}$ associated with the contact V-line bundle $\mathcal{L}$ is a smooth compact manifold.*

Thus, for good $\mathbb{Q}$-factorial Fano contact varieties, $\mathcal{S}$ desingularizes $\mathcal{Z}$. As discussed in the Appendix this happens precisely when all the leaf holonomy groups inject into the group S^1 of the bundle. Notice also that in this case $\mathcal{S}$ is necessarily compact. We now are ready for:

THEOREM 2.2.9: *A good $\mathbb{Q}$-factorial Fano contact variety $\mathcal{Z}$ is the twistor space associated to a compact 3-Sasakian manifold if and only if it admits a compatible Kähler-Einstein metric h.*

PROOF: Let $\mathcal{Z}$ be a good $\mathbb{Q}$-factorial Fano contact variety with a compatible Kähler-Einstein metric h. Choose the scale of h so that the scalar curvature is $8(2n+1)(n+1)$. Let $\pi:\mathcal{S}\to\mathcal{Z}$ denote the principal orbifold circle bundle associated to $\mathcal{L}$. It is a smooth compact submanifold embedded in the dual of the contact V-line bundle $\mathcal{L}^{-1}$. The Kähler-Einstein metric h has Ricci form $\rho=4(n+1)\omega$, where ω is the Kähler form on $\mathcal{Z}$, and ρ represents the first Chern class of K^{-1}. Let η^1 be the connection in $\pi:\mathcal{S}\to\mathcal{Z}$ with curvature form $2\pi^*\omega$. Then the Riemannian metric $g_{\mathcal{S}}$ on $\mathcal{S}$ can be defined by $g_{\mathcal{S}}=\pi^*h+(\eta^1)^2$. It is standard (see the proof in Example 1 of Section 4.2 in [B-F-G-K]) that $g_{\mathcal{S}}$ is Sasakian-Einstein. As in Proposition 2.2.4 of [Sw] the V-bundle $\mathcal{L}\otimes\Lambda^{(1,0)}\mathcal{Z}$ has a section θ such that the Kähler-Einstein metric h decomposes as $h=|\theta|^2+h_D$, where h_D is a metric in the V-bundle D. Let us write $\pi^*\theta=\eta^+$. Since $\mathcal{S}$ is a circle bundle in $\mathcal{L}^{-1}$, the contact bundle $\mathcal{L}$ trivializes when pulled back to $\mathcal{S}$. This together with the condition that $\theta\wedge(d\theta)^n$ is nowhere vanishing on $\mathcal{Z}$ implies that η^+ is a nowhere vanishing complex valued 1-form on $\mathcal{S}$. So the metric $g_{\mathcal{S}}$ on $\mathcal{S}$ can be written as

$$g_{\mathcal{S}}=(\eta^1)^2+|\eta^+|^2+\pi^*h_D.$$

We claim that this metric is 3-Sasakian. To see this consider the total space M of the dual of the contact V-line bundle minus its 0 section which is $\mathcal{S}\times\mathbb{R}^+$. Put the cone metric dr^2+r^2g on M. The natural $\mathbb{C}^*$ action on M induces homotheties of this metric. Now using a standard Weitzenböck argument, LeBrun [Le 3] shows that M has a parallel holomorphic symplectic structure and his argument works just as well in our case. Let ϑ denote the pullback of the contact form θ to M which is a holomorphic 1-form on M that is homogeneous of degree 1 with respect to the $\mathbb{C}^*$ action. Thus $\Upsilon=d\vartheta$ is a holomorphic symplectic form on M which is parallel with respect to the Levi-Civita connection of the cone metric. Hence, (M,dr^2+r^2g) is hyperkähler. Furthermore, if $\{I^a\}_{a=1}^3$ denote hyperkähler endomorphisms on M, ϑ^2,ϑ^3 are the real and imaginary parts of ϑ, and ϑ^1 is the pullback of η^1 to M, then LeBrun shows that

$$\vartheta^1I^1=\vartheta^2I^2=\vartheta^3I^3.$$

It then follows from our previous work [B-G-M 1] that g is 3-Sasakian. But by construction $\mathcal{Z}$ is the space of leaves of the foliation generated by ξ^1, so $\mathcal{Z}$ must be the twistor space of the compact 3-Sasakian manifold $\mathcal{S}$. ∎

2.3 The Three Dimensional 3-Sasakian Foliation

Next we consider the three dimensional foliation $\mathcal{F}_3$ discussed in Section 1.2.

PROPOSITION 2.3.1: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold such that the characteristic vector fields ξ^a are complete. Let $\mathcal{F}_3$ denote the the canonical three dimensional foliation on $\mathcal{S}$. Then*

(i) *The metric g is bundle-like.*

(ii) *The leaves of $\mathcal{F}_3$ are totally geodesic spherical space forms $\Gamma\backslash S^3$ of constant curvature one, where $\Gamma \subset Sp(1) = SU(2)$ is a finite subgroup.*

(iii) *The 3-Sasakian structure on $\mathcal{S}$ restricts to a 3-Sasakian structure on each leaf.*

(iv) *The generic leaves are either $SU(2)$ or $SO(3)$.*

PROOF: The proof of (i), (ii), and (iii) follow from the basic relations for 3-Sasakian manifolds as in Proposition 2.1.1. To prove (iv) we notice that the foliation $\mathcal{F}_3$ is regular restricted to the generic stratum $\mathcal{S}_0$. By (ii) and regularity there is a finite subgroup $\Gamma \subset SU(2)$ such that the leaves of this restricted foliation are all diffeomorphic to $\Gamma\backslash S^3$, which is 3-Sasakian by (iii). Now the regularity of $\mathcal{F}_3$ on $\mathcal{S}_0$ implies that its leaves must all be regular with respect to the foliation generated by ξ^1. But a result of Tanno [Tan 1] says that the only regular 3-Sasakian 3-manifolds have $\Gamma = \mathrm{id}$ or $\mathbb{Z}_2$, in which case (iv) follows. ∎

EXAMPLE 2.3.2: Consider the 3-Sasakian lens space $L(p;q) = \mathbb{Z}_p\backslash S^7$ of Example 2.2.6. If p is odd then $-\mathrm{id}$ is not an element of $\mathbb{Z}_p$ so the generic leaf of the foliation $\mathcal{F}_3$ is S^3. The singular stratum consists of two leaves both of the form $\mathbb{Z}_p\backslash S^3$ with leaf holonomy group $\mathbb{Z}_p$. These two leaves are described by $u_2 = 0$ and $u_1 = 0$, respectively. If p is even then $-\mathrm{id}$ is an element of $\mathbb{Z}_p$, so the generic leaf is $SU(2)/\mathbb{Z}_2 = SO(3)$, and the leaf holonomy of the two singular leaves is $\mathbb{Z}_{\frac{p}{2}}$.

The next theorem was first proved by Ishihara [Ish 2] in the regular case using slightly different methods. First we need to describe our structures in the orbifold category. Recall that a quaternionic Kähler structure on a Riemannian manifold M is defined by

DEFINITION 2.3.3: *A Riemannian orbifold $\mathcal{O}$ is called a quaternionic Kähler orbifold if there is a rank 3 subbundle $\mathcal{G}$ of the endomorphism bundle End TM of TM which is preserved by the Levi-Civita connection and is locally generated by almost complex structures I, J, K that satisfy the algebra of the quaternions, and the action of the local uniformizing groups preserves the bundle $\mathcal{G}$. An alternative definition which works only in dimension greater than 4 is that $\mathcal{O}$ is a Riemannian orbifold whose holonomy group is a subgroup of $Sp(n)\cdot Sp(1)$.*

It is well-known that the strata of a quaternionic Kähler orbifold are not necessarily quaternionic Kähler [G-L]. The strata will be quaternionic Kähler if the local uniformizing groups act trivially on the fibres of $\mathcal{G}$ [D-Sw]. The group of the bundle $\mathcal{G}$ is $SO(3)$ with the adjoint representation. Thus, for each local uniformizing system on $\mathcal{O}$ there is a group homomorphism $\psi_i : \Gamma_i \longrightarrow SO(3)$.

THEOREM 2.3.4: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold of dimension $4n+3$ such that the characteristic vector fields ξ^a are complete. Then the space of leaves $\mathcal{S}/\mathcal{F}_3$ has the structure of a quaternionic Kähler orbifold $(\mathcal{O}, g_{\mathcal{O}})$ of dimension $4n$ such that the natural projection $\pi : \mathcal{S} \longrightarrow \mathcal{O}$ is a principal V-bundle with group $SU(2)$ or $SO(3)$ and a Riemannian orbifold submersion such that the scalar curvature of $g_{\mathcal{O}}$ is $16n(n+2)$.*

PROOF: We can split $T\mathcal{S} = \mathcal{V}_3 \oplus \mathcal{H}$, where $\mathcal{V}_3$ is the subbundle spanned by the characteristic vector fields $\{\xi^1,\xi^2,\xi^3\}$ and the "horizontal" bundle is the orthogonal complement $\mathcal{H}=\mathcal{V}_3^\perp$. Let $h\Phi^a=\Phi^a\mid_{\mathcal{H}}$ be the restriction of characteristic endomorphisms. One can easily see that

$$h\Phi^a\circ h\Phi^b=-\delta^{ab}\mathbf{1}+\sum_c\epsilon^{abc}h\Phi^c.$$

It follows that $\mathcal{H}$ is pointwise a quaternionic vector space and $\mathcal{O}$ is a compact quaternionic orbifold. We must show that the metric $g_{\mathcal{O}}$ obtained from g by the orbifold Riemannian submersion $\pi:\mathcal{S}\to\mathcal{O}$ has its holonomy group reduced to a subgroup of $Sp(n)\cdot Sp(1)$. This can be done by constructing a parallel 4-form on $\mathcal{O}$. Consider $\phi^a=d\eta^a$ and

$$\bar{\Phi}^a=\phi^a+\sum_{b,c}\epsilon^{abc}\eta^b\wedge\eta^c.$$

It is easy to see that the 4-form $\Omega=\sum_a\bar{\Phi}^a\wedge\bar{\Phi}^a$ is horizontal and $Sp(1)$-invariant. It follows that there is a unique 4-form $\hat{\Omega}$ on the orbifold $\mathcal{O}$ invariant under the action of the local uniformizing groups such that $\pi^*\hat{\Omega}=\Omega$. One can show that $\hat{\Omega}$ is parallel on $\mathcal{O}$ using standard tensor computation with O'Neill formulas [see B-G-M 1, G-L for details]. In the case $n=1$ the parallelism of the 4-form does not further restrict Riemannian geometry of $\mathcal{O}$. However, one can show that $(\mathcal{O},g_{\mathcal{O}})$ is a compact self-dual Einstein orbifold. Self-duality follows easily from the fact that $\mathcal{O}$ is quaternionic. The fact that the metric is Einstein is a simple computation and in the regular case can be found in [Tan 2]. ∎

There is an important inversion theorem of Theorem 2.3.4 originally in the regular case due to Konishi [Kon]. By now there are several proofs of this, all of them related. Given a quaternionic Kähler orbifold $\mathcal{O}$ one can construct the Salamon twistor space $\mathcal{Z}$ and then get $\mathcal{S}$ from the inversion theorem of [B-G 1]. Another approach would be to construct the orbifold version of Swann's bundle [Sw] on $\mathcal{O}$ and then use the results of [B-G-M 1] to obtain $\mathcal{S}$. Here our proof is essentially that of Konishi's, only slightly modified to handle the orbifold situation.

THEOREM 2.3.5: *Let $(\mathcal{O},g_{\mathcal{O}})$ be a quaternionic Kähler orbifold of dimension $4n$ with positive scalar curvature $16n(n+2)$. Then there is a principal $SO(3)$ V-bundle over $\mathcal{O}$ whose total space $\mathcal{S}$ admits a 3-Sasakian structure with scalar curvature $2(2n+1)(4n+3)$.*

PROOF: Let $\mathcal{G}$ denote the V-subbundle of End $T\mathcal{O}$ describing the quaternionic structure. Let $\{\tilde{U}_i\}$ be local uniformizing neighborhoods that cover $\mathcal{O}$ and $\mathcal{I}^a_i$ a local framing of $\mathcal{G}$ on $\tilde{U}_i$ that satisfies

$$\mathcal{I}^a_i\circ\mathcal{I}^b_i=-\delta^{ab}\mathrm{id}+\epsilon^{abc}\mathcal{I}^c_i.$$

Since $\mathcal{O}$ is quaternionic Kähler there are 1-forms τ^a_i on each $\tilde{U}_i$ such that

$$\nabla\mathcal{I}^a_i=\epsilon^{abc}\tau^b_i\otimes\mathcal{I}^c_i.$$

Now the structure group of the V-bundle End $T\mathcal{O}$ is $Sp(n)\cdot Sp(1)$, and that of the V-subbundle $\mathcal{G}$ is $SO(3)$. Let $\pi:\mathcal{S}\longrightarrow\mathcal{O}$ denote the principal $SO(3)$ V-bundle associated to $\mathcal{G}$. The local 1-forms τ^a_i are the components of an $\mathfrak{so}(3)$ connection $\tau_i=\sum_{a=1}^3\tau^a_ie_a$, where $\{e_a\}$ denotes the standard basis of $\mathfrak{so}(3)$ which satisfies the Lie bracket relations $[e_a,e_b]=2\epsilon^{abc}e_c$. The local connection forms satisfy the well-known relations

$$\tau_i=\mathrm{ad}_{g_{ij}}\tau_j+g_{ij}^{-1}dg_{ij}$$

in $\tilde{U}_i \cap \tilde{U}_j$ for some smooth map $g_{ij} : \tilde{U}_i \cap \tilde{U}_j \longrightarrow SO(3)$. Furthermore, from [Ish 2] one checks that the curvature forms

2.3.6 $$\omega_i^a = d\tau_i^a + \epsilon^{abc}\tau_i^b \wedge \tau_i^c$$

satisfy the relation $2g_{\mathcal{O}}(\mathcal{I}_i^a X, Y) = \omega_i^a(X, Y)$. Now on each $\tilde{U}_i$ there exists a smooth local section $\sigma_i : \tilde{U}_i \longrightarrow \mathcal{S}$ and on $\mathcal{S}$ there is a global 1-form η^a such that $\tau_i^a = \sigma_i^* \eta^a$. On $\mathcal{S}$ we define a Riemannian metric by

$$g = \pi^* g_{\mathcal{O}} + \sum_{a=1}^{3} \eta^a \otimes \eta^a.$$

By construction the vector fields ξ^a generating the $SO(3)$ action on $\mathcal{S}$ are dual to the forms η^a with respect to this metric, viz $g_{\mathcal{S}}(X, \xi^a) = \eta^a(X)$ for any vector field X on $\mathcal{S}$. Also by construction the vector fields ξ^a are Killing fields with respect to the metric g. Now define the $(1,1)$ tensor field $\Phi^a = \nabla \xi^a$. Since the ξ^a are mutually orthogonal vector fields of unit length on $\mathcal{S}$, one easily checks that $\Phi^a \xi^b = -\epsilon^{abc}\xi^c$. Thus Φ^a splits as

$$\Phi^a = h\Phi^a + \epsilon^{abc}\xi^b \otimes \eta^c.$$

One then checks using 2.3.6 that on each open set $\pi^{-1}(U_i)$, $h\Phi^a$ equals the horizontal component of $(\sigma_i)_* \mathcal{I}_i^a$. From this one then shows that

$$\Phi^a \circ \Phi^b - \xi^a \otimes \eta^b \;=\; -\epsilon^{abc}\Phi^c - \delta^{ab}\mathrm{id}.$$

The result then follows by Propositions 1.2.2 and 1.2.3. ∎

We mention that from the discussion in the Appendix it follows that if the homomorphisms $\psi_i : \Gamma_i \longrightarrow SO(3)$ are injective the total space $\mathcal{S}$ will be a smooth 3-Sasakian manifold.

Konishi's construction gives an $SO(3)$ bundle over $\mathcal{O}$. In the case that $\mathcal{O}$ is a smooth manifold there is a well-known obstruction [Sal 1] to lifting this bundle to an $Sp(1)$ bundle, the Marchiafava-Romani class ε. Actually ε is the obstruction to lifting the principal $Sp(n)\cdot Sp(1)$ frame bundle to an $Sp(n) \times Sp(1)$ bundle [Ma-Ro, Sal 1]. This obstruction also occurs when $\mathcal{O}$ is an orbifold as long as one uses Haefliger's orbifold cohomology (see Appendix). The class ε is the image of the connecting homomorphism

2.3.7 $$\delta : H^1_{orb}(\mathcal{O}, \mathcal{G}) \longrightarrow H^2_{orb}(\mathcal{O}, \mathbb{Z}_2),$$

where $\mathcal{G}$ is the sheaf of germs of smooth orbifold maps from open sets of $\mathcal{O}$ to the group $Sp(n)\cdot Sp(1)$. If following Salamon [Sal 1] we write $T\mathcal{O} \otimes \mathbb{C} \simeq \mathbf{E} \otimes \mathbf{H}$, then ε is the second Stiefel-Whitney class w_2 of the bundle $S^2(\mathbf{H})$ over $\mathcal{O}$. We have

Proposition 2.3.8: *The principal $SO(3)$ V-bundle constructed in Theorem 2.3.5 lifts to a principal $Sp(1)$ V-bundle if and only if $\varepsilon \in H^2_{orb}(\mathcal{O}, \mathbb{Z}_2)$ vanishes. Moreover, when $\varepsilon = 0$ the 3-Sasakian structure on the total space $\mathcal{S}$ of the $SO(3)$ V-bundle lifts to the total space $\mathcal{S}'$ of the $Sp(1)$ V-bundle.*

Thus, in the case that $\varepsilon = 0$ there are precisely two 3-Sasakian orbifolds $\mathcal{S}, \mathcal{S}'$ corresponding to the quaternionic Kähler orbifold $\mathcal{O}$. Let $\mathcal{Z}$ denote the twistor space of the orbifold $\mathcal{O}$. Then likewise, since $S^2 \simeq SO(3)/S^1 \simeq Sp(1)/S^1$ the two 3-Sasakian orbifolds $\mathcal{S}$ and $\mathcal{S}'$ have the same twistor space $\mathcal{Z}$. When $\mathcal{O}$ is a smooth manifold a result of Salamon [Sal 1] says that $\varepsilon = 0$ if and only if the quaternionic Kähler manifold $\mathcal{O}$ is quaternionic projective space. If we impose the condition that the orbifolds $\mathcal{S}$ and $\mathcal{S}'$ are smooth manifolds, there is a similar result.

THEOREM 2.3.9: *If two 3-Sasakian manifolds $\mathcal{S}$ and $\mathcal{S}'$ are associated to the same quaternionic Kähler orbifold $\mathcal{O}$ or equivalently the same twistor space $\mathcal{Z}$, then both $\mathcal{S}$ and $\mathcal{S}'$ have the same universal covering space $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ is a standard 3-Sasakian sphere.*

PROOF: We work with the twistor space $\mathcal{Z}$. Now $\mathcal{S}$ and $\mathcal{S}'$ are unit circle bundles in the line V-bundles $\mathcal{L}^{-1}$ and $\mathcal{L}'^{-1}$ respectively. Moreover, since $\mathcal{S}'$ is a double cover of $\mathcal{S}$, it follows that $\mathcal{L} = \mathcal{L}'^2$. Consider the universal orbifold cover $\tilde{\mathcal{Z}}$ of $\mathcal{Z}$ with $\pi_1^{orb}(\tilde{\mathcal{Z}}) = 0$. Pull back the V-bundles $\mathcal{L}$ and $\mathcal{L}'$ to V-bundles $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}'$ on $\tilde{\mathcal{Z}}$ respectively. These bundles satisfy $\tilde{\mathcal{L}}' = \widetilde{\mathcal{L}^{\frac{1}{2}}}$. By construction and naturality of the covering maps $\tilde{\mathcal{L}}$ is the contact line bundle on $\tilde{\mathcal{Z}}$. Moreover, $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}'$ which are the total spaces of the pullbacks of $\mathcal{S}$ and $\mathcal{S}'$ to $\tilde{\mathcal{Z}}$ are both smooth manifolds since they cover smooth manifolds. Thus, by Proposition 2.2.4 $\tilde{\mathcal{Z}} \simeq \mathbb{P}^{2n+1}$. It follows that $\tilde{\mathcal{S}}' \simeq S^{4n+3}$. ∎

REMARK 2.3.10: Konishi also considers the case when the quaternionic Kähler manifold has negative scalar curvature. This gives a Sasakian 3-structure on $\mathcal{S}$ with indefinite signature $(3, 4n)$.

Finally, we give some general results concerning the curvature of any 3-Sasakian manifold. Since the curvature of any Riemannian manifold is completely determined by its sectional curvature and the sectional curvature of any Sasakian manifold [Bl, Y-K] is completely determined by the Φ-sectional curvature, we essentially give the latter. This shows that the local geometry of any 3-Sasakian manifold determines and is determined by that of its associated quaternionic Kähler orbifold.

PROPOSITION 2.3.11: *Let $(\mathcal{S}, g, \xi^a)$ be a 3-Sasakian manifold and let K and $\check{K}$ denote the sectional curvatures of g and its transverse component $\check{g}$, respectively. Then if X is any horizontal vector field of unit length on $\mathcal{S}$, we have $K(X, \Phi^a X) = \check{K}(X, \Phi^a X) - 3$.*

PROOF: We first notice that [Bes: 9.29c] gives $K(X, \Phi^a X) = \check{K}(X, \Phi^a X) - 3|A_X \Phi^a X|^2$ where A is O'Neill's tensor [Bes] which is essentially the curvature of the $\mathfrak{sp}(1)$ valued orbifold connection [B-G-M 1]. One then shows that for any horizontal vector fields X, Y on $\mathcal{S}$ we have

2.3.12
$$A_X Y \;=\; \sum_{a=1}^{3} g(\Phi^a X, Y)\xi^a,$$

and the identity follows. ∎

2.4 The Second Einstein Metric

Of course, by an Einstein metric we actually mean a homothety class of Einstein metrics. In this section we shall show by using a theorem of Berard-Bergery [Bes] that every 3-Sasakian manifold has at least two distinct homothety classes of Einstein metrics. The method involves the canonical variation [Bes] associated with Riemannian submersions. Due to the local nature of the calculations involved this construction holds equally well for orbifold Riemannian submersions. The canonical variation is constructed as follows [Bes]:

Let $\pi : M \longrightarrow B$ be an orbifold Riemannian submersion with g the Riemannian metric on M. Let $\mathcal{V}$ and $\mathcal{H}$ denote the vertical and horizontal subbundles of the tangent bundle TM.

For each real number $t > 0$ we construct a one parameter family g_t of Riemannian metrics on M by defining

2.4.1 $$g_t|\mathcal{V} = tg|\mathcal{V}, \qquad g_t|\mathcal{H} = g|\mathcal{H}, \qquad g_t(\mathcal{V},\mathcal{H}) = 0.$$

So for each $t > 0$ we have an orbifold Riemannian submersion with the same base space. Furthermore, if the fibers of g are totally geodesic, so are the fibers of g_t. We apply the canonical variation to the orbifold Riemannian submersion $\pi : \mathcal{S}\longrightarrow\mathcal{O}$. The metric as well as other objects on $\mathcal{O}$ will be denoted with a check such as $\check{g}$.

THEOREM 2.4.2: *Every 3-Sasakian manifold admits a second Einstein metric of positive scalar curvature.*

PROOF: We apply the canonical variation to the orbifold Riemannian submersion $\pi : \mathcal{S}\longrightarrow\mathcal{O}$. According to the Bérard-Bergery Theorem [Bes: 9.73] there are several conditions to check. First, the connection $\mathcal{H}$ must be a Yang-Mills connection. The condition for this is [Bes]:

$$\sum_i g((\nabla_{X_i}A)_{X_i}X,\xi^a) = 0$$

for each $a = 1,2,3$ and where X_i is a local orthonormal frame of $\mathcal{H}$, X is any horizontal vector field, and A is O'Neill's tensor. Actually we can use standard computations together with 1.1.3 and 2.3.11 to prove the stronger condition $(\nabla_{X_i}A)_{X_i}X = 0$. Second $|A|^2$ must be constant. To compute this notice that using [Bes] and 2.3.11 we find

2.4.3 $$g(A_{X_i},A_{X_j}) = 3\delta_{ij}, \qquad g(A\xi^a,A\xi^b) = 4n\delta^{ab}.$$

This gives $|A|^2 = 12n$. The final condition to be satisfied is $(\check{\lambda})^2 - \hat{\lambda}(12n+18) > 0$, where $\check{\lambda}$ and $\hat{\lambda}$ are the Einstein constants for $\mathcal{O}$ and the fibers, respectively, and we have made use of 2.4.3. Since in our case $\check{\lambda} = 4(n+2)$ and $\hat{\lambda} = 2$, we see that the inequality is satisfied. ∎

The scalar curvature of any metric g_t in the canonical variation of the metric g is given by the formula $s_t = 16n(n+2) + 6/t - 12nt$ [Bes]. Moreover, the value of t that gives the second Einstein metric is $t_0 = \frac{\hat{\lambda}}{\check{\lambda}-\hat{\lambda}} = \frac{1}{2n+3}$. The ratio of the two metrics depends only on the homothety class and is given by

2.4.4 $$\frac{s_{\frac{1}{2n+3}}}{s_1} = 1 + \frac{6(n+1)}{(2n+3)(2n+1)}.$$

In the special case $n = 1$ that is dim $\mathcal{S} = 7$, both the 3-Sasakian metric and the second Einstein metric have weak G_2 holonomy [F-K-M-S, G-Sal]. See Theorem 5.2.9 below.

2.5 Invariants and the Classification of 3-Sasakian Structures

We consider the question of equivalence of 3-Sasakian manifolds. A 3-Sasakian structure $\{\xi^a,\eta^a,\Phi^a\}_{a=1}^3$ on a manifold $(\mathcal{S},g)$ is determined completely by the metric g and the characteristic vector fields ξ^a.

DEFINITION 2.5.1: *Two 3-Sasakian manifolds $(\mathcal{S},g)$ and $(\mathcal{S}',g')$ are said to be isomorphic if there exist a diffeomorphism $F : \mathcal{S}\longrightarrow\mathcal{S}'$ and an $\phi \in Sp(1)$ such that $F^*g' = g$ and $\acute{\xi}^a = (Ad_\phi)_*F_*\xi^a$, where Ad denotes the adjoint action of $Sp(1)$ on its Lie algebra $\mathfrak{sp}(1)$.*

In practice we shall always choose a basis $\acute{\xi}^a$ of the 3-Sasakian structure on $\mathcal{S}'$ so that $F_*\xi^a = \acute{\xi}^a$. Now given such a diffeomorphism $F : \mathcal{S}\longrightarrow\mathcal{S}'$ it is clear that the corresponding foliations are F-related, that is that $F_*\mathcal{F}_1 = \mathcal{F}_1'$ and $F_*\mathcal{F}_3 = \mathcal{F}_3'$. This induces a commutative diagram of orbifold diffeomorphisms

2.5.2
$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{F} & \mathcal{S}' \\ \downarrow & & \downarrow \\ \mathcal{Z} & \xrightarrow{F_1} & \mathcal{Z}' \\ \downarrow & & \downarrow \\ \mathcal{O} & \xrightarrow{F_3} & \mathcal{O}'. \end{array}$$

This implies that if L_x is the leaf of $\mathcal{F}_3$ at $x \in \mathcal{O}$, then $F(L_x)$ is the leaf at $F_3(x) \in \mathcal{O}'$, that is, $L'_{F_3(x)} = F(L_x)$. Let $G(L)$ denote the leaf holonomy group of the leaf L. Then we have $G(L'_{F_3(x)}) \approx G(L_x)$. More generally let $\mathcal{G}(\mathcal{S})$ denote the holonomy groupoid [Moo-Sch] of the foliation $\mathcal{F}_3$, that is the set of triples $(x, y, [\alpha])$ where $x, y \in \mathcal{S}$ lie on the same leaf L_x of $\mathcal{F}_3$ and $[\alpha]$ is the holonomy equivalence class of all piecewise smooth paths from x to y lying in L_x. Multiplication in the groupoid $\mathcal{G}(\mathcal{S})$ is defined on pairs of triples $(x, y, [\alpha]), (x', y', [\alpha'])$ precisely when $y = x'$ and then $(x, y, [\alpha])\cdot(x', y', [\alpha']) = (x, y', [\alpha'\alpha])$. Furthermore, the subgroup of triples $(x, x, [\alpha])$ with x fixed is identified with the holonomy group $G(L_x)$. With this structure, $\mathcal{G}(\mathcal{S})$ is a locally compact topological groupoid [Moo-Sch]. (Actually $\mathcal{G}(\mathcal{S})$ is a smooth manifold of dimension $4n + 6$ but we do not use this here). We have

Proposition 2.5.3: *Let $F : \mathcal{S}\longrightarrow\mathcal{S}'$ be an isomorphism of 3-Sasakian manifolds. Then F induces an isomorphism $F_* : \mathcal{G}(\mathcal{S})\longrightarrow\mathcal{G}(\mathcal{S}')$ of topological groupoids.*

The groupoid $\mathcal{G}(\mathcal{S})$ will be studied in a forthcoming work. For now we are interested in the unordered list $(\Gamma_1, \Gamma_2, \cdots)$ of holonomy groups in $\mathcal{G}(\mathcal{S})$ up to abstract isomorphism. This list is finite if $\mathcal{S}$ is complete and it provides important invariants of a 3-Sasakian manifold. Since the leaves of the foliation $\mathcal{F}_3$ are all spherical space forms, the groups Γ_i are all either subgroups of $Sp(1)$ or all subgroups of $SO(3)$, depending on whether the Marchiafava-Romani class ε of the quaternionic Kähler orbifold $\mathcal{O}$ is 0 or 1, respectively. Notice that it follows from its definition and 2.5.2 above that the class ε is an invariant of the 3-Sasakian structure on the manifold $\mathcal{S}$. Indeed, ε can be identified with a certain secondary characteristic class of the foliation $\mathcal{F}_3$. Thus, the Marchiafava-Romani class splits the isomorphism classes $\mathbb{S}$ of 3-Sasakian manifolds into the disjoint union $\mathbb{S}_0 + \mathbb{S}_1$ depending on whether ε is 0 of 1. A further rough classification scheme is given by

Definition 2.5.4: *$\mathcal{S}$ is said to be:*

(1) regular if all the Γ_i are the identity.

(2) of cyclic type if all the Γ_i are cyclic.

(3) of dihedral type if all the Γ_i are either cyclic or dihedral or binary dihedral with at least one Γ_i non-Abelian.

(4) of polyhedral type if at least one of the Γ_i is one of the polyhedral groups, tetrahedral, octahedral, or icosahedral (or the corresponding binary double covers) groups.

The definition of regular here coincides with that of Definition 1.2.8. In general 3-Sasakian dimension $4n+3$ the only known examples of 3-Sasakian manifolds of polyhedral or dihedral type are the spherical space forms $\Gamma\backslash S^{4n+3}$ and $\Gamma\backslash\mathbb{RP}^{4n+3}$, where Γ is a binary polyhedral or a binary dihedral group in the first case and a polyhedral or a dihedral group in the second. The action is that induced by the diagonal action of Γ on the quaternionic vector space $\mathbb{H}^{n+1}$. However, in dimension 7 there exist 3-Sasakian manifolds of dihedral or polyhedral type [B-G 3, G-Ni] which are not spherical space forms. All other known non-regular 3-Sasakian manifolds are of cyclic type and are discussed in detail in Section 7. The following is essentially due to Salamon:

Theorem 2.5.5: *Let $\mathcal{S}$ be a complete regular 3-Sasakian manifold with $\varepsilon = 0$. Then $\mathcal{S} \simeq S^{4n+3}$ or $\mathbb{RP}^{4n+3}$.*

For more results about regular 3-Sasakian manifolds see Section 4.4 below. Next we consider an important infinitesimal rigidity result. In the regular case this rigidity is a simple consequence of the results of LeBrun [Le 2] and Nagatomo [N] (see [G-Sal]). In the general case it was recently proved by Pedersen and Poon [Pe-Po 2].

Theorem 2.5.6: *Complete 3-Sasakian manifolds are infinitesimally rigid.*

Outline of Proof: The deformation theory of 3-Sasakian manifolds is tied to the deformation theory of hypercomplex manifolds studied previously in [Pe-Po 1]. Let $\mathcal{S}$ be a complete 3-Sasakian manifold. Then the compact manifold $S^1\times\mathcal{S}$ has a natural hypercomplex structure [B-G-M 2]. Thus, its twistor space W is compact and fibers holomorphically over $\mathbb{CP}^1$. Moreover, there is a holomorphic foliation on W whose leaves are elliptic Hopf surfaces, and whose space of leaves is the twistor space $\mathcal{Z}$ associated to $\mathcal{S}$. The geometry of the corresponding deformation theory is as follows. Deformations $(\mathcal{S}_t, g_t)$ of the 3-Sasakian structure $(\mathcal{S}_0, g_0)$ on $\mathcal{S}$ correspond to deformations of the hypercomplex structure on $S^1\times\mathcal{S}$ of the form $S^1\times\mathcal{S}_t$. In turn these deformations correspond to deformations of the holomorphic fibration $p: W\longrightarrow\mathbb{CP}^1$. Thus, there are natural projections:

2.5.7
$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "\Phi"] & \\ \mathbb{CP}^1 & & \mathcal{Z}, \end{tikzcd}$$

where each fiber of p is a divisor in W diffeomorphic to $S^1\times\mathcal{S}$ and Φ is an orbifold submersion whose leaves are elliptic Hopf surfaces. Now the product map $p\times\Phi: W\longrightarrow\mathbb{CP}^1\times\mathcal{Z}$ is an orbifold submersion whose leaves are elliptic curves. The differential of $p\times\Phi$ induces the exact sequence of sheaves

$$0\longrightarrow\mathcal{O}_W\longrightarrow\Theta_W\longrightarrow\Phi^*\Theta_{\mathcal{Z}}\oplus p^*\Theta_{\mathbb{CP}^1}\longrightarrow 0,$$

where $\mathcal{O}_W$ denotes the structure sheaf of W and Θ denotes the holomorphic tangent sheaf. Then using standard techniques together with the Kodaira-Baily vanishing theorem and the orbifold version of the Akizuki-Nagano vanishing theorem, Pedersen and Poon show that the virtual parameter space for 3-Sasakian deformations lies in

$$H^0(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})\otimes H^1(F,\mathcal{O}_F)\oplus H^0(\mathcal{Z},\Theta_{\mathcal{Z}})\otimes H^1(F,\mathcal{O}_F)$$

2.5.8
$$\oplus H^1(\mathcal{Z},\Theta_{\mathcal{Z}})\otimes H^0(F,\mathcal{O}_F)\oplus H^1(W,p^*\Theta_{\mathbb{CP}^1}),$$

where F is the generic elliptic Hopf surface $S^1 \times Sp(1)$. One then analysis each summand of 2.5.8 to show that there are no 3-Sasakian deformations. For example, possible deformations lying in the last summand vanish by results of Horikawa, while 3-Sasakian deformations lying in the second and third summands must preserve the complex contact structure on $\mathcal{Z}$. There are no such deformations in the third summand by the Kodaira-Baily vanishing theorem. Elements in the second summand correspond to complex contact transformations that are invariant under the $U(1) \times U(1)$ action coming from a discrete quotient of the $\mathbb{C}^*$ principal action on $\mathcal{L}$, and there are no such elements. Finally, elements of the first summand correspond to scale changes in the S^1 factor of $S^1 \times Sp(1)$ and these hypercomplex deformations do not come from 3-Sasakian ones. ∎

While this theorem says that there is no "infinitesimal moduli", there may well be discrete moduli of 3-Sasakian structures. Indeed, we believe that the work of Kruggel's [Kru 3] can be used with the aid of a computer to construct distinct 3-Sasakian structures on the same manifold. See Remark 7.4.9 below.

3. Homogeneous Spaces

In this section we classify Sasakian-Einstein and 3-Sasakian homogeneous spaces. We begin with the Sasakian-Einstein case.

3.1 Homogeneous Sasakian-Einstein Manifolds

As a Sasakian vector field ξ is Killing, every Sasakian, and, hence, Sasakian-Einstein manifold $\mathcal{S}$ has non-trivial isometries. Recall the following well-known terminology. Let G be a complex semi-simple Lie group. A maximal solvable complex subgroup B is called a Borel subgroup, and B is unique up to conjugacy. Any complex subgroup P that contains B is called a parabolic subgroup. Then the homogeneous space G/P is called a generalized flag manifold. A well-known result of Wang [Ahk] says that every simply-connected homogeneous Kähler manifold is a generalized flag manifold.

Definition 3.1.1: *A compact Sasakian-Einstein manifold $\mathcal{S}$ is called a homogeneous Sasakian-Einstein manifold if there is a transitive group K of isometries on $\mathcal{S}$ that preserve the Sasakian structure, that is, if $\phi^k \in$ Diff $\mathcal{S}$ corresponds to $k \in K$, then $\phi^k_*\xi = \xi$. (This implies that both Φ and η are also invariant under the action of K.)*

Note that K is a compact Lie group by compactness of $\mathcal{S}$. The following is a result of [B-G 2].

Theorem 3.1.2: *Let $\mathcal{S}$ be a compact quasi-regular homogeneous Sasakian-Einstein manifold. Then $\mathcal{S}$ is an S^1-bundle over a generalized flag manifold G/P. Conversely, given any generalized flag manifold G/P there is a circle bundle $\pi : \mathcal{S} \longrightarrow G/P$ whose total space $\mathcal{S}$ is a homogeneous Sasakian-Einstein manifold.*

Proof: As in Proposition 4.6 of [B-G-M 2], $\mathcal{S}$ is regular. By Proposition 2.1.2 $\mathcal{S}$ fibers over a simply-connected Fano variety $\mathcal{Z}$ with a Kähler-Einstein metric of positive scalar curvature. Since the action of K commutes with ξ it sends fibers to fibers, and thus acts transitively on $\mathcal{Z}$. But by Wang's theorem [Akh], $\mathcal{Z} = G/P$ for some complex semi-simple Lie group G and some parabolic subgroup $P \subset G$. Now K preserves the Kähler-Einstein structure, and thus the complex structure. So $K \subset G$. In fact G is just the complexification of K its maximal compact subgroup [W].

Conversely, by a theorem of Matsushima [Bes] every G/P admits K invariant Kähler-Einstein metric, where K is the maximal compact subgroup of G. Moreover, there is a subgroup $U \subset K$ such that $G/P = K/U$. Then by the Kobayashi construction described in the previous section there is a circle bundle over G/P whose total space $\mathcal{S}$ admits a

Sasakian-Einstein metric. By the construction one easily sees that this metric is homogeneous. ∎

The following corollary lists all the possible G/P in the first three dimensions:

COROLLARY 3.1.3: *Let $\mathcal{S}$ be a compact homogeneous Sasakian-Einstein manifold of dimension $2n+1$. Then $\mathcal{S}$ is a circle bundle over*

(i) $\mathbb{CP}^1$ *when* $n = 1$,

(ii) $\mathbb{CP}^2$ *or* $\mathbb{CP}^1 \times \mathbb{CP}^1$ *when* $n = 2$,

(iii) $\mathbb{CP}^3$, $\mathbb{CP}^2 \times \mathbb{CP}^1$, $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$, *the complex flag* $F_{3,2,1} = SU(3)/T^2$, *and the real Grassmannian* $Gr_2(\mathbb{R}^5)$ *when* $n = 3$.

REMARK 3.1.4: Note that $(\mathcal{S}, g)$ does not have to be simply-connected. For each G/P and each $k \in \mathbb{Z}^+$ we get a homogeneous S^1 bundle over G/P with fundamental group $\pi_1 = \mathbb{Z}_k$. It can be obtained as a discrete $\mathbb{Z}_k$-quotient of the unique simply-connected model of such $\mathcal{S}$.

3.2 Homogeneous 3-Sasakian Manifolds

Every 3-Sasakian manifold $(\mathcal{S}, g)$ has a nontrivial isometry group $I(\mathcal{S}, g)$ of dimension at least three. We first recall some of the results about $I(\mathcal{S}, g)$.

DEFINITION 3.2.1: *Let $I_0(\mathcal{S}, g) \subset I(\mathcal{S}, g)$ be the subgroup of the isometry group which preserves the 3-Sasakian structure, that is if $\phi^k \in \mathit{Diff}\, \mathcal{S}$ corresponds to $k \in I_0(\mathcal{S}, g)$ then $\phi^k_* \xi^a = \xi^a$, for all $a = 1, 2, 3$. Then $I_0(\mathcal{S}, g)$ is called the group of 3-Sasakian isometries and when it acts transitively on $(\mathcal{S}, g)$ the space $\mathcal{S}$ is said to be a 3-Sasakian homogeneous space.*

LEMMA 3.2.2: *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold and $X \in \mathfrak{i}$ be a Killing vector field on $\mathcal{S}$. Let $\mathcal{L}_X$ denote the Lie derivative with respect to X. Then the following conditions are equivalent*

$$(i)\ \ \mathcal{L}_X \Phi^a = 0, \quad a = 1, 2, 3, \qquad (ii)\ \ \mathcal{L}_X \eta^a = 0, \quad a = 1, 2, 3, \qquad (iii)\ \ \mathcal{L}_X \xi^a = 0, \quad a = 1, 2, 3.$$

Furthermore, if any (hence, all) of the conditions above is satisfied, then for any vector field Y on $\mathcal{S}$ we have $X\eta^a(Y) = \eta^a([X, Y])$.

The above lemma gives alternative characterizations of the Lie algebra of $I_0(\mathcal{S}, g)$ and it easily follows from the definition and properties of the 3-Sasakian structure. As its immediate consequence we get the following theorem [Tan 1]:

THEOREM 3.2.3: *Let $(\mathcal{S}, g)$ be a complete 3-Sasakian manifold which is not of constant curvature. $\mathfrak{i}$ and $\mathfrak{i}_0$ denote the Lie algebras of $I(\mathcal{S}, g)$ and $I_0(\mathcal{S}, g)$, respectively. Then as Lie algebras $\mathfrak{i} = \mathfrak{i}_0 \oplus \mathfrak{sp}(1)$, where $\mathfrak{sp}(1)$ is the Lie algebra generated by $\{\xi^1, \xi^2, \xi^3\}$.*

Notice that any of the first three conditions in Lemma 3.2.2 can be used to describe the Lie subalgebra $\mathfrak{i}_0 \subset \mathfrak{i}$. Moreover, the equivalence of conditions (iii) and (i) says that the Lie algebra $\mathfrak{c}(\mathfrak{sp}(1))$ of the centralizer of $Sp(1)$ in $I(\mathcal{S}, g)$ is precisely $\mathfrak{i}_0$. Globally, on the group level we obtain:

PROPOSITION 3.2.4: *Let $(\mathcal{S}, g)$ be a complete 3-Sasakian manifold. Then both the isometry groups $I(\mathcal{S}, g)$ and $I_0(\mathcal{S}, g)$ are compact. Furthermore, if $(\mathcal{S}, g)$ is not of constant curvature then either $I(\mathcal{S}, g) = I_0(\mathcal{S}, g) \times Sp(1)$ or $I(\mathcal{S}, g) = I_0(\mathcal{S}, g) \times SO(3)$. Finally, if $(\mathcal{S}, g)$ does have constant curvature then $I(\mathcal{S}, g)$ strictly contains either $I_0(\mathcal{S}, g) \times Sp(1)$ or $I_0(\mathcal{S}, g) \times SO(3)$ as a proper subgroup and $I_0(\mathcal{S}, g)$ is the centralizer of $Sp(1)$ or $SO(3)$.*

Proof: The first assertion follows from Corollary 1.2.6 and a standard result of Myers and Steenrod (cf. [Bes]). Next, since $I_0(\mathcal{S},g)$, $Sp(1)$, and $SO(3)$ are all compact, the direct sum on the Lie algebra level given in Theorem 3.2.3 also gives a direct product of Lie groups. The last assertion follows immediately from lemma 3.2.2. ∎

Proposition 3.2.5: *Let $(\mathcal{S},g)$ be a 3-Sasakian homogeneous space. Then all leaves are diffeomorphic and $\mathcal{S}/\mathcal{F}_3$ is a quaternionic Kähler manifold where the natural projection $\pi : \mathcal{S}\rightarrow\mathcal{S}/\mathcal{F}_3$ is a locally trivial Riemannian fibration. Furthermore, $I_0(\mathcal{S},g)$ acts transitively on the space of leaves $\mathcal{S}/\mathcal{F}_3$.*

Proof: Let $\psi : I_0(\mathcal{S},g)\times\mathcal{S}\rightarrow\mathcal{S}$ denote the action map so that, for each $a\in I_0(\mathcal{S},g)$, $\psi_a=\psi(a,\cdot)$ is a diffeomorphism of $\mathcal{S}$ to itself. Proposition 3.2.4 implies that the isometry group $I(\mathcal{S},g)$ contains $I_0(\mathcal{S},g)\times Sp(1)$ where either $Sp(1)$ acts effectively or its $\mathbb{Z}_2$ quotient $SO(3)\simeq Sp(1)/\mathbb{Z}_2$ acts effectively. Since the Killing vector fields ξ^a for $a=1,2,3$ are both the infinitesimal generators of the group $Sp(1)$ and a basis for the vertical distribution $\mathcal{V}$, it follows that $Sp(1)$ acts transitively on each leaf with isotropy subgroup of a point some finite subgroup $\Gamma\subset Sp(1)$. Now let p_1 and p_2 be any two points of $\mathcal{S}$ and let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the corresponding leaves through p_1 and p_2, respectively. Since $I_0(\mathcal{S},g)$ acts transitively on $\mathcal{S}$, there exists an $a\in I_0(\mathcal{S},g)$ such that $\psi_a(p_1)=p_2$. Now ψ_a restricted to $\mathcal{L}_1$ maps $\mathcal{L}_1$ diffeomorphically onto its image, and, since the $Sp(1)$ factor acts transitively on each leaf and commutes with $I_0(\mathcal{S},g)$, the image of ψ_a lies in $\mathcal{L}_2$. But the same holds for the inverse map $\psi_{a^{-1}}$ with $\mathcal{L}_1$ and $\mathcal{L}_2$ interchanged, so the leaves must be diffeomorphic. Thus, the leaf holonomy is trivial and $\pi:\mathcal{S}\rightarrow\mathcal{S}/\mathcal{F}_3=\mathcal{O}$ is a locally trivial Riemannian fibration. The fact that the space of leaves $\mathcal{O}$ is a quaternionic Kähler manifold now follows from Ishihara's theorem 2.3.3. Finally, the constructions above shows directly that $I_0(\mathcal{S},g)$ acts transitively on $\mathcal{O}$. ∎

The following classification theorem is now immediate from Proposition 3.2.5, the result of Alekseevsky which states that all homogeneous quaternionic Kähler manifolds of positive scalar curvature are symmetric [Al 2], and Proposition 1.2.10:

Theorem 3.2.6: *Let $\mathcal{S}$ be a 3-Sasakian homogeneous space. Then $\mathcal{S}=G/H$ is precisely one of the following:*

$$\frac{Sp(n+1)}{Sp(n)},\quad \frac{Sp(n+1)}{Sp(n)\times\mathbb{Z}_2},\quad \frac{SU(m)}{S\big(U(m-2)\times U(1)\big)},\quad \frac{SO(k)}{SO(k-4)\times Sp(1)},$$
$$\frac{G_2}{Sp(1)},\quad \frac{F_4}{Sp(3)},\quad \frac{E_6}{SU(6)},\quad \frac{E_7}{\mathrm{Spin}(12)},\quad \frac{E_8}{E_7}.$$

Here $n\geq 0$, $Sp(0)$ denotes the trivial group, $m\geq 3$, and $k\geq 7$. Hence, there is one-to-one correspondence between the simple Lie algebras and the simply-connected 3-Sasakian homogeneous manifolds.

Below we give the fundamental diagram $\diamondsuit(G/H)$ for each 3-Sasakian homogeneous space of Theorem 3.2.6:

3.2.7

$$\begin{tikzcd}
& \mathbb{R}_+\times G/H \arrow[dl] \arrow[dr] \arrow[dd] & \\
G/H\cdot U(1) \arrow[dr] & & G/H, \arrow[ll] \arrow[dl] \\
& G/H\cdot Sp(1) &
\end{tikzcd}$$

where $G/H\cdot Sp(1)$ are precisely the Wolf spaces [Wol].

REMARK 3.2.8: Note that a homogeneous 3-Sasakian manifold is necessarily simply-connected with the exception of the real projective space. This is in sharp contrast with the Sasakian-Einstein case. Also notice that a 3-Sasakian manifold can be Riemannian homogeneous (i.e., the full isometry group acts transitively) but not 3-Sasakian homogeneous. This is true for the lens spaces $\Gamma\backslash S^3$, with $|\Gamma|>2$. Observe that $\mathbb{Z}_k\backslash S^3,\ k>2$, is a homogeneous Sasakian-Einstein manifold but not 3-Sasakian homogeneous.

Theorem 3.2.6 does not specify what is the 3-Sasakian metric on the coset G/H. In Section 6.2 we will describe a quotient construction of the 3-Sasakian homogeneous spaces with $G=SU(n+1)$ and $G=SO(n+1)$. Here we quote a theorem of Bielawski [Bi 1], which gives an explicit description of these metrics in all cases.

THEOREM 3.2.9: *Let $\mathcal{S}=G/H$ be one of the spaces in Theorem 3.2.6. and let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ be the corresponding decomposition of the Lie algebras. Then there is a natural decomposition $\mathfrak{m}=\mathfrak{sp}(1)\oplus\mathfrak{m}'$ and the metric g on $\mathcal{S}$ is given in terms of the scalar product on $\mathfrak{m}$*

$$\|m\|^2=-<\sigma,\sigma>-\frac{1}{2}<m',m'>,$$

where $\sigma\in\mathfrak{sp}(1)$, $m'\in\mathfrak{m}'$, and $<\cdot,\cdot>$ is the Killing form on $\mathfrak{g}$. In particular, the metric g is not naturally reductive with respect to the homogeneous structure on $\mathcal{S}$.

REMARK 3.2.10: In the case when $\mathcal{S}$ is of constant curvature the canonical metric on S^{4n+3} (or $\mathbb{RP}^{4n+3}$) is *not* the standard homogeneous metric on the homogeneous space $Sp(n+1)/Sp(n)$ (or $Sp(n+1)/Sp(n)\times\mathbb{Z}_2$) with respect to the reductive decomposition $\mathfrak{sp}(n+1)\simeq\mathfrak{sp}(n)+\mathfrak{m}$. It is, of course, the standard homogeneous metric with respect to the naturally reductive decomposition $\mathfrak{o}(4n+4)\simeq\mathfrak{o}(4n+3)+\mathfrak{m}$. This is quite special to the sphere and orthogonal group. In general the 3-Sasakian homogeneous metrics are not naturally reductive with respect to any reductive decomposition.

4. 3-Sasakian Cohomology

In this section we will describe some cohomological properties of 3-Sasakian manifolds $\mathcal{S}$. We prove a vanishing theorem and then derive a relation between the Betti numbers of $\mathcal{S}$ and the Betti numbers of the associated orbifolds $\mathcal{Z}$ and $\mathcal{O}$. We conclude with various implications of these relations in the case $\mathcal{S}$ is regular.

4.1 Sasakian Manifolds and Harmonic Theory

We start by recalling some old results about harmonic forms on Sasakian manifolds due to Tachibana [Tach]. Let $(\mathcal{S},g)$ be a compact Sasakian manifold of dimension $2m+1$ with Sasakian structure $\{\xi,\eta,\Phi\}$ and let $\Omega^p(\mathcal{S})$ be the space of smooth p-forms on $\mathcal{S}$. Furthermore, let $\mathcal{H}^p(\mathcal{S})=\{u\in\Omega^p(\mathcal{S}):du=0=d*u\}$ denote the finite-dimensional space of harmonic p-forms. By Hodge theory any harmonic form u is necessarily invariant under the isometry group $I(\mathcal{S},g)$ The tensor Φ extends to an endomorphism of $\Omega^p(\mathcal{S})$ by setting

4.1.1
$$(\mathbf{\Phi}u)(X_1,X_2,\ldots,X_p)=\sum_{i=1}^{p}u(X_1,\ldots,\Phi X_i,\ldots,X_p).$$

With this notation we have [Tach]

THEOREM 4.1.2: *Let $u\in\mathcal{H}^p(\mathcal{S})$, $p\leq m$. Then $\xi\lfloor u=0$, and $\mathbf{\Phi}(u)\in\mathcal{H}^p(\mathcal{S})$.*

Proof: The first statement is easy to prove in the case $p = 1$. Indeed, let $u = \alpha + f\eta$ be a closed invariant 1-form, where $\alpha(\xi) = 0$, and f is a function. The vanishing of the Lie derivative of u along ξ implies that $0 = d(\xi\lfloor u) = df$, so that f is a constant and $0 = d\alpha + f d\eta$. Then

$$0 = \int_{\mathcal{S}} d(\alpha \wedge (d\eta)^{m-1} \wedge \eta) = -\int_{\mathcal{S}} f(d\eta)^m \wedge \eta.$$

Since $(d\eta)^m \wedge \eta$ is a non-zero multiple of the volume form of $\mathcal{S}$, we obtain $f = 0$ and $\xi\lfloor u = 0$. The general case of the original proof uses an explicit computation in local coordinates and we omit it here. The second statement follows immediately from 4.1.1 and the fact that Φ preserves horizontal subspaces. ∎

Let us define the following $\mathcal{I} : \mathcal{H}^p(\mathcal{S}) \to \mathcal{H}^p(\mathcal{S})$ endomorphism for $p \le m$:

4.1.3
$$(\mathcal{I}u)(X_1, \ldots, X_p) = u(\Phi X_1, \ldots, \Phi X_p)$$

The basic identity 1.1.8(i) together with Theorem 4.1.2 shows that $\mathcal{I}u$ is a linear combination of $(\mathbf{\Phi})^k u$ for $0 \le k \le p$. Thus $\mathcal{I}$ also maps $\mathcal{H}^p(\mathcal{S})$ into itself. The following proposition is now a simple consequence of the definition 4.1.3 and Theorem 4.1.2 [Bl, Bl-Go]:

Proposition 4.1.4: *Let $\mathcal{I} : \mathcal{H}^p(\mathcal{S}) \to \mathcal{H}^p(\mathcal{S})$ and $p \le m$. Then $\mathcal{I} \circ \mathcal{I} = (-1)^p$. In particular, when p is odd, $\mathcal{I}$ defines an almost complex structure on the vector space $\mathcal{H}^p(\mathcal{S})$.*

Corollary 4.1.5: *Let $(\mathcal{S}, g)$ be a compact Sasakian manifold of dimension $2m+1$. Then the Betti numbers b_p for p odd and $p \le m$ are even.*

In the case of compact Sasakian-Einstein manifolds this and the fact that the fundamental group is finite are the only known general topological restrictions on $\mathcal{S}$. Under some additional curvature conditions we can get further restrictions. For example, it is known [Bl] that a compact simply-connected Sasakian manifold of positive sectional curvature is isometric to a sphere. For other similar results see [Bl, Go] and references therein.

4.2 A Vanishing Theorem

Now, let $(\mathcal{S}, g)$ be a compact 3-Sasakian manifold of dimension $4n+3$ and 3-Sasakian structure $\{\xi^a, \eta^a, \Phi^a\}$. Throughout this section we shall suppose that $p \le 2n+1$. Referring to the splitting of the tangent bundle of $\mathcal{S}$ into $T\mathcal{S} = \mathcal{V}_3 \oplus \mathcal{H}$, we shall say that a p-form $u \in \Omega^p(\mathcal{S})$ has *bidegree* $(i, p-i)$ if it is a section of the subbundle of $\bigwedge^p T^*\mathcal{S}$ isomorphic to the dual of $\bigwedge^i \mathcal{V}_3 \otimes \bigwedge^{p-i} \mathcal{H}$. In particular, u is called *3-horizontal* if it has bidegree $(0, p)$, or equivalently if $\xi^a\lfloor u = 0$ for $a = 1, 2, 3$. An element $\omega \in \Omega^p(\mathcal{S})$ is called *invariant* if $h^*\omega = \omega$ for all $h \in Sp(1)$. In the regular case, there is a principal $Sp(1)$-bundle $\pi : \mathcal{S} \to \mathcal{O}$, and ω is both 3-horizontal and invariant if and only if it is the pullback $\pi^*\hat{\omega}$ of a form $\hat{\omega}$ on the quaternionic Kähler base $\mathcal{O}$. Now the curvature forms $\bar{\Phi}^a$ defined in 2.3.4 are horizontal with respect to the foliation $\mathcal{F}_3$. The Killing fields ξ^a transform according to the adjoint representation of $Sp(1)$, and the same is true of the associated triples η^a, $d\eta^a$, and Φ^a. For example, if $h \in Sp(1)$, we may write

4.2.1
$$h_*\Phi^a = \sum_b h^{ab}\Phi^b, \quad a = 1, 2, 3,$$

where h^{ab} are components of the image of h in $Sp(1)/\mathbb{Z}_2 \cong SO(3)$. The 3-forms

$$\Upsilon=\eta^1\wedge\eta^2\wedge\eta^3, \qquad \Theta=\sum_a \eta^a\wedge\bar{\Phi}^a=\sum_a \eta^a\wedge d\eta^a+6\Upsilon \tag{4.2.2}$$

have respective bidegrees $(3,0),(1,2)$, and are clearly invariant. Their exterior derivatives are

$$d\Upsilon=\eta^1\wedge\eta^2\wedge\bar{\Phi}^3+\eta^2\wedge\eta^3\wedge\bar{\Phi}^1+\eta^3\wedge\eta^1\wedge\bar{\Phi}^2, \qquad d\Theta=\Omega+2d\Upsilon, \tag{4.2.3}$$

where the 4-form Ω is defined in section 2.3. In fact, Ω is the canonical 4-form determined by the quaternionic structure of Proposition 1.2.4 of the subbundle $\mathcal{H}$, and is the pullback of the fundamental 4-form $\hat{\Omega}$ on the quaternionic Kähler orbifold $\mathcal{O}$ (see section 2.2).

Theorem 4.1.2(i) implies that any harmonic p-form with $p\leq 2n+1$ on the compact 3-Sasakian manifold $\mathcal{S}$ is 3-horizontal. Apply 4.1.1 so as to obtain $\mathbf{\Phi}^a:\mathcal{H}^p(\mathcal{S})\to\mathcal{H}^p(\mathcal{S}),\quad a=1,2,3,\quad p\leq 2n+1$, and 4.1.3 to get

$$(\mathcal{I}^a u)(X_1,X_2,\ldots,X_p)=u(\Phi^a X_1,\Phi^a X_2,\ldots,\Phi^a X_p). \tag{4.2.4}$$

Now, using the basic identities of Proposition 1.2.4 we can generalize Proposition 4.1.4 to get the following result due to Kuo [Kuo]:

Proposition 4.2.5: *Let $\mathcal{I}^a:\mathcal{H}^p(\mathcal{S})\to\mathcal{H}^p(\mathcal{S})$, $a=1,2,3$, and $p\leq 2n+1$. Then*

$$\mathcal{I}^b\circ\mathcal{I}^a=(-\delta^{ab})^p\mathbf{I}+\sum_c(\epsilon^{abc})^p\mathcal{I}^c. \tag{4.2.6}$$

In particular, when p is odd, $\{\mathcal{I}^1,\mathcal{I}^2,\mathcal{I}^3\}$ defines an almost quaternionic structure on the vector space $\mathcal{H}^p(\mathcal{S})$.

We are now ready to prove the main theorem (Vanishing Theorem) of this section:

Theorem 4.2.7: *Let $u\in\mathcal{H}^p(\mathcal{S})$, $p\leq 2n+1$.*

(i) If p is odd then $u\equiv 0$.

(ii) If p is even then $\mathcal{I}^a u=u$ for $a=1,2,3$.

Proof: Let $u\in\mathcal{H}^p(\mathcal{S})$. We shall in fact show that $\mathcal{I}^1u=\mathcal{I}^2u$ irrespective of whether p is even or odd; the result then follows from the identities 4.2.6 and symmetry between the indices 1,2,3. By 4.2.1, we may choose an isometry $h\in Sp(1)$ so that $h_*\Phi^1=\Phi^2$. Both u and $\mathcal{I}^1u$ are harmonic, so $h^*u=u$ and

$$(\mathcal{I}^1u)(X_1,\ldots,X_p)=(h_*(\mathcal{I}^1u))(X_1,\ldots,X_p)=u((h_*\Phi^1)(X_1),\ldots,(h_*\Phi^1)(X_p))=$$

$$=u(\Phi^2X_1,\ldots,\Phi^2X_p)=(\mathcal{I}^2u)(X_1,\ldots,X_p).$$

∎

Corollary 4.2.8: *Let $(\mathcal{S},g)$ be a compact 3-Sasakian manifold of dimension $4n+3$. Then the odd Betti numbers b_{2k+1} of $\mathcal{S}$ are all zero for $0\leq k\leq n$.*

We should point out that Corollary 4.2.8 does not apply to compact Sasakian or even Sasakian-Einstein manifolds. In [B-G 2] the authors construct examples of Sasakian-Einstein manifolds with certain non-vanishing odd Betti numbers within the range given in Corollary 4.2.8. For example, in dimension 7 there are circle bundles over Fermat hypersurfaces in $\mathbb{CP}^3$, as well as circle bundles over certain complete intersections that admit Sasakian-Einstein structures and have $b_3 \neq 0$. These are the only known examples of Sasakian-Einstein manifolds which cannot admit any 3-Sasakian structure.

4.3 3-Sasakian Cohomology As Primitive Cohomology

We are going to consider connection between the cohomology of $\mathcal{S}$ and that of $\mathcal{Z}$ and $\mathcal{O}$. We will use the vanishing theorem and orbifold Gysin sequence arguments for the diagram of orbifold bundles of $\diamondsuit(\mathcal{S})$:

4.3.1
$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \mathcal{Z}. \\ \Big\downarrow & \swarrow & \\ \mathcal{O} & & \end{array}$$

Proposition 4.3.2: *Let $\mathcal{S}$ be a compact 3-Sasakian manifold of dimension $4n+3$ and $\mathcal{Z} = \mathcal{S}/S^1$ be the twistor space. Then $b_p(\mathcal{S}) = b_p(\mathcal{Z}) - b_{p-2}(\mathcal{Z})$, for $p \leq 2n+1$. In particular, all odd Betti numbers of $\mathcal{Z}$ vanish.*

Proof: The result follows form the rational Gysin sequence applied to the orbifold fibration $S^1 \to \mathcal{S} \to \mathcal{Z}$. First, note that the bundle $S^1 \longrightarrow \mathcal{S} \longrightarrow \mathcal{Z}$ is a circle V-bundle over a compact Kähler-Einstein orbifold $\mathcal{Z}$. As explained in Section 2, up to a possible $\mathbb{Z}_2$ cover, $\mathcal{S}$ is the total space of the unit circle bundle in the dual of the contact line V-bundle on $\mathcal{Z}$, and the Kähler-Einstein metric of $\mathcal{Z}$ arises in accordance with the orbifold version of the Kobayashi's theorem [Kob, B-G 1] . It follows that the connecting homomorphism δ is given by wedging with a non-zero multiple of the Kähler form of $\mathcal{Z}$. When $\mathcal{Z}$ is smooth this is well-known to be injective so long as $p \leq 2n+2$. However, the Lefschetz decomposition is equally true for compact orbifolds and the result still holds in this more general situation [B-G 1]. The Gysin sequence therefore reduces to a series of short exact sequences up to and including $H^{2n+1}(\mathcal{S})$, and the proposition follows. ∎

Proposition 4.3.3: *Let $\mathcal{S}$ be a compact 3-Sasakian manifold of dimension $4n+3$ and let $\mathcal{O} = \mathcal{S}/\mathcal{F}_3$. Then $b_{2p}(\mathcal{S}) = b_{2p}(\mathcal{O}) - b_{2p-4}(\mathcal{O})$, for $p \leq 2n+1$.*

Proof: The result follows form the Gysin sequence applied to the orbifold fibration $\mathcal{L} \to \mathcal{S} \to \mathcal{O}$. Since the principal orbit of the $Sp(1)$ action (or generic leaf $\mathcal{L}$) is either S^3 or $SO(3)$ the usual Gysin sequence argument applies as well in this situation (see the Appendix). We have

$$\cdots \to H^i(\mathcal{S},\mathbb{Q}) \to H^{i-3}(\mathcal{O},\mathbb{Q}) \xrightarrow{\delta} H^{i+1}(\mathcal{O},\mathbb{Q}) \to H^{i+1}(\mathcal{S},\mathbb{Q}) \to H^{i-2}(\mathcal{S},\mathbb{Q}) \to \cdots$$

and the statement of the proposition follows easily from the vanishing of the odd Betti numbers of $\mathcal{S}$. ∎

Recall that the vector space of primitive harmonic p-forms $\mathcal{H}^p_0(\mathcal{Z},\mathbb{Q})$ of the orbifold $\mathcal{Z}$ is isomorphic to the cokernel of the injective mapping $L_{\mathcal{Z}} : \mathcal{H}^{p-2}(\mathcal{Z}) \hookrightarrow \mathcal{H}^p(\mathcal{Z}), \quad p \leq 2n$ defined by wedging with the Kähler 2-form. We define the primitive Betti numbers $b^0_p(\mathcal{Z})$ of $\mathcal{Z}$ as the dimension of $\mathcal{H}^p_0(\mathcal{Z})$. Proposition 4.3.2 says that the primitive Betti numbers of $\mathcal{Z}$ are the usual Betti numbers of $\mathcal{S}$ and it follows from the fact that for, $0 \leq r \leq 2n+1$, an r-form on $\mathcal{S}$ is harmonic if and only if it is the lift of a primitive harmonic form on $\mathcal{Z}$ [B-G 1]. Similarly, the vector space of primitive harmonic p-forms $\mathcal{H}^p_0(\mathcal{O},\mathbb{Q})$ of the orbifold $\mathcal{O}$ is isomorphic to the cokernel of the injective mapping $L_{\mathcal{O}} : \mathcal{H}^{p-4}(\mathcal{O}) \hookrightarrow \mathcal{H}^p(\mathcal{O}), \quad p \leq 2n+2$ defined by wedging with the quaternionic Kähler 4-form Ω. The injectivity of this mapping is well-known in the smooth case [Bon, Fuj, Kra] and it extends to the orbifold case. We define the primitive Betti numbers $b^0_p(\mathcal{O})$ of $\mathcal{O}$ as the dimension of $\mathcal{H}^p_0(\mathcal{O})$. Proposition 4.3.3 says that the primitive Betti numbers of $\mathcal{O}$ are the usual Betti numbers of $\mathcal{S}$. Again, Proposition 4.3.3 is a consequence of the fact that an r-form on $\mathcal{S}$ is harmonic if and only if it is the lift of a primitive harmonic form on $\mathcal{O}$, $0 \leq r \leq 2n+1$.

4.4 Regular 3-Sasakian Cohomology, Finiteness, and Rigidity

In this Section we shall assume that $\mathcal{S}$ is regular and, hence, both $\mathcal{Z}$ and $\mathcal{O}$ are smooth. In this instance, using the results of the previous section, one can easily translate all the results about strong rigidity of positive quaternion Kähler manifolds [Le 1, Le-Sal, Sal 3] (see the chapter in this volume on Quaternionic Kähler Manifolds by S. Salamon) to compact regular 3-Sasakian manifolds. In particular, we get

PROPOSITION 4.4.1: *Let $\mathcal{S}$ be a compact regular 3-Sasakian manifold of dimension $4n+3$. Then $\pi_1(\mathcal{S}) = 0$ unless $\mathcal{S} = \mathbb{RP}^{4n+3}$ and*

$$\pi_2(\mathcal{S}) = \begin{cases} \mathbb{Z} & \text{iff } \mathcal{S} = SU(n+2)/S(U(n)\times U(1)), \\ finite & \text{otherwise.} \end{cases}$$

Furthermore, up to isometries, for each $n \geq 1$ there are only finitely many regular 3-Sasakian manifolds $\mathcal{S}$.

PROOF: Using the long exact homotopy sequence for the vertical map in 4.3.1, this follows from the strong rigidity theorem of LeBrun and Salamon [Le-Sal,Le 1] for positive quaternionic Kähler manifolds, and Salamon's theorem that a positive quaternionic Kähler manifold with vanishing Marchiafava-Romani class must be $\mathbb{HP}^n$. ∎

n	Relation on Betti numbers or coefficients thereof
2	$b_2 = b_4$
3	$b_2 = b_6$
4	$2b_2 + b_4 = b_6 + 2b_8$
5	$5b_2 + 4b_4 = 4b_8 + 5b_{10}$
6	$5b_2 + 5b_4 + 2b_6 = 2b_8 + 5b_{10} + 5b_{12}$
7	$7b_2 + 8b_4 + 5b_6 = 5b_{10} + 8b_{12} + 7b_{14}$
8	$28b_2 + 35b_4 + 27b_6 + 10b_8 = 10b_{10} + 27b_{12} + 35b_{14} + 28b_{16}$
9	$12b_2 + 16b_4 + 14b_6 + 8b_8 = 8b_{12} + 14b_{14} + 16b_{16} + 12b_{18}$
10	$15, 21, 20, 14, 5$
16	$40, 65, 77, 78, 70, 55, 35, 12$
28	$126, 225, 299, 350, 380, 391, 385, 364, 330, 285, 231, 170, 104, 35$

Table 1: Betti number relations in lower dimensions

PROPOSITION 4.4.2: *The Betti numbers of a regular compact 3-Sasakian manifold $\mathcal{S}$ of dimension $4n+3$ satisfy*

(i) $b_2 \leq 1$, *with equality iff* $\mathcal{S} = SU(l+2)/S(U(l)\times U(1))$,

(ii) $\sum_{k=1}^{n} k(n+1-k)(n+1-2k)b_{2k} = 0.$

Proof: (i) follows from Proposition 4.4.1 and (ii) for Salamon's relation on Betti numbers of $\mathcal{O}$ via Theorem 4.3.3. ∎

The following is a 3-Sasakian version of a theorem of Salamon [G-Sal]:

Proposition 4.4.3: *Let $\mathcal{S}$ be a regular compact 3-Sasakian manifold of dimension $4n+3$. If $n = 3, 4$ and $b_4 = 0$, then $\mathcal{S}$ is either a sphere S^{4n+3} or a real projective space $\mathbb{RP}^{4n+3}$.*

The linear Betti number relations in Proposition 4.4.2(ii) exhibit an interesting symmetry of the coefficients which, for lower values of n, are listed in Table 1.

One can compute the Poincaré polynomials of all known regular 3-Sasakian manifolds, that is 3-Sasakian homogeneous space of Theorem 3.2.6. We get [G-Sal]

Proposition 4.4.4: *The Poincaré polynomials of the homogeneous 3-Sasakian manifolds are as given in Table 2.*

G	H	$P(G/H,t)$
$SU(n+2)$	$SU(n)\times_{\mathbb{Z}_n} T^1$	$\sum_{i=0}^{n}(t^{2i}+t^{4n+3-2i})$
$SO(2k+3)$	$SO(2k-1)\times SU(2)$	$\sum_{i=0}^{k-1}(t^{4i}+t^{8k-1-4i})$
$Sp(n+1)$	$Sp(n)$	$1+t^{4n+3}$
$SO(2l+4)$	$SO(2l)\times SU(2)$	$t^{2l}+t^{6l+3}+\sum_{i=0}^{l}(t^{4i}+t^{8l+3-4i})$
E_6	$SU(6)$	$1+t^6+t^8+t^{12}+t^{14}+t^{20}+\cdots$
E_7	$\mathrm{Spin}(12)$	$1+t^8+t^{12}+t^{16}+t^{20}+t^{24}+t^{32}+\cdots$
E_8	E_7	$1+t^{12}+t^{20}+t^{24}+t^{32}+t^{36}+t^{44}+t^{56}+\cdots$
F_4	$Sp(3)$	$1+t^8+t^{23}+t^{31}$
G_2	$SU(2)$	$1+t^{11}$

Table 2: Betti numbers of 3-Sasakian homogeneous spaces

We conclude this section with a translation of two well-known classification results for positive quaternionic Kähler manifolds.

Theorem 4.4.5: *Let $(\mathcal{S}, g)$ be a compact regular 3-Sasakian manifold of dimension $4n+3$. If $n < 3$ then then $\mathcal{S} = G/H$ is homogeneous, and hence one of the spaces listed in Theorem 3.2.6.*

The $n = 0$ case is trivial and it was an observation made by Tanno [Tan 2]. The $n = 1$ case is based on [Hit 1, Fr-Kur] and it was first observed in [Fr-Kat 2, B-G-F-K]. The $n = 2$ case is based on [Po-Sal] and was stated in [B-G-M 1].

5. Killing Spinors and G_2-Structures

In this section we discuss some additional properties of Sasakian and Sasakian-Einstein manifolds which are connected with spin structure and eigenvalues of the Dirac operator

5.1 Killing Spinors

DEFINITION 5.1.1: *Let (M,g) be a complete n-dimensional Riemannian spin manifold, and let $S(M)$ be the spin bundle of M and ψ a smooth section of $S(M)$. We say that ψ is a Killing spinor if*

5.1.2
$$\nabla_X \psi = \alpha \cdot X \cdot \psi, \qquad \forall X \in \Gamma(TM),$$

where ∇ is the Levi-Civita connection of g and $X\cdot\psi$ denotes the Clifford product of X and ψ. We say that ψ is imaginary when $\alpha \in \mathrm{Im}(\mathbb{C}^*)$, ψ is parallel if $\alpha = 0$ and ψ is real if $\alpha \in \mathrm{Re}(\mathbb{C}^*)$.

From the point of view of Einstein geometry the importance of Killing spinors is an immediate consequence of the following theorem of Friedrich [Fr]:

THEOREM 5.1.2: *Let (M,g) be an n-dimensional; complete Riemannian spin manifold with a Killing spinor. Then M is Einstein with Einstein constant $\lambda = 4(n-1)\alpha^2$. In particular, when $\alpha \in \mathrm{Re}(\mathbb{C}^*)$, M is compact of positive scalar curvature.*

On the other hand, Friedrich showed that if M is a compact spin manifold of positive scalar curvature and R_0 is the minimum of the scalar curvature, then for all eigenvalues β of the Dirac operator D one has $\beta^2 \geq \frac{1}{4}\frac{nR_0}{n-1}$ [Fr 1]. If the equality holds than it follows that the corresponding eigenspinor must be a Killing spinor with $\alpha = \pm\frac{1}{2}[\frac{R_0}{n(n-1)}]^{1/2}$. We have the following important property of manifolds with Killing spinors [B-G-F-K]:

THEOREM 5.1.3: *Let (M^n,g) be a connected Riemannian spin manifold admitting a non-trivial Killing spinor with $\alpha \neq 0$. Then (M,g) is locally irreducible. Furthermore, if M is locally symmetric, or $n \leq 4$, then M is a space of constant sectional curvature equal $4\alpha^2$.*

From now on we will be interested only in the case of real Killing spinors. It was Friedrich and Kath [Fr-Kat 1-2] who first noticed that in some low odd dimensions the existence of real Killing spinors leads naturally to the existence of Sasakian-Einstein or 3-Sasakian structures. Later, the problem found a simple classification in terms of the holonomy of the associated metric cone $C(M)$ [Bär]. First, we have the following definition:

DEFINITION 5.1.4: *We say that M is of type (p,q) if it carries exactly p linearly independent real Killing spinors with $\alpha > 0$ and exactly q linearly independent real Killing spinors with $\alpha < 0$, or vice versa.*

For, example, the standard sphere S^n is of type $(2^{[n/2]}, 2^{[n/2]})$. Bär shows that when M admits a real Killing spinor then the cone $(C(M),\bar{g})$ has a parallel spinor. In particular, $C(M)$ is always Ricci-flat and, when M is simply-connected, then only a few holonomy groups $\mathrm{Hol}(\bar{g})$ are possible [Wan 3]:

THEOREM 5.1.5: *Let (M^n,g) be a simply-connected Riemannian spin manifold admitting a non-trivial Killing spinor and let $\mathrm{Hol}(\bar{g})$ be the holonomy group of the metric cone $(C(M),\bar{g})$. Then there are only the following 6 possibilities for the triple $\big(n, \mathrm{Hol}(\bar{g}), (p,q)\big)$:*

(1) n *arbitrary,*	$\mathrm{Hol}(\bar{g}) = \mathrm{id}$,	$(p,q) = (2^{[n/2]}, 2^{[n/2]})$,
(2) $n = 2m+1, m$ *even,*	$\mathrm{Hol}(\bar{g}) = SU(m+1)$,	$(p,q) = (1,1)$,

(3) $n = 4m+3$, $\mathrm{Hol}(\bar{g}) = SU(2m+2)$, $(p,q) = (2,0)$,
(4) $n = 4m+3$, $\mathrm{Hol}(\bar{g}) = Sp(m+1)$, $(p,q) = (m+2,0)$,
(5) $n = 7$, $\mathrm{Hol}(\bar{g}) = \mathrm{Spin}(7)$, $(p,q) = (1,0)$,
(6) $n = 6$, $\mathrm{Hol}(\bar{g}) = G_2$, $(p,q) = (1,1)$.

The first case is special as M is the n-dimensional round sphere. Since M is assumed to be simply-connected, in the next two cases, by Proposition 1.1.9, M must be Sasakian-Einstein. In the case (4), by Definition 1.2.1, M is 3-Sasakian. Specifically, we get the following theorem [Bär]:

THEOREM 5.1.6: *Let (M^n, g) be a complete simply-connected Riemannian spin manifold admitting a non-trivial Killing spinor with $\alpha > 0$ or $\alpha < 0$. If $n = 2m+1$, $m \geq 2$ even, then there are two possibilities:*

(i) $(M,g) = (S^n, g_{can})$,

(ii) (M,g) *is of type* $(1,1)$ *and it is a Sasakian-Einstein manifold.*

Conversely, if (M,g) is a complete simply-connected Sasakian-Einstein manifold of dimension $4m+1$, then M carries Killing spinors with $\alpha > 0$ and $\alpha < 0$.

REMARK 5.1.7: Note that in the converse statement we do not need to assume that M is spin. When $\pi_1(M) = 0$ this is automatic by Corollary 1.1.11. When $\pi_1(M) \neq 0$ then the 'if' part of Theorem 5.1.6 can be generalized and we still get two possibilities: (i) either M is a spin spherical space form, or (ii) it is of type $(1,1)$ with a Sasakian-Einstein structure and $\mathrm{Hol}(\bar{g}) = SU(m+1)$ [Wan 3].

THEOREM 5.1.8: *Let (M^n, g) be a complete simply-connected Riemannian spin manifold admitting a non-trivial Killing spinor with $\alpha > 0$ or $\alpha < 0$. If $n = 4m+3$, $m \geq 2$, then there are three possibilities:*

(i) $(M,g) = (S^n, g_{can})$,

(ii) (M,g) *is a Sasakian-Einstein manifolds of type* $(2,0)$*, but* (M,g) *is not 3-Sasakian,*

(iii) (M,g) *is of type* $(m+2,0)$ *and it is 3-Sasakian.*

Conversely, if (M,g) is a complete simply-connected 3-Sasakian manifold, of dimension $4m+3$ which is not of constant curvature, then M carries $(m+2)$ linearly independent Killing spinors with $\alpha > 0$. If (M,g) is a complete simply-connected Sasakian-Einstein manifold of dimension $4m+3$ which is not 3-Sasakian then M carries 2 linearly independent Killing spinors with $\alpha > 0$.

REMARK 5.1.9: Note that in Theorem 5.1.8(ii) we are not excluding the possibility of M having another 3-Sasakian structure with a different metric g'. We are only saying that the holonomy group $\mathrm{Hol}(\bar{g}) = SU(2m+2)$ rather than $Sp(m+1) \subset SU(2m+2)$, which, by definition, means that g cannot be 3-Sasakian. However, we are not aware of any such example. We have excluded $\dim(M) = 7$ because in this case we have one more possibility due to Theorem 5.1.5 and we want to discuss the associated geometry in more detail later. Again, one can generalize Theorems 5.1.8 to $\pi_1(M) \neq 0$. For the full list of possible holonomy groups $\mathrm{Hol}(\bar{g})$ see [Wan 2]. The coresponding M are then only locally Sasakian-Einstein or locally 3-Sasakian [Or-Pi, Pi]. The problem of the existence of Killing spinors on a Sasakian-Einstein or 3-Sasakian manifold with $\pi_1(M) \neq 0$ is, however, more subtle.

COROLLARY 5.1.10: *Let $(\mathcal{S}, g)$ be a compact Sasakian-Einstein manifold of dimension $2m+1$. Then $\mathcal{S}$ is locally symmetric if and only if $\mathcal{S}$ is of constant curvature. Moreover, $(\mathcal{S}, g)$ is locally irreducible as a Riemannian manifold.*

PROOF: If necessary, go to the universal cover $\tilde{\mathcal{S}}$. This is a compact simply-connected Sasakian-Einstein manifold; hence, it admits a non-trivial Killing spinor by Theorems 5.1.6 and 5.1.8. The statement then follows from the Theorem 5.1.3. ∎

COROLLARY 5.1.11: *Let $(\mathcal{S}, g)$ be a compact Sasakian-Einstein manifold of dimension $2m+1$. Then $\mathrm{Hol}(g) = SO(2m+1)$.*

PROOF: Let us consider universal cover $\tilde{\mathcal{S}}$. This is a compact simply-connected Sasakian-Einstein manifold; hence, it admits a non-trivial Killing spinor. By the previous corollary, it can be symmetric if only if it is isomorphic to a space of constant curvature, that is, a sphere. Then $\mathcal{S}$ is a spherical space form and $\mathrm{Hol}(g) = SO(2m+1)$. Assume $\mathcal{S}$ is not locally symmetric. By Corollary 5.1.10 $\mathcal{S}$ is locally Riemannian irreducible, so for dimensional reasons and Berger's famous classification theorem [Ber], the only possibilities for the restricted holonomy group $\mathrm{Hol}^0(g)$ are $SO(2m+1)$ and G_2 in dimension 7. But G_2 holonomy implies Ricci-flat and, hence, not Sasakian-Einstein. Hence, the restricted holonomy group $\mathrm{Hol}^0(g) = SO(2m+1)$. Since $\mathcal{S}$ is orientable this coincides with the holonomy group $\mathrm{Hol}(g)$. ∎

5.2 G_2-Structures

Recall, that geometrically G_2 is defined to be the Lie group acting on $\mathbb{R}^7$ and preserving the 3-form

5.2.1

$$\varphi = \alpha_1\wedge\alpha_2\wedge\alpha_3+\alpha_1\wedge(\alpha_4\wedge\alpha_5-\alpha_6\wedge\alpha_7)+\alpha_2\wedge(\alpha_4\wedge\alpha_6-\alpha_7\wedge\alpha_5)+\alpha_3\wedge(\alpha_4\wedge\alpha_7-\alpha_5\wedge\alpha_6),$$

where $\{\alpha_i\}_{i=1}^7$ is a fixed orthonormal basis of the dual of $\mathbb{R}^7$. A G_2 structure on a 7-manifold M is, by definition, a reduction of the structure group of the tangent bundle to G_2. This is equivalent to the existence of a global 3-form $\varphi \in \Omega^3(M)$ which may be written locally as 5.2.1. Such a 3-form defines an associated Riemannian metric, an orientation class, and a spinor field of constant length. The following terminology is due to Gray [Gra 2]:

DEFINITION 5.2.2: *Let (M, g) be a complete 7-dimensional Riemannian manifold. We say that that (M, g) has weak holonomy G_2 if there exist a global 3-form $\varphi \in \Omega^3(M)$ which locally can be written in terms of a local orthonormal basis as in 5.2.1, and $d\varphi = c\star\varphi$, where $\star$ is the Hodge star operator associated to g and c is a constant whose sign is fixed by an orientation convention.*

The equation $d\varphi = c\star\varphi$ implies that φ is 'nearly parallel' in the sense that only a 1-dimensional component of $\nabla\varphi$ is different from zero [Fe-Gra]. Thus, a weak holonomy G_2 structure is sometimes called a *nearly parallel G_2 structure*. The case of $c = 0$ is somewhat special. In particular, it is known [Sal 4] that the condition $d\varphi = 0 = d\star\varphi$ is equivalent to the condition that φ be parallel, i.e., $\nabla\varphi = 0$ which is equivalent to the condition that the metric g has holonomy group contained in G_2. For a discussion of this very interesting and very difficult case, see the article by D. Joyce in this volume. The following theorem provides the connection with the previous discussion on Killing spinors [Bär]

THEOREM 5.2.3: *Let (M, g) be a complete 7-dimensional Riemannian manifold with weak holonomy G_2. Then the holonomy group $\mathrm{Hol}(\bar{g})$ of the metric cone $(C(M), \bar{g})$ is contained in $\mathrm{Spin}(7)$. In particular, $C(M)$ is Ricci-flat and M is Einstein with positive Einstein constant $\lambda = 6$.*

REMARK 5.2.4: The sphere S^7 with its constant curvature metric is isometric to the isotropy irreducible space $Spin(7)/G_2$. The fact that G_2 leaves invariant (up to constants) a unique 3-form and a unique 4-form on $\mathbb{R}^7$ implies immediately that this space has weak holonomy G_2.

DEFINITION 5.2.5: *Let (M,g) be a complete 7-dimensional Riemannian manifold. We say that g is a proper G_2-metric if* $\mathrm{Hol}(\bar{g}) = \mathrm{Spin}(7)$.

THEOREM 5.2.6: *Let (M^7,g) be a complete simply-connected Riemannian spin manifold of dimension 7 admitting a non-trivial Killing spinor with $\alpha > 0$ or $\alpha < 0$. Then there are four possibilities:*

(i) *(M,g) is of type $(1,0)$ and it is a proper G_2-manifold,*

(ii) *(M,g) is of type $(2,0)$ and it is a Sasakian-Einstein manifold, but (M,g) is not 3-Sasakian,*

(iii) *(M,g) is of type $(3,0)$ and it is 3-Sasakian,*

(iv) *$(M,g) = (S^7, g_{can})$ and is of type $(8,8)$.*

Conversely, if (M,g) is a compact simply-connected proper G_2-manifold then it carries a Killing spinor with $\alpha > 0$. If (M,g) is a compact simply-connected Sasakian-Einstein 7-manifold which is not 3-Sasakian then M carries 2 linearly independent Killing spinors with $\alpha > 0$. Finally, if (M,g) is a 3-Sasakian 7-manifold, which is not of constant curvature, then M carries 3 linearly independent Killing spinors with $\alpha > 0$.

REMARK 5.2.7 The four possibilities of the Theorem 5.2.6 correspond to the sequence of inclusions
$$\mathrm{Spin}(7) \supset SU(4) \supset Sp(2) \supset \{\mathrm{id}\}.$$

All of the corresponding cases are examples of weak holonomy G_2 metrics. If we exclude the trivial case when the associated cone is flat, we have three types of the weak holonomy G_2 geometries. Following [F-K-M-S] we use the number of linearly independent Killing spinors to classify the types of weak holonomy G_2 geometries. We call these type I, II, and III corresponding to cases (i), (ii), and (iii) of Theorem 5.2.6, respectively.

REMARK 5.2.8 In the case $\pi_1(M) \neq 0$, then M is either a spin spherical space form or $\mathrm{Hol}(\bar{g})$ equals to $SU(4)$, $SU(4) \rtimes \mathbb{Z}_2$, $Sp(2)$, or $\mathrm{Spin}(7)$ of type $(2,0), (1,0), (3,0), (1,0)$, respectively. Hence, we have just one more possible geometry for M [Wan 3]. Note that in the case $\mathrm{Hol}(\bar{g}) = SU(4) \rtimes \mathbb{Z}_2$, the cone $C(M)$ is not Kähler so that M cannot be Sasakian, but it is locally so.

Recall that S^7, regarded as the space $Sp(2)/Sp(1)$ and fibering over S^4, admits a 'squashed' Einstein metric which does not have constant curvature. This metric also has weak holonomy G_2 since the associated cone metric has holonomy equal to $Spin(7)$ and therefore S^7 with this metric is a proper G_2-manifold. We can generalize this example to get [G-Sal, F-K-M-S]:

THEOREM 5.2.9: *Let $(\mathcal{S},g)$ be a 7-dimensional 3-Sasakian manifold. Then the metric g has weak holonomy G_2. Moreover, the second Einstein metric g' given by Theorem 2.4.2 has weak holonomy G_2. In fact g' is a proper G_2 metric.*

PROOF: For the second Einstein metric g' we have three mutually orthonormal 1-forms $\alpha^1 = \sqrt{t}\eta^1, \quad \alpha^2 = \sqrt{t}\eta^2, \quad \alpha^3 = \sqrt{t}\eta^3$, where t is the parameter of the canonical variation discussed in section 2.4. Let $\{\alpha^4, \alpha^5, \alpha^6, \alpha^7\}$ be local 1-forms spanning the annihilator of $\mathcal{V}_3$ in $T^*\mathcal{S}$ such that
$$\bar{\Phi}^1 = 2(\alpha^4 \wedge \alpha^5 - \alpha^6 \wedge \alpha^7),$$
$$\bar{\Phi}^2 = 2(\alpha^4 \wedge \alpha^6 - \alpha^7 \wedge \alpha^5),$$
$$\bar{\Phi}^3 = 2(\alpha^4 \wedge \alpha^7 - \alpha^5 \wedge \alpha^6).$$

Then the set $\{\alpha^1, \cdots \alpha^7\}$ forms a local orthonormal coframe for the metric g'. In terms of the 3-forms Υ and Θ of 4.2.2 we have $\varphi = \frac{1}{2}\sqrt{t}\Theta + \sqrt{t}^3\Upsilon$. One easily sees that this is of the type of Equation 5.2.1 and, therefore, defines a compatible G_2-structure. Moreover, a straightforward computation gives

$$d\varphi = \frac{1}{2}\sqrt{t}\Omega + \sqrt{t}(t+1)d\Upsilon, \qquad \star\varphi = -\frac{1}{2}td\Upsilon - \frac{1}{24}\Omega.$$

Thus, $d\varphi = c \star \varphi$ is solved with $\sqrt{t} = 1/\sqrt{5}$, and $c = -12/\sqrt{5}$. So g' has weak holonomy G_2. That g' is a proper G_2 metric is due to [F-K-M-S]. The idea is to use Theorem 5.2.6. Looking at the four possibilities given in that theorem, we see that it suffices to show that g' is not Sasakian-Einstein. The details are in [F-K-M-S]. ∎

EXAMPLES 5.2.10: 3-Sasakian 7-manifolds are plentiful and examples will be discussed in next section. These give, by Theorem 5.2.9, many examples of type I and type III geometries. Examples of simply-connected type I geometries that do not arise via Theorem 5.2.9 are the homogeneous Aloff-Wallach spaces $N_{k,l} = SU(3)/T^1_{k,l}$, with $\gcd(k,l) = 1$ and $(k,l) \neq (1,1)$ [C-M-Sw, B-G-F-K] together with the homogeneous real Stiefel manifold $SO(5)/SO(3)$ [Bry]. All the known type II geometries are the 3 homogeneous examples from the list of Corollary 3.1.3(iii) (not 3-Sasakian) and the inhomogeneous simply-connected circle bundles over $P_k \times \mathbb{CP}^1$, where P_k is the del Pezzo surface with $2 < k < 9$ [F-K-M-S]. Actually, $N_{1,1}$ has three Einstein metrics. One is 3-Sasakian and is denoted by $\mathcal{S}(1,1,1)$ in section 7.4 below. The second is the proper G_2 metric of Theorem 5.9, while the third Einstein metric also has weak holonomy G_2 most likely of type I but we could not positively exclude type II as a possibility [C-M-Sw].

6. The Quotient Construction

In this section we give a general 3-Sasakian reduction procedure which constructs new 3-Sasakian manifolds from a given 3-Sasakian manifold $\mathcal{S}$ with a non-trivial 3-Sasakian isometry group $I_0(\mathcal{S}, g)$ [B-G-M 2]. Actually, this is a reduction that is associated with a quadruple of spaces of the fundamental diagram $\diamondsuit(\mathcal{S})$. At the level of the hyperkähler cone $C(\mathcal{S})$ the reduction was discovered by Lindström and Roček [L-R] in the context of supersymmetric σ-model and later rigorously described by Hitchin et al. in [H-K-L-R]. In the case of the quaternionic Kähler base $\mathcal{O}$ the reduction was discovered by the second author and H.B. Lawson [G, G-L]. The lift of the quaternionic Kähler quotient to the twistor space $\mathcal{Z}$ was described by Hitchin [Hit 2]. In this section we restrict ourselves to describing the general procedure of reduction together with the homogeneous case arising from reduction by a circle group, as well as a brief discription of the singular case. The large class of 3-Sasakian toric manifolds obtained by reduction is relagated to a separate section, namely section 7. It should also be understood that every 3-Sasakian reduction gives as well a reduction procedure for each of the spaces of the fundamental diagram $\diamondsuit(\mathcal{S})$.

6.1. The 3-Sasakian Moment Map

Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold with a nontrivial group $I_0(\mathcal{S}, g)$ of 3-Sasakian isometries. By the Definition 1.2.1, $C(\mathcal{S}) = \mathcal{S} \times \mathbb{R}^+$ is a hyperkähler manifold with respect to the cone metric $\bar{g}$. The isometry group $I_0(\mathcal{S}, g)$ extends to a group $I_0(C(\mathcal{S}), \bar{g}) \cong I_0(\mathcal{S}, g)$ of isometries on $C(\mathcal{S})$ by defining each element to act trivially on $\mathbb{R}^+$. Furthermore, it follows easily from the definition of the complex structures I^a given in equation 1.1.7

that these isometries $I_0(C(\mathcal{S}),\bar{g})$ are hyperkähler; that is, they preserve the hyperkähler structure on $C(\mathcal{S})$. Recall [H-K-L-R] shows that any subgroup $G\subset I_0(M,\bar{g})$ gives rise to a hyperkähler moment map $\mu : M\longrightarrow\mathfrak{g}^*\otimes\mathbb{R}^3$, where $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}^*$ is its dual. Thus, we can define a 3-Sasakian moment map

6.1.1
$$\mu_{\mathcal{S}} : \mathcal{S}\longrightarrow\mathfrak{g}^*\otimes\mathbb{R}^3$$

by restriction $\mu_{\mathcal{S}}=\mu\,|\,\mathcal{S}$. We denote the components of $\mu_{\mathcal{S}}$ with respect to the standard basis of $\mathbb{R}^3$, which we have identified with the imaginary quaternions, by $\mu^a_{\mathcal{S}}$. Recall that ordinarily moment maps determined by Abelian group actions (in particular, those associated to 1-parameter groups) are only specified up to an arbitrary constant. This is not the case for 3-Sasakian moment maps since we require that the group $Sp(1)$ generated by the Sasakian vector fields ξ^a acts on the level sets of $\mu_{\mathcal{S}}$. However, we shall see that 3-Sasakian moment maps are given by a particularly simple expression.

PROPOSITION 6.1.2: *Let $(\mathcal{S},g)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ by 3-Sasakian isometries. Let τ be an element of the Lie algebra $\mathfrak{g}$ of G and let X^τ denote the corresponding infinitesimal isometry. Then there is a unique 3-Sasakian moment map $\mu_{\mathcal{S}}$ such that the zero set $\mu^{-1}_{\mathcal{S}}(0)$ is invariant under the group $Sp(1)$ generated by the vector fields ξ^a. This moment map is given by*

6.1.3
$$<\mu^a_{\mathcal{S}},\tau> = \frac{1}{2}\eta^a(X^\tau).$$

Furthermore, the zero set $\mu^{-1}_{\mathcal{S}}(0)$ is G invariant.

PROOF: Using the Definition 1.2.1 we can define the 2-forms $\omega^a_{\mathcal{S}}$ on $\mathcal{S}$ as the restriction of the hyperkähler 2-forms ω^a. Then any 3-Sasakian moment map $\mu^a_{\mathcal{S}}(\tau)$ determined by $\tau\in\mathfrak{g}$ satisfies $2d\mu^a_{\mathcal{S}}(\tau)=2X^\tau\rfloor\omega^a_{\mathcal{S}}=-X^\tau\rfloor d\eta^a$. As X^τ is a 3-Sasakian infinitesimal isometry, Lemma 3.2.2 implies that $2<\mu^a_{\mathcal{S}},\tau>$ differs from $\eta^a(X^\tau)$ by a constant depending on a and τ. One then uses the invariance of the zero set $\mu^{-1}_{\mathcal{S}}(0)$ to show that these constants must vanish. See [B-G-M 2] for details. ∎

Henceforth by the 3-Sasakian moment map, we shall mean the moment map $\mu_{\mathcal{S}}$ determined in Proposition 6.1.2. Hence, the Definition 1.2.1 and Proposition 6.1.2 imply

THEOREM 6.1.5: *Let $(\mathcal{S},g)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ smoothly and properly by 3-Sasakian isometries. Let $\mu_{\mathcal{S}}$ be the corresponding 3-Sasakian moment map and assume both that 0 is a regular value of $\mu_{\mathcal{S}}$ and that G acts freely on the submanifold $\mu^{-1}_{\mathcal{S}}(0)$. Furthermore, let $\iota:\mu^{-1}_{\mathcal{S}}(0)\longrightarrow\mathcal{S}$ and $\pi:\mu^{-1}_{\mathcal{S}}(0)\longrightarrow\mu^{-1}_{\mathcal{S}}(0)/G$ denote the corresponding embedding and submersion. Then $(\mathcal{S}/\!\!/\!\!/G=\mu^{-1}_{\mathcal{S}}(0)/G,\check{g})$ is a smooth 3-Sasakian manifold of dimension $4(n-\dim\mathfrak{g})+3$ with metric $\check{g}$ and characteristic vector fields $\check{\xi}^a$ determined uniquely by the two conditions $\iota^*g=\pi^*\check{g}$ and $\pi_*(\xi^a\,|\,\mu^{-1}_{\mathcal{S}}(0))=\check{\xi}^a$.*

We conclude this part with the following fact concerning 3-Sasakian isometries whose proof can be found in [B-G-M2].

PROPOSITION 6.1.6: *Assume that the hypothesis of Theorem 6.1.5 holds. In addition assume that $(\mathcal{S},g)$ is complete and hence compact. Let $C(G)\subset I_0(\mathcal{S},g)$ denote the centralizer of G in $I_0(\mathcal{S},g)$ and let $C_0(G)$ denote the subgroup of $C(G)$ given by the connected component of the identity. Then $C_0(G)$ acts on the submanifold $\mu^{-1}_{\mathcal{S}}(0)$ as isometries with respect to the restricted metric ι^*g and the 3-Sasakian isometry group*

$I_0(\mathcal{S}/\!\!/\!\!/G, \breve{g})$ of the quotient $(\mathcal{S}/\!\!/\!\!/G, \breve{g})$ determined in Theorem 6.1.5 contains an isomorphic copy of $C_0(G)$. Furthermore, if $C_0(G)$ acts transitively on $\mathcal{S}/\!\!/\!\!/G$, then $\mathcal{S}/\!\!/\!\!/G$ is a 3-Sasakian homogeneous space.

It should be mentioned that it is not required that the isometry group $I_0(\mathcal{S}/\!\!/\!\!/G, \breve{g})$ acts effectively.

6.2 Regular Quotients And Classical Homogeneous Metrics

We now apply the reduction procedure given in Theorem 6.1.4 to the round unit sphere S^{4n+3} to explicitly construct the Riemannian metrics for the 3-Sasakian homogeneous manifolds arising from the simple classical Lie algebras. These metrics are precisely the ones associated to the three infinite families appearing in Theorem 3.2.6. The quotient construction applied to $\diamondsuit(\mathcal{S}^{4n+3})$ explicitly describes all metrics in the fundamental diagrams $\diamondsuit(G/H)$, where G is either the special unitary $SU(n+1)$ or the orthogonal group $SO(n+1)$. To carry out this reduction we must set some conventions. We describe the unit sphere S^{4n+3} by its embedding in flat space and we represent an element $\mathbf{u} = (u_1, \cdots, u_{n+1}) \in \mathbb{H}^{n+1}$ as a column vector. The quaternionic components of this vector are denoted by $\mathbf{u}^0$ for the real component and by $\mathbf{u}^a$ for the three imaginary components so that we can write $\mathbf{u} = \mathbf{u}^0 + i\mathbf{u}^1 + j\mathbf{u}^2 + k\mathbf{u}^3$ using the quaternionic units $\{i, j, k\}$. We also define quaternionic conjugate $\bar{\mathbf{u}} = \mathbf{u}^0 - i\mathbf{u}^1 - j\mathbf{u}^2 - k\mathbf{u}^3$.

Now, the infinitesimal generators of the subgroup $Sp(1) \subset \mathbb{H}^*$ acting by the right multiplication on $\mathbf{u}$ are the defining vector fields ξ^a for the Sasakian 3-structure. These vector fields are given by

6.2.1
$$\xi^a_r \;=\; \mathbf{u}^0 \cdot \frac{\partial}{\partial \mathbf{u}^a} - \mathbf{u}^a \cdot \frac{\partial}{\partial \mathbf{u}^0} - \epsilon^{abc}\mathbf{u}^b \cdot \frac{\partial}{\partial \mathbf{u}^c},$$

where the dot indicates sum over the vector components u_i and the subscript r means that these vector fields are the generators of the right action.

We will first consider $G = U(1)$ acting on the sphere S^{4n+3} as follows

6.2.2
$$\varphi_t(\mathbf{u}) = \tau\mathbf{u}, \qquad \tau = e^{2\pi i t}, \qquad \mathbf{u} \in S^{4n+3}.$$

Note that this action is actually free on S^{4n+3} and hence it will be automatically free on the level set of the moment map. To compute the moment map we identify the imaginary quaternions $\mathbb{R}^3$ with the Lie algebra $\mathfrak{sp}(1)$ in equation 6.1.1 and the Lie algebra of $U(1)$ with $\mathbb{R}$, so the moment map is $\mu_{\mathcal{S}} : S^{4n+3} \longrightarrow \mathbb{R} \otimes \mathfrak{sp}(1)$ and it can easily be computed

6.2.3
$$\mu_{\mathcal{S}}(\mathbf{u}) = \sum_{\alpha=1}^{n+1} \bar{u}_\alpha i u_\alpha.$$

One can easily identify the zero-level set of the moment map with the Stiefel manifold of complex 2-frames in $\mathbb{C}^{n+1}$, and the following proposition is then an immediate consequence of Theorem 6.1.5.

Proposition 6.2.4: *Let $N = \mu_{\mathcal{S}}^{-1}(0)$ and $\iota : N \hookrightarrow S^{4n+3}$ be the inclusion. Then ι is an embedding and $(N, \iota^* g_{can})$ is the complex homogeneous Stiefel manifold $V^{\mathbb{C}}_{2,n+1} = SU(n+1)/SU(n-1)$ of 2-frames in $\mathbb{C}^{n+1}$. Hence, the 3-Sasakian quotient $S^{4n+3}/\!\!/\!\!/U(1) =$*

$V^{\mathbb{C}}_{2,n+1}/U(1) = SU(n+1)/S(U(n-1)\times U(1))$ *with the 3-Sasakian metric* $\breve{g}$ *given by inclusion* ι *and submersion* $\pi : N \longrightarrow N/U(1)$*, i.e.,* $\iota^* g_{can} = \pi^*\breve{g}$*.*

REMARK 6.2.5: A similar construction can be carried out for the $Sp(1)$-action on S^{4n+3} defined by the left multiplication of $\mathbf{u}$ by a unit quaternion σ, i.e.,

6.2.6 $$\varphi_\sigma(\mathbf{u}) = \sigma\mathbf{u}, \qquad \sigma\bar{\sigma} = 1, \qquad \mathbf{u}\in S^{4n+3}.$$

This action is free on S^{4n+3} and the zero-level set of the corresponding moment map can be easily identified with the real Stiefel manifold $V^{\mathbb{R}}_{4,n+1} \simeq SO(n+1)/SO(n-3)$ of 4-frames in $\mathbb{R}^{n+1}$ with $n\geq 4$. Hence, the reduced space $S^{4n+3}/\!/\!/Sp(1) = \frac{SO(n+1)}{SO(n-3)\times Sp(1)}$. For the more detailed and uniform description of the geometry of these two quotients see [B-G-M 2].

6.3 The Structure of Singular Quotients

In this section we will describe a more general situation, when the zero-level set of the 3-Sasakian moment map 6.1.1 is not necessarily smooth and the group action on the level set is not necessarily locally free.

Let G be a Lie group acting smoothly and properly on a manifold $\mathcal{S}$ and let $H\subset G$ be a subgroup. Using standard notation we will denote by $\mathcal{S}_H\subset\mathcal{S}$ the set of points in $\mathcal{S}$ where the stability group is exactly equal to H and by $\mathcal{S}_{(H)}\subset\mathcal{S}$ the set of points with stabilizer conjugate to H in G. It follows than that the normalizer $N(H)$ of H in G acts freely on $\mathcal{S}_{(H)}$. Then we have the following theorem due to Dancer and Swann [D-Sw]:

THEOREM 6.3.1: *Let* $(\mathcal{S},g)$ *be a 3-Sasakian manifold with a connected compact Lie group* G *acting on* $\mathcal{S}$ *smoothly and properly by 3-Sasakian isometries. Let* $\mu_{\mathcal{S}}$ *be the corresponding 3-Sasakian moment map. Then the quotient* $\mu_{\mathcal{S}}^{-1}(0)/G$ *is a union of the smooth, 3-Sasakian manifolds* $(\mathcal{S}_{(H)}\cap\mu_{\mathcal{S}}^{-1}(0))/G$*, where* (H) *runs over the conjugacy classes of stabilizers of points in* $\mathcal{S}$*.*

Quite often $\mathcal{S}_{(H)}$ does not meet the zero locus of the moment map. Then the stratum $(\mathcal{S}_{(H)}\cap\mu_{\mathcal{S}}^{-1}(0))/G$ is empty.

EXAMPLE 6.3.2: We start with the 3-Sasakian sphere S^{4n+3} in the notation of the previous section. But now we consider a different circle action $U(1)$, namely

$$\varphi^{p,q;m}_t(\mathbf{u}) = (\tau^p u_1,\ldots,\tau^p u_m,\tau^q u_{m+1},\ldots,\tau^q u_{n+1}), \qquad \tau = e^{2\pi i t}, \qquad \mathbf{u}\in S^{4n+3},$$

where $0\leq m\leq n+1$ and $p,q\in\mathbb{Z}$. Let $\mathcal{S}(p,q;m) = S^{4n+3}/\!/\!/U(1)$ be the quotient.

(i) First, let p,q be relatively prime positive integers bigger than 1 and $2\leq m\leq n-1$. Then, the stratified manifold $\mathcal{S}(p,q;m)$ consists of 3 strata. The stratum of the highest dimension corresponding to $H=\{\mathrm{id}\}$ is an open incomplete 3-Sasakian manifold. The two strata of lower dimension are easily seen to be the homogeneous spaces: one with $H=\mathbb{Z}_q$ is the homogeneous 3-Sasakian space of $SU(n+1-m)$ and the one with $H=\mathbb{Z}_p$ is the homogeneous 3-Sasakian space of $SU(m)$. In this case, $\mathcal{S}(p,q;m)$ is actually a compact 3-Sasakian orbifold and the stratification of Theorem 6.3.1 coincides with the orbifold stratification.

(ii) Consider $\mathcal{S}(0,p;m)$, where $p>1$ and $2\leq m\leq n-1$. There are two strata now: the stratum of the highest dimension corresponds to $H=\mathbb{Z}_p$ and the second stratum is just the sphere S^{4m-1} with $H = U(1)$. The space $\mathcal{S}(0,p;m)$ is not an orbifold but, as pointed out in [D-Sw], it does have a length space structure.

(iii) Consider $\mathcal{S}(0,1;n)$. Here H is either $U(1)$ or trivial but the set S^{4n+3}_{id} does not meet the zero locus of the moment map. Hence, there is only one stratum and $\mathcal{S}(0,1;n) = S^{4n-1}$.

(iv) Finally, consider $\mathcal{S}(0,1;n-1)$. The stability group H is either $U(1)$ or trivial. The stratum corresponding to $H = U(1)$ is the sphere S^{4n-5}. We leave it as an exercise to the reader to show that $\mathcal{S}(0,1;n-1)$ is an orbifold and that it can be identified with $S^{4n-1}/\mathbb{Z}_2$, where $(w_1,....,w_n) \in S^{4n-1}$, where $\mathbb{Z}_2$ acts on the last quaternionic coordinate by multiplication by ± 1.

7. Toric 3-Sasakian Manifolds

In this section we shall describe the quotient construction of large families of 3-Sasakian manifolds $\mathcal{S}(\Omega)$. They all have the property that $I_0(\mathcal{S}(\Omega), g(\Omega)) \supset T^m$, where $\dim(\mathcal{S}(\Omega)) = 4m-1$, and following the ideas of [Bi-D] we shall call such 3-Sasakian manifolds *toric* (See 7.6.1 for a precise definition). We also describe some interesting geometric and topological properties of such spaces. Up until now all known examples of 3-Sasakian manifolds are either homogeneous or toric or discrete quotients of them.

7.1 Toral Reductions of Spheres

Using the notation of the previous section we start with the unit $(4n+3)$-dimensional sphere embedded in the quaternionic vector space $\mathbb{H}^{n+1}$. The subgroup of the full isometry group $O(4n+3)$ that preserves the quaternionic structure is $Sp(n+1)\cdot Sp(1)$ acting by

7.1.1 $$\varphi_{\mathbb{A},\sigma}(\mathbf{u}) = \mathbb{A}\mathbf{u}\sigma^{-1},$$

where $\mathbb{A} \in Sp(n+1)$ is the quaternionic $(n+1)\times(n+1)$ matrix of the quaternionic representation of $Sp(n+1)$, and $\sigma \in Sp(1)$ is a unit quaternion. As the diagonal $\mathbb{Z}_2$ acts trivially this is indeed an $Sp(n+1)\cdot Sp(1)$ action. The group $Sp(n+1)$ is the subgroup of $Sp(n+1)\cdot Sp(1)$ which preserves the 3-Sasakian structure on S^{4n+3}, so we have $I_0(S^{4n+3}, g_{can}) = Sp(n+1)$. We shall consider the maximal torus $T^{n+1} \subset Sp(n+1)$ and its subgroups. Every quaternionic representation of a k-torus T^k on $\mathbb{H}^{n+1}$ can be described by an exact sequence $0 \longrightarrow T^k \xrightarrow{f_\Omega} T^{n+1} \longrightarrow T^{n+1-k} \longrightarrow 0$. The monomorphism f_Ω can be represented by the matrix

7.1.2 $$f_\Omega(\tau_1,\ldots,\tau_k) = \begin{pmatrix} \prod_{i=1}^k \tau_i^{a_1^i} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \prod_{i=1}^k \tau_i^{a_{n+1}^i} \end{pmatrix},$$

where $(\tau_1,..,\tau_k) \in S^1\times\cdots\times S^1 = T^k$ are the complex coordinates on T^k, and $a_j^i \in \mathbb{Z}$ are the coefficients of a $k\times(n+1)$ integral *weight* matrix $\Omega = (a_\alpha^l)_{\alpha=1,\ldots,n+1}^{l=1,\ldots,k} \in \mathcal{M}_{k,n+1}(\mathbb{Z})$.

Let $\{e_l\}_{l=1}^k$ denote the standard basis for $\mathfrak{t}_k^* \simeq \mathbb{R}^k$. Then the 3-Sasakian moment map $\mu_\Omega : S^{4n+3} \longrightarrow \mathfrak{t}_k^* \otimes \mathbb{R}^3$ of the k-torus action defined by $\varphi_{(\tau_1,\ldots,\tau_k)}(\mathbf{u}) = f_\Omega(\tau_1,\ldots,\tau_k)\mathbf{u}$, is given by $\mu_\Omega = \sum_l \mu_\Omega^l e_l$ with

7.1.3 $$\mu_\Omega^l(\mathbf{u}) = \sum_\alpha \bar{u}_\alpha i a_\alpha^l u_\alpha.$$

Let us further denote the triple $(T^k, f_\Omega, \varphi_{(\tau_1,\ldots,\tau_k)})$ by $T^k(\Omega)$.

Definition 7.1.4: *$N(\Omega) = \mu_\Omega^{-1}(0)$ and $\mathcal{S}(\Omega) = S^{4n+3}/\!/\!/T^k(\Omega) = N(\Omega)/T^k(\Omega)$.*

Let S_H^{4n+3} denote all the points on the sphere where the stability group $H \subset T^k$ is exactly H. Because T^k is Abelian $S_H^{4n+3} = S_{(H)}^{4n+3}$. Furthermore, let $K_H = T^k/H$ and denote by $\mathcal{S}(\Omega; H) = S_H^{4n+3} \cap N(\Omega)/K_H$. Following Theorem 6.3.1 we have

Proposition 7.1.5: *The quotient $\mathcal{S}(\Omega) = \bigcup_H \mathcal{S}(\Omega; H)$ is a disjoint union of 3-Sasakian manifolds, where each stratum $\mathcal{S}(\Omega; H)$ is smooth.*

We will be interested in the case when $\mathcal{S}(\Omega)$ is a compact orbifold (all stability groups H for which $S_H^{4n+3} \cap N(\Omega)$ are non-empty are discrete) or a compact smooth manifold (there is only one stratum). Necessary and sufficient conditions for this to happen can be expressed in terms of properties of the matrix Ω. First observe that, without loss of generality, we can assume that the rank of Ω equals k. Otherwise, one simply has an action of a torus of lower dimension and the whole problem reduces to considering another weight matrix Ω with fewer rows.

We introduce the following terminology: Consider the $\binom{n}{k}$ minor determinants

7.1.6 $$\Delta_{\alpha_1\ldots\alpha_k} = \det \begin{pmatrix} a_{\alpha_1}^1 & \ldots & a_{\alpha_k}^1 \\ \vdots & & \vdots \\ a_{\alpha_1}^k & \ldots & a_{\alpha_k}^k \end{pmatrix}$$

obtained by deleting $n+1-k$ columns of Ω.

Definition 7.1.7: *Let $\Omega \in \mathcal{M}_{k,n+1}(\mathbb{Z})$ be the weight matrix.*

(i) If $\Delta_{\alpha_1\cdots\cdots\alpha_k} \neq 0, \forall \quad 1 \leq \alpha_1 < \cdots < \alpha_k \leq n+1$, then we say that Ω is non-degenerate.

Suppose Ω is non-degenerate and let g be the kth determinantal divisor, i.e., the gcd of all the k by k minor determinants $\Delta_{\alpha_1\cdots\cdots\alpha_k}$. Then Ω is said to be admissible if in addition we have

(ii) $\gcd(\Delta_{\alpha_2\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\alpha_k}) = g$ for all sequences of length (k+1) such that $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n+1$.

7.2 Equivalence Problem and Admissibility

Before we show how these properties of the matrix Ω impact on the geometry of the quotient $\mathcal{S}(\Omega)$ we need to discuss the notion of the equivalence of T^k-actions on S^{4n+3} and obtain a normal form for admissible weight matrices. We are free to change bases of the Lie algebra $\mathfrak{t}_k$. This can be done by the group of unimodular matrices $GL(k,\mathbb{Z})$. Moreover, if we fix a maximal torus T^{n+1} of $Sp(n+1)$, its normalizer, the Weyl group $\mathcal{W}(Sp(n+1)) \simeq \Sigma_{n+1} \rtimes (\mathbb{Z}_2)^{n+1}$, preserves the 3-Sasakian structure on S^{4n+3} and intertwines the T^k actions. Thus, there is an induced action of $GL(k,\mathbb{Z}) \times \mathcal{W}(Sp(n+1))$ on the set of weight matrices $\mathcal{M}_{k,n+1}(\mathbb{Z})$. The group $GL(k,\mathbb{Z})$ acts on $\mathcal{M}_{k,n+1}(\mathbb{Z})$ by matrix multiplication from the left, and the Weyl group $\mathcal{W}(Sp(n+1))$ acts by permutation and overall sign changes of the columns. Actually we want a slightly stronger notion of equivalence than that described above. If the ith row of Ω has a gcd d_i greater than one, then by reparameterizing the one-parameter subgroup $\tau'_i = \tau_i^{d_i}$ we obtain $\tau_i^{a^j_\alpha} = (\tau'_i)^{b^i_\alpha}$, where $\gcd\{b^i_\alpha\}_\alpha = 1$. So the action obtained by using the matrix whose ith row is divided by its gcd d_i is the same as the original action. The integers d_i all divide the kth determinantal divisor g. We say that a non-degenerate matrix Ω is in *reduced* form (or simply *reduced*) if $g = 1$. The following easy lemma says that among non-degenerate matrices it is sufficient to consider matrices in a reduced form.

LEMMA 7.2.1: *Every non-degenerate weight matrix Ω is equivalent to a matrix in a reduced form.*

Henceforth, we shall only consider matrices in a reduced form.

DEFINITION 7.2.2: *Let $\mathcal{A}_{k,n+1}(\mathbb{Z}) \subset \mathcal{M}_{k,n+1}(\mathbb{Z})$ denote the subset of reduced admissible matrices. This subset is invariant under the action of $GL(k,\mathbb{Z}) \times \mathcal{W}(Sp(n+1))$, so the set $\mathcal{A}_{k,n+1}(\mathbb{Z})/GL(k,\mathbb{Z}) \times \mathcal{W}(Sp(n+1))$ of equivalence classes $[\Omega]$ is well defined. We let $\mathcal{E}_{k,n+1}(\mathbb{Z}) \subset \mathcal{A}_{k,n+1}(\mathbb{Z})$ denote a fundamental domain for the action.*

Our interest in $\mathcal{A}_{k,n+1}(\mathbb{Z})$ is the following:

THEOREM 7.2.3: *Let $\mathcal{S}(\Omega)$ be the quotient space of definition 7.1.6. Then*

(i) if Ω is non-degenerate, $\mathcal{S}(\Omega)$ is an orbifold.

(ii) If Ω is degenerate, then either $\mathcal{S}(\Omega)$ is a singular stratified space which is not an orbifold or it is an orbifold obtained by reduction of a lower dimensional sphere $S^{4n-4r-1}$ by a torus $T^{k-r}(\Omega')$ or a finite quotient of such, where $1 \le r \le k$ and Ω' is non-degenerate. (When $r = k$ the quotient is the sphere $S^{4n-4k-1}$).

(iii) Assuming that Ω is non-degenerate $\mathcal{S}(\Omega)$ is a smooth manifold if and only if Ω is admissible.

One can easily see that the non-degeneracy of Ω is not necessary for the quotient space $\mathcal{S}(\Omega)$ to be smooth or a compact orbifold (see Example 6.3.3(iv)). However, Theorem 7.2.5(ii) shows that then we can reformulate the whole problem in terms of another quotient and a new non-degenerate weight matrix Ω' and can be found in [B-G-M 7]. Theorem 7.2.5(iii) shows then the importance of admissible matrices in the construction and it easily follows from the fact that non-degeneracy implies that at most $n-k$ quaternionic coordinates u_j can simultaneously vanish on $N(\Omega)$ [B-G-M-R 1].

REMARK 7.2.4: Our discussion shows clearly that, if $\Omega, \Omega' \in \mathcal{A}_{k,n+1}(\mathbb{Z})$ such that $[\Omega] = [\Omega']$ then the quotients $\mathcal{S}(\Omega) \simeq \mathcal{S}(\Omega')$ are equivalent as 3-Sasakian manifolds. We believe that the converse of this is also true, though we will establish it later only in certain cases.

7.3.3 Combinatorics and Admissibility

In general Theorem 7.2.3 is not yet an existence theorem, since $\mathcal{A}_{k,n+1}(\mathbb{Z})$ could be empty. Indeed, for many pairs (k,n) this is the case and we shall demonstrate this next.

Let $\Omega \in \mathcal{A}_{k,n+1}(\mathbb{Z})$. Since Ω is reduced there is a k by k minor determinant that is odd. By permuting columns if necessary this minor can be taken to be the first k columns. Now consider the mod 2 reduction $\mathcal{M}_{k,n+1}(\mathbb{Z}) \longrightarrow \mathcal{M}_{k,n+1}(\mathbb{Z}_2)$. We have the following commutative diagram

7.3.1
$$\begin{array}{ccc} GL(k,\mathbb{Z})\times\mathcal{M}_{k,n+1}(\mathbb{Z}) & \longrightarrow & \mathcal{M}_{k,n+1}(\mathbb{Z}) \\ \downarrow & & \downarrow \\ GL(k,\mathbb{Z}_2)\times\mathcal{M}_{k,n+1}(\mathbb{Z}_2) & \longrightarrow & \mathcal{M}_{k,n+1}(\mathbb{Z}_2). \end{array}$$

Let $\tilde{\Omega}\in\mathcal{A}_{k,n+1}(\mathbb{Z}_2)$ denote the mod 2 reduction of $\Omega\in\mathcal{A}_{k,n+1}(\mathbb{Z})$. Since the first k by k minor determinant of Ω is odd, the mod 2 reduction of this minor in $\tilde{\Omega}$ is invertible. Thus, we can use the $GL(k,\mathbb{Z}_2)$ action to put $\tilde{\Omega}$ in the form

7.3.2
$$\tilde{\Omega}=\begin{pmatrix} 1 & 0 & \dots & 0 & a^1_{k+1} & \dots & a^1_{n+1} \\ 0 & 1 & \dots & 0 & a^2_{k+1} & \dots & a^2_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & 1 & a^k_{k+1} & \dots & a^k_{n+1} \end{pmatrix}$$

with $a^i_j\in\mathbb{Z}_2$.

Lemma 7.3.3: *The set $\mathcal{A}_{k,n+1}(\mathbb{Z})$ is empty for $n>k+1$ and $k>4$.*

Proof: The second admissibility condition is equivalent to the condition that every k by $k+1$ submatrix of $\tilde{\Omega}$ has rank k. By considering $k-1$ of the first k columns and 2 of last $n+1-k$ columns, this condition implies $(a^j_l,a^j_m)\neq(0,0)$ for all $j=1,\cdots,k$, and $k+1\le l<m\le n+1$. Similarly, by considering $k-2$ of the first k columns and 3 of last $n+1-k$ columns 7.3.2 implies

7.3.4
$$\begin{pmatrix} a^i_l & a^i_m & a^i_r \\ a^j_l & a^j_m & a^j_r \end{pmatrix}\neq\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} a^i_l & a^i_m & a^i_r \\ a^j_l & a^j_m & a^j_r \end{pmatrix}\neq\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

where the last inequality is understood to be up to column permutation. Hence, it follows that, up to column and row permutations, that any four triples of the last $n-k$ columns of an admissible $\tilde{\Omega}$ must have the form

7.3.5
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

So we see that we cannot add another row without violating the above conditions. It follows that $k\le 4$. ∎

Similar analysis shows that

Lemma 7.3.6: *The set $\mathcal{A}_{k,n+1}(\mathbb{Z})$ is empty if $k>1$ and $n-k\ge 4$.*

Remark 7.3.7: In view of the above lemmas and the fact that in the remainder of this section we will be interested only in the smooth and compact quotients we are left with the following possibilities:

(i) Trivial case of $n=k$. Then there are many admissible matrices Ω but $\dim(\mathcal{S}(\Omega))=3$ and it follows that $\mathcal{S}(\Omega)=S^3/\mathbb{Z}_p$, where $p=p(\Omega)$ depends on Ω. This case is of little interest.

(ii) Bi-quotient geometry with $k=1$ and $n>1$ arbitrary. Here Ω is just a row vector $\mathbf{p}$. The admissibility condition means that the entries are non-zero and pairwise relatively prime. The quotient $\mathcal{S}(\mathbf{p})$ turns out to be a bi-quotient of the unitary group $U(n+1)$ and we shall discuss its geometry and topology in the next subsection.

(iii) The most interesting, 7-dimensional case of $k=n-1$. Here one easily sees that there are many admissible matrices and we analyze the geometry and topology of the quotients in a separate subsection.

(iv) "Special" quotients: $(k,n)=\{(2,4),(2,5),(3,5),(3,6),(4,6),(4,7)\}$. These quotients are 11- or 15-dimensional and we give examples of admissible weight matrices in each case. We shall show also that they provide counterexamples to certain Betti number relations that are satisfied in the regular case [G-Sal].

7.4 3-Sasakian Structures on Bi-Quotients

When $k=1$ we have $\Omega=\mathbf{p}=(p_1,...,p_{n+1})$ and we shall write $\mathcal{S}(\Omega)=\mathcal{S}(\mathbf{p})$, $N(\Omega)=N(\mathbf{p})$, $f_\Omega=f_{\mathbf{p}}$, and $\tau_1=\tau$. The quotients $\mathcal{S}(\mathbf{p})$ are generalizations of the homogeneous examples discussed in Section 6.2. We get

$$\mathcal{A}_{1,n+1}(\mathbb{Z})=\{\mathbf{p}\in(\mathbb{Z})^{n+1}\ \mid\ p_i\neq 0\ \forall i=1,...,n+1 \text{ and } \gcd(p_i,p_j)=1\qquad \forall i\neq j\},$$

$$\mathcal{E}_{1,n+1}(\mathbb{Z})=\{\mathbf{p}\in\mathbb{Z}^{n+1}\ \mid\ 0<p_1\leq\cdots\leq p_{n+1} \text{ and } \gcd(p_i,p_j)=1\qquad \forall i\neq j\}.$$

Note that $\mathcal{E}_{1,n+1}(\mathbb{Z})$ can be identified with a certain integral lattice in the positive Weyl chamber in $\mathbf{t}^*_{n+1}$.

First, by studying the geometry of the foliations in the diagram $\diamondsuit(\mathcal{S}(\mathbf{p}))$ [B-G-M 6] one can solve the equivalence problem in this case. We get [B-G-M 3]:

PROPOSITION 7.4.1: *Let $n\geq 2$ and $\mathbf{p},\mathbf{q}\in\mathcal{A}_{1,n+1}(\mathbb{Z})$ so the quotients $\mathcal{S}(\mathbf{p})$ and $\mathcal{S}(\mathbf{q})$ are smooth manifolds. Then $\mathcal{S}(\mathbf{p})\simeq\mathcal{S}(\mathbf{q})$ are 3-Sasakian equivalent if and only if $[\mathbf{p}]=[\mathbf{q}]$.*

It is easy to see that for $\mathbf{p}\in\mathcal{A}_{1,n+1}(\mathbb{Z})$ the zero locus of the moment map $N(\mathbf{p})$ is always diffeomorphic to the Stiefel manifold $V^{\mathbb{C}}_{2,n+1}$ of complex 2-frames in $\mathbb{C}^{n+1}$. Hence, the quotient $\mathcal{S}(\mathbf{p})=V^{\mathbb{C}}_{2,n+1}/S^1$. We first observe that one can identify $V^{\mathbb{C}}_{2,n+1}$ with the homogeneous space $U(n+1)/U(n-1)$. Using this identification we have

PROPOSITION 7.4.2: *For each $\mathbf{p}\in\mathcal{E}_{1,n+1}(\mathbb{Z})$, there is an equivalence $\mathcal{S}(\mathbf{p})\simeq U(1)_{\mathbf{p}}\backslash U(n+1)/U(n-1)$ as smooth $U(1)_{\mathbf{p}}\times U(n-1)$-spaces, where the action of $U(1)_{\mathbf{p}}\times U(n-1)\subset U(n+1)_L\times U(n+1)_R$ is given by the formula*

$$\varphi^{\mathbf{p}}_{\tau,\mathbb{B}}(\mathbb{W})=f_{\mathbf{p}}(\tau)\mathbb{W}\begin{pmatrix}\mathbb{I}_2 & \mathbb{O}\\ \mathbb{O} & \mathbb{B}\end{pmatrix}. \tag{7.4.3}$$

Here $\mathbb{W}\in U(n+1)$ and $(\tau,\mathbb{B})\in S^1\times U(n-1)$.

Note that the identification $\mathcal{S}(\mathbf{p})\simeq U(1)_{\mathbf{p}}\backslash U(n+1)/U(n-1)$ is only true after assuming that all the weights are positive, as the right-hand side is not invariant under such sign changes. Proposition 7.4.2 shows that, in a way, the quotients $\mathcal{S}(\mathbf{p})$ can be though of as "bi-quotient deformation" of the homogeneous model $\mathcal{S}(\mathbf{1})$. Now let $\iota_{\mathbf{p}}:N(\mathbf{p})\hookrightarrow S^{4n+3}$ be the inclusion and $\pi_{\mathbf{p}}:N(\mathbf{p})\to\mathcal{S}(\mathbf{p})$ be the Riemannian submersion of the moment map. Then the metric $g(\mathbf{p})$ is the unique metric on $\mathcal{S}(\mathbf{p})$ that satisfies $\iota^*_{\mathbf{p}}g_{can}=\pi^*_{\mathbf{p}}g(\mathbf{p})$. Using the geometry of the inclusion $\iota_{\mathbf{p}}$ one can show the following [B-G-M 3,6]

Theorem 7.4.4: *Let $I_0(\mathcal{S}(\mathbf{p}), g(\mathbf{p}))$ be the group of 3-Sasakian isometries of $\big(\mathcal{S}(\mathbf{p}), g(\mathbf{p})\big)$ and let k be the number of 1's in $\mathbf{p}$. Then the connected component of I_0 is $S(U(k) \times U(1)^{n+1-k})$, where we define $U(0) = \{e\}$. Thus, the connected component of the isometry group is the product $S(U(k) \times U(1)^{n+1-k}) \times SO(3)$ if the sums $p_i + p_j$ are even for all $1 \leq i, j \leq n+1$, and $S(U(k) \times U(1)^{n+1-k}) \times Sp(1)$ otherwise.*

In the case that $\mathbf{p}$ has no repeated 1's, the cohomogeneity can easily be determined, viz. [B-G-M 3]

Corollary 7.4.5 : *If the number of 1's in $\mathbf{p}$ is 0 or 1 then the dimension of the principal orbit in $\mathcal{S}(\mathbf{p})$ equals $n+3$ and the cohomogeneity of $g(\mathbf{p})$ is $3n-4$. In particular, the 7-dimensional $\mathcal{S}(\mathbf{p})$ the family $(\mathcal{S}(\mathbf{p}), g(\mathbf{p}))$ contains metrics of cohomogeneity 0,1, and 2.*

Combining Proposition 7.4.2 with techniques developed by Eschenburg [Esch 1-2] in the study of certain 7-dimensional bi-quotients of $SU(3)$ one can compute the integral cohomology ring of $\mathcal{S}(\mathbf{p})$ [B-G-M 2]:

Theorem 7.4.6: *Let $n \geq 2$ $\mathbf{p} \in \mathcal{E}_{1,n+1}(\mathbb{Z})$. Then, as rings,*

$$H^*\big(\mathcal{S}(\mathbf{p}), \mathbb{Z}\big) \;\cong\; \left(\frac{\mathbb{Z}[b_2]}{[b_2^{n+1} = 0]} \otimes E[f_{2n+1}]\right)\Big/\mathcal{R}(\mathbf{p}).$$

Here the subscripts on b_2 and f_{2n+1} denote the cohomological dimension of each generator. Furthermore, the relations $\mathcal{R}(\mathbf{p})$ are generated by $\sigma_n(\mathbf{p})b_2^n = 0$ and $f_{2n+1}b_2^n = 0$, where $\sigma_n(\mathbf{p}) = \sum_{j=1}^{n+1} p_1 \cdots \hat{p_j} \cdots p_{n+1}$ is the n^{th} elementary symmetric polynomial in the entries of $\mathbf{p}$.

Notice that Theorem 7.4.6 shows that $H^{2n}(\mathcal{S}(\mathbf{p}); \mathbb{Z}) \;=\; \mathbb{Z}_{\sigma_n(\mathbf{p})}$ and hence has the following corollary.

Corollary 7.4.7: *The quotients $(\mathcal{S}(\mathbf{p}), g(\mathbf{p}))$ give infinitely many homotopy inequivalent simply-connected compact inhomogeneous 3-Sasakian manifolds in dimension $4n-1$ for every $n \geq 2$. In fact, there are infinite families that are not homotopy equivalent to any homogeneous space.*

Remark 7.4.8: Corollary 7.4.7 shows that the finiteness results for regular 3-Sasakian manifolds discussed in Section 4.4 fail for non-regular 3-Sasakian manifolds. Moreover, combining our results with a well-known finiteness theorem of Anderson [An] we have

Corollary 7.4.9: *For each $n \geq 2$ there are infinitely many 3-Sasakian $4n-1$-manifolds with arbitrarily small injectivity radii.*

When $n = 2$ the spaces $\mathcal{S}(\mathbf{p}) = \mathcal{S}(p_1, p_2, p_3)$ give a subfamily of the more general bi-quotients of $U(3)$ studied by Eschenburg [Esch 1-2]. This large collection of spaces contains not only our 3-Sasakian subfamily, but also the well-known Aloff-Wallach spaces [Al-Wa] which are of much interest since they admit Einstein metrics of positive sectional curvature [Wan 1]. These two subfamilies intersect at the homogeneous 3-Sasakian manifold $\mathcal{S}(1,1,1)$, that is $\mathcal{S}(1,1,1)$ is diffeomorphic to the Aloff-Wallach space $N_{1,1}$ mentioned in 5.2.10. Then following Eschenburg [Esch 1] we can make use of the Cheeger ρ^*-topology on the space of Riemannian manifolds to show the existence of an infinite number of 3-Sasakian manifolds that admit metrics of positive sectional curvature. More precisely [B-G-M 2],

Corollary 7.4.10: *For all sufficiently large odd positive integers c, the 3-Sasakian manifolds $\mathcal{S}(c, c+1, c+2)$ admits a metric of positive sectional curvature.*

In the next subsection we give a result in the opposite direction. We shall exhibit an infinite family of 3-Sasakian manifolds that cannot admit any metric whose sectional curvature is bounded from below by a fixed arbitrary negative number.

We end this subsection with a discussion of topological and differential invariants of the 7-manifolds $\mathcal{S}(p_1, p_2, p_3)$. Homotopy invariants for Eschenburg space have been worked out independently by Kruggel [Kru 1,2] and Milgram [Mil]. The homeomorphism and diffeomorphism classification was first done for a certain subclass of Eschenburg spaces which include some of the $\mathcal{S}(p_1, p_2, p_3)$ by Astey, Micha, and Pastor [A-M-P]. Later Kruggel [Kru 3] obtained the diffeomorphism and homeomorphism classification of all Eschenburg's bi-quotients by computing the Kreck-Stolz invariants [K-S 1]. This, in principle, gives a complete differential topological description of the 7-dimensional family $\mathcal{S}(p_1, p_2, p_3)$. Using this classification together with the help of a computer program, one would expect to find examples $\mathcal{S}(\mathbf{p})$ and $\mathcal{S}(\mathbf{q})$ with $[\mathbf{p}] \neq [\mathbf{q}]$ such that the quotients are homeomorphic, but not diffeomorphic, as well as examples that are diffeomorphic, but not 3-Sasakian equivalent. The later would show that a smooth 7-manifold can admit more than one inequivalent 3-Sasakian structure. In the case of the former, such exotic structures are known to exist for the family of Aloff-Wallach spaces [K-S 2], but the examples involve large integers and were obtained with help of a computer program. The analysis of the above mentioned invariants for our family $\mathcal{S}(p_1, p_2, p_3)$ proves even harder due to the positivity of the weights. For a fixed $\sigma_2 = p_1p_2 + p_2p_3 + p_3p_1$ there are only finitely many positive integer solution $\mathbf{p} \in \mathcal{E}_{1,3}(\mathbb{Z})$.

7.5 7-dimensional Toric 3-Sasakian Manifolds

In this case we can easily see that there are many examples of admissible weight matrices Ω. The simplest family of examples is given by matrices of the form

7.5.1
$$\Omega = \begin{pmatrix} 1 & 0 & \dots & 0 & a_1 & b_1 \\ 0 & 1 & \dots & 0 & a_2 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_k & b_k \end{pmatrix},$$

for which we have

Proposition 7.5.2: *Let k be a positive integer, and let $\Omega \in \mathcal{M}_{k,k+2}(\mathbb{Z})$ be as in 7.5.1. Then $\Omega \in \mathcal{A}_{k,k+2}(\mathbb{Z})$ if and only if $(\mathbf{a}, \mathbf{b}) \in (\mathbb{Z}^*)^k \oplus (\mathbb{Z}^*)^k$ and the components (a^i, b^i) are pairs of relatively prime integers for $i = 1, \cdots, k$ such that if for some pair i, j $a^i = \pm a^j$ or $b^i = \pm b^j$ then we must have $b^i \neq \pm b^j$ or $a^i \neq \pm a^j$, respectively.*

Proposition 7.5.2 shows that $\mathcal{A}_{k,k+2}(\mathbb{Z})$ is never empty and we have many examples of compact smooth 7-dimensional quotients $\mathcal{S}(\Omega)$ for arbitrary $k > 1$. Some of these examples were first mentioned in [B-G-M 1] and the idea of the quotient is based on the result of [G-Ni]. As we shall not present here the complete solution to the equivalence problem, we shall further assume that $\Omega \in \mathcal{A}_{k,k+2}(\mathbb{Z})$ is arbitrary and shall determine some important topological properties of the quotients $\mathcal{S}(\Omega)$. More explicitly,

Theorem 7.5.3: *Let $\Omega \in \mathcal{A}_{k,k+2}(\mathbb{Z})$ Then $\pi_1(\mathcal{S}(\Omega)) = 0$ and $\pi_2(\mathcal{S}(\Omega)) = \mathbb{Z}^k$.*

Because of Corollary 4.2.8 and Poincare duality, Theorem 7.5.3 completely determines the rational homology of the 3-Sasakian 7-manifolds $\mathcal{S}(\Omega)$. The proof given below is a compilation with some simplifications of the proofs in [B-G-M-R 1, B-G-M 8], while some of the more tedious details are left to those references.

PROOF: First note that the groups $T^{k+2}\times Sp(1)$ and $T^2\times Sp(1)$ act as isometry groups on $N(\Omega)$ and $\mathcal{S}(\Omega)$, respectively. Let us define the following quotient spaces:

$$Q(\Omega) = N(\Omega)/T^{k+2}\times Sp(1) \qquad B(\Omega) = N(\Omega)/Sp(1).$$

We have the following commutative diagram

7.5.4
$$\begin{array}{ccc} N(\Omega) & \longrightarrow & B(\Omega) \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{S}(\Omega) & \longrightarrow & Q(\Omega). \end{array}$$

The top horizontal arrow and the left vertical arrow are principal bundles with fibers $Sp(1)$ and T^k, respectively. The remaining arrows are not fibrations. The right vertical arrow has generic fibers T^{k+2}, while the lower horizontal arrow has generic fibers $T^2{\cdot}Sp(1)$ homeomorphic either to $T^2\times\mathbb{RP}^3$ or $T^2\times S^3$ depending on Ω. The dimension of the orbit space $Q(\Omega)$ is 2.

The difficulty is in proving that both $N(\Omega)$ and $B(\Omega)$ are 2-connected. Once this is accomplished the result follows by applying the long exact homotopy sequence to the left vertical arrow in diagram 7.5.4.

LEMMA 7.5.5: *Both $N(\Omega)$ and $B(\Omega)$ are 2-connected.*

PROOF: The idea is to construct a stratification giving a Leray spectral sequence whose differentials can be analyzed. Let us define the following subsets of $N(\Omega)$: (Recall that, in this case, at most one quaternionic coordinate can vanish.)

7.5.6
$$\begin{aligned} N_0(\Omega) &= \{\mathbf{u}\in N(\Omega)|\ u_\alpha = 0 \text{ for some } \alpha = 1,\cdots,k+2\}, \\ N_1(\Omega) &= \{\mathbf{u}\in N(\Omega)|\ \text{for all } \alpha = 1,\cdots,k+2,\ u_\alpha\neq 0 \text{ and} \\ &\qquad \text{there is a pair } (u_\alpha,u_\beta) \text{ that lies on the same} \\ &\qquad \text{complex line in } \mathbb{H}\}, \\ N_2(\Omega) &= \{\mathbf{u}\in N(\Omega)|\ \text{for all } \alpha = 1,\cdots,k+2,\ u_\alpha\neq 0 \text{ and} \\ &\qquad \text{no pair } (u_\alpha,u_\beta) \text{ lies on the same complex line} \\ &\qquad \text{in } \mathbb{H}\}. \end{aligned}$$

Clearly, $N(\Omega) = N_0(\Omega)\sqcup N_1(\Omega)\sqcup N_2(\Omega)$ and $N_2(\Omega)$ is a dense open submanifold of $N(\Omega)$. This stratification is compatible with the diagram 7.5.4 and induces corresponding stratifications

7.5.7 $$B(\Omega) = B_0(\Omega)\sqcup B_1(\Omega)\sqcup B_2(\Omega) \quad Q(\Omega) = Q_0(\Omega)\sqcup Q_1(\Omega)\sqcup Q_2(\Omega).$$

The $B_i(\Omega)$ fiber over the $Q_i(\Omega)$ whose fibers are tori T^{k+i}. The strata are labeled by the dimension of the cells in the resulting CW decomposition of $Q(\Omega)$. Using known results about cohomogeneity 2 actions [Bre] one can easily prove:

LEMMA 7.5.8:

(i) The orbit space $Q(\Omega)$ is homeomorphic to the closed disc $\bar{D}^2$, and the subset of singular orbits $Q_1(\Omega)\sqcup Q_0(\Omega)$ is homeomorphic to the boundary $\partial\bar{D}^2\simeq S^1$.

(ii) *$Q_2(\Omega)$ is homeomorphic to the open disc D^2.*

(iii) *$Q_1(\Omega)$ is homeomorphic to the disjoint union of $k+2$ copies of the open unit interval.*

(iv) *$Q_0(\Omega)$ is a set of $k+2$ points.*

Next one can easily show that $\pi_1(B(\Omega))$ is Abelian; hence, $\pi_1(B(\Omega)) = H_1(B(\Omega))$. Now we claim that $H_1(B(\Omega)) = H_2(B(\Omega)) = 0$, and since the fibers of the top horizontal arrow of 7.5.4 are S^3's this together with Hurewicz will prove Lemma 7.5.5. To prove this claim we define $Y_0 = Q_0(\Omega)$, $Y_1 = Q_0(\Omega) \cup Q_1(\Omega)$, and $Y_2 = Q(\Omega)$. Then, we filter $B(\Omega)$ by $X_i = \pi^{-1}(Y_i)$ to obtain the increasing filtration

$$X_0 = B_0(\Omega), \quad X_1 = B_0(\Omega) \cup B_1(\Omega), \quad \text{and} \quad X_2 = B(\Omega).$$

The Leray spectral sequence associated to this filtration has E^1 term given by

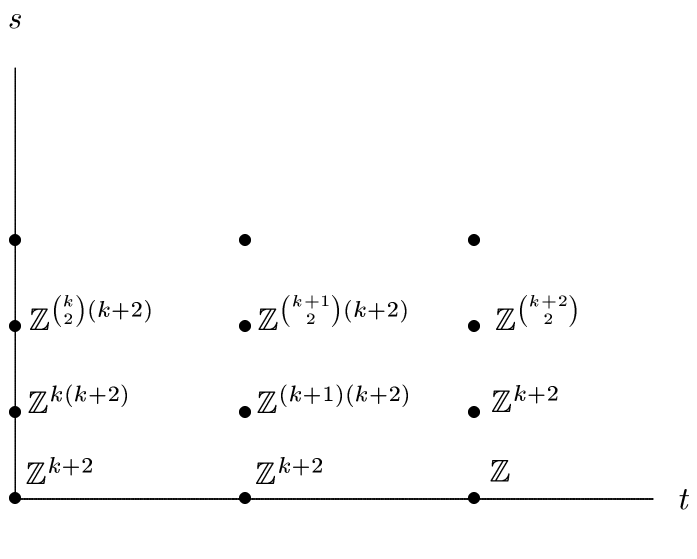
$$E^1_{s,t} \;\cong\; H_{s+t}(X_t, X_{t-1}; \mathbb{Z})$$

with differential $d_1 : H_{s+t}(X_t, X_{t-1}; \mathbb{Z}) \longrightarrow H_{s+t-1}(X_{t-1}, X_{t-2}; \mathbb{Z})$, where we use the convention that $X_{-1} = \emptyset$.

To compute these E^1 terms notice that all the pairs (X_t, X_{t-1}) are relative manifolds so that one can apply the Alexander-Poincaré duality theorem. Hence, by 7.5.7

$$\begin{aligned} H_s(X_0; \mathbb{Z}) \;&\cong\; H_s(\sqcup_{k+2} T^k; \mathbb{Z}); \\ H_s(X_1, X_0; \mathbb{Z}) \;&\cong\; H^{k+2-s}(\sqcup_{k+2} T^{k+1}; \mathbb{Z}); \\ H_s(X_2, X_1; \mathbb{Z}) \;&\cong\; H^{k+4-s}(T^{k+2}; \mathbb{Z}), \end{aligned}$$

where $\sqcup_j T^l$ means the disjoint union of j copies of T^l. Hence, the $E^1_{s,t}$ term of the spectral sequence is described by the diagram



$E^1_{s,t}$

The computation of the differentials is fairly tedious and we refer the reader to [B-G-M 8] for details. Suffice it to say here that after making certain choices Lemma 7.5.8 can be used to represent $Q(\Omega)$ topologically as a polygon

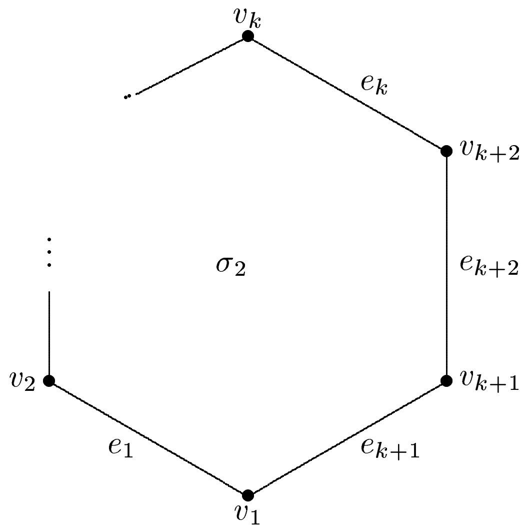


Diagram 7.5.9

The d_1 differential can then be computed and the result is that the $E^2_{s,t}$ term has zeros for $(s,t) = (1,0),(2,0),(1,1),(1,2),(0,1),(0,2)$. Then $E^2_{s,t} = E^\infty_{s,t}$ which converges to $H_{s+t}(B(\Omega),\mathbb{Z})$, so this proves Lemma 7.5.5 and hence, Theorem 7.5.3. ∎

By further analysis of the differentials it should be possible to determine the torsion in $H_3(\mathcal{S}(\Omega),\mathbb{Z})$. This should be given in terms of symmetric functions of the invariants $|\Delta_{\alpha_1,\cdots,\alpha_k}|$.

Remark 7.5.10: It was pointed out to the authors by Karsten Grove that if one takes the metric geometry into account, the internal angles in Diagram 7.5.9 are all less than 90 degrees. This indicates the presence of hyperbolic geometry.

Now, using Theorem 7.5.3 with Propositions 4.3.2, 4.3.3 gives:

Proposition 7.5.11: *Let $\Omega \in \mathcal{A}_{k,k+2}(\mathbb{Z})$ so that $\mathcal{S}(\Omega)$ is a smooth manifold. Let $\mathcal{Z}(\Omega)$ and $\mathcal{O}(\Omega)$ be the associated twistor space and quaternionic Kähler orbifold, respectively. Then we have*

$$b_2(\mathcal{S}(\Omega)) = b_2(\mathcal{O}(\Omega)) = b_2(\mathcal{Z}(\Omega)) - 1 = k.$$

This shows that inequality $b_2 \leq 1$ in Proposition 4.4.2 does not hold for non-regular 3-Sasakian manifolds. Finally we give several interesting corollaries of our work.

Corollary 7.5.12: *There exists a simply-connected 3-Sasakian 7-manifold for every rational homology type allowed by Corollary 4.2.8.*

Our next corollary follows from the results of this section and remarkable theorem of Gromov [Gro]:

Corollary 7.5.13: *For any non-positive real number κ there are infinitely many 3-Sasakian 7-manifolds which do not admit metrics whose sectional curvatures are all greater than or equal to κ.*

For such an infinite family of 3-Sasakian 7-manifolds, the appearance of negative curvature is foretold by Remark 7.5.10. Corollary 7.5.13 can be contrasted with Corollary 7.4.7.

COROLLARY 7.5.14— *There exist 7-manifolds with arbitrary second Betti number having metrics of weak holonomy G_2.*

Of course, Corollary 7.5.13 also applies to these weak holonomy G_2 metrics.

COROLLARY 7.5.15: *There exist $\mathbb{Q}$-factorial contact Fano 3-folds X with $b_2(X) = l$ for any positive integer l.*

This corollary should be contrasted to the smooth case, where Mori and Mukai have proven that $b_2 \leq 10$ [Mo-Mu]:

COROLLARY 7.5.16: *If the second Betti number $b_2(\mathcal{S}(\Omega)) = k > 3$, the 3-Sasakian manifolds $\mathcal{S}(\Omega)$ are not homotopy equivalent to any homogeneous space.*

This corollary can be compared to Corollary 7.4.7. Finally we have

COROLLARY 7.5.17: *There exist compact, T^2-symmetric, self dual Einstein orbifolds of positive scalar curvature with arbitrary second Betti number.*

Again this should be contrasted to the smooth case where we must have $b_2 \leq 1$.

7.6 Higher Dimensional Toric 3-Sasakian Manifolds

We begin with the definition of a toric 3-Sasakian manifolds which is motivated by the hyperkähler case [Bi-D].

DEFINITION 7.6.1: *A 3-Sasakian manifold (orbifold) of dimension $4m-1$ is said to be a toric 3-Sasakian manifold (orbifold) if it admits an effective action of a m-torus T^m that preserves the 3-Sasakian structure.*

The importance of toric 3-Sasakian manifolds is underlined by the following recent Delzant-type theorem of Bielawski:

THEOREM 7.6.2 [Bi 3]: *Let $\mathcal{S}$ be a toric 3-Sasakian manifold of dimension $4n-1$. Then $\mathcal{S}$ is isomorphic as a 3-Sasakian T^n-manifold to a 3-Sasakian quotient of a sphere by a torus, that is to a $\mathcal{S}(\Omega)$ for some Ω.*

This theorem includes the degenerate case when the quotient is a sphere or a discrete quotient of such. The Betti numbers of a 3-Sasakian orbifold obtained by a toral quotient of a sphere were computed by Bielawski [Bi 2] using different techniques than the ones employed in Section 7.5:

THEOREM 7.6.3 [Bi 2]: *Let $\Omega \in \mathcal{M}_{k,n+1}(\mathbb{Z})$ be non-degenerate so that $\mathcal{S}(\Omega)$ is a compact 3-Sasakian orbifold of dimension $4(n-k)+3$. Then we have*

7.6.7
$$b_{2i} = \binom{k+i-1}{k}, \qquad i < n+1-k.$$

Furthermore, the Betti number constraints of Proposition 4.4.2(ii) can hold for $\mathcal{S}(\Omega)$ if and only if $k=1$.

Combining Theorems 7.6.2 and 7.6.3 with Lemmas 7.3.3 and 7.3.6 which give obstructions to smoothness gives the somewhat surprising result [B-G-M 7],

THEOREM 7.6.4: *Let $\mathcal{S}$ be a toric 3-Sasakian manifold.*

(i) If the dimension of $\mathcal{S}$ is 19 or greater, then $b_2(\mathcal{S}) \leq 1$.

(ii) If the dimension of $\mathcal{S}$ is 11 or 15, then $b_2(\mathcal{S}) \leq 4$.

(iii) If $b_2(\mathcal{S}) > 4$, then the dimension of $\mathcal{S}$ is 7.

A corollary due to Bielawski [Bi 3] is:

COROLLARY 7.6.5: *Let $\mathcal{S}$ be a regular toric 3-Sasakian manifold. Then $\mathcal{S}$ is one of the 3-Sasakian homogeneous spaces* $S^{4n-1}, \mathbb{RP}^{4n-1}$ *or* $\frac{SU(n)}{S\big(U(n-2)\times U(1)\big)}$.

Next we give an explicit construction of toric 3-Sasakian manifolds not eliminated by Theorem 7.6.4. It is enough to show that $\mathcal{A}_{4,8}$ and $\mathcal{A}_{4,7}$ are not empty as the rest follow by deletion of rows of the corresponding $\Omega \in \mathcal{A}_{4,*}$. We shall present two three parameter families of solutions, namely

7.6.6 $$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1+2^l \\ 1 & 16 & 1+2^m \\ -1 & 3 & 2c \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1+2^{l'} & -1 \\ 1 & 16 & 1+2^{2n} & 3 \\ -1 & 3 & 2c' & -1 \end{pmatrix},$$

where $l, l', m, n \in \mathbb{Z}^+$, and $c, c' \in \mathbb{Z}$.

With the aid of MAPLE symbolic manipulation program, we find

LEMMA 7.6.7 [B-G-M 7]: *Let* $\Delta = 2(31c + 6 + 19\cdot 2^{l-1} - 7\cdot 2^{m-1})$.

(i) $\Omega_1 = (\,\mathbb{I}_4 \quad A_1\,)$ *is admissible if and only if* $c \neq 0$ *and is not divisible by* 3, *and* $\Delta \neq 0$ *and is not divisible by* $7, 19$ *nor* 31.

(ii) $\Omega_2 = (\,\mathbb{I}_4 \quad A_2\,)$ *is admissible if and only if* c *and all minor determinants of* A_2 *are non-vanishing, and* $c' \not\equiv 0 \pmod 3$, $l' \not\equiv 0 \pmod 4$, $c' \not\equiv 5 \pmod 7$, *and* $11, 19, 37, 71$ *do not divide* $\det A = 19\cdot 2^{2n} - 63 - 148c' - 11\cdot 2^{l'}$, *and the following conditions hold:*

$$\begin{aligned} \gcd(3, 4c' + 2^{l'} + 1, 2c' - 2^{l'} - 1) &= 1 \\ \gcd(7, 2^{2n+1} - 2^{l'} + 1, 3\cdot 2^{l'} + 2^{2n} + 4) &= 1 \\ \gcd(19, 2^{2n} - 2^{l'+4} - 15, 3\cdot 2^{l'} + 2^{2n} + 4) &= 1 \\ \gcd(25, 32c' - 3\cdot 2^{2n} - 3, 6c' + 2^{2n} + 1) &= 1. \end{aligned}$$

The conditions in this proposition guarantee that the quotient spaces denoted by $\mathcal{S}(c, l, m)$ and $\mathcal{S}(c', l', n)$ are smooth manifolds of dimension 11 and 15, respectively.

It is routine to verify that the three parameter infinite family given by

7.6.8 $$c \equiv 14 \pmod{21}, \qquad l \not\equiv 1 \pmod 5, \qquad m \not\equiv \alpha(c) \pmod{18},$$

where $2^{\alpha(c)} = 22(31c + 6) \pmod{18}$ satisfies the conditions in (i) of Lemma 7.6.3. This gives examples in dimension 11. (Notice that as 2 is a primitive root of 19 the equation defining $\alpha(c)$ has a unique solution $\pmod{18}$ for each value of c.) Similarly, it is straightforward to verify that the infinite family given by

7.6.9 $$c' = 2, \qquad l' = 1, \qquad n \equiv 21 \pmod{90},$$

satisfies the conditions (ii) of Lemma 7.6.7. We have arrived at:

THEOREM 7.6.10 [B-G-M 7]: *There exist toric 3-Sasakian manifolds* $\mathcal{S}$ *of dimensions* 11 *and* 15 *with* $b_2(\mathcal{S}) = 2, 3, 4$. *Consequently, the Betti number relations of Proposition 4.4.2 do not hold generally. More explicitly there are compact 11-dimensional 3-Sasakian manifolds for which* $b_2 \neq b_4$, *and compact 15-dimensional 3-Sasakian manifolds for which* $b_2 \neq b_6$.

8. Open Problems and Questions

We conclude this section with a short list of interesting problems. Some minor questions do appear in the text but these are usually of more technical ones. There, quite often, we simply could not provide a complete answer only because of the time constraint imposed by the fact that this chapter is a part of a collection of articles. Here we try to concentrate on, what we believe, are more fundamental questions.

PROBLEM 8.1: *Classify all compact simply-connected Sasakian-Einstein manifolds in dimension 5.*

All the known examples are regular and regular spaces were classified in [B-G-F-K]. Can one find irregular examples? Consider the connected sum $\mathcal{S}_k = S^5\#^k(S^2\times S^3)$. Now, $\mathcal{S}_k$ admits a Sasakian-Einstein metric for $k = 0,1,3,4,5,6,7,8$. How about $k = 2$ and $k > 8$? In this case $\mathcal{S}_k$ would necessarily have to be a Seifert fibered space with the space of leaves a positive scalar curvature Kähler-Einstein orbifold (and not a smooth manifold) X_k with $b_2(X_k) = k$. If one could construct such structures for each remaining k, by the result of Smale, every compact simply-connected 5-manifold with no 2-torsion would admit an Einstein metric of positive scalar curvature. The same problem in any dimension $2m+1$, $m > 2$ appears to be much more involved as it would necessarily have to include the classification of 3-Sasakian 7-manifolds [B-G 2].

PROBLEM 8.2: *Classify all compact simply-connected 3-Sasakian manifolds in dimension 7.*

Again, regular examples were classified in [B-G-M 1, Fr-Kat 2]. This appears to be a difficult problem. Its solution would amount to a classification of good self-dual and Einstein orbifold of positive scalar curvature which, in smooth case, was done in [Hi 1, Fr-Kur]. Certainly, a more modest, partial classification could be in reach. In particular, in terms of Definition 2.5.4 one easily sees that all toric 3-Sasakian manifolds are regular or of cyclic type. Is the converse true? That is:

QUESTION 8.3: *Is every 3-Sasakian 7-manifold of cyclic type toric (which includes discrete quotients of a spheres as a degenerate case)?*

In terms of the classification by symmetries one can ask:

QUESTION 8.4: *Is every compact 3-Sasakian 7-manifold of cohomogeneity ≤ 2 toric?*

QUESTION 8.5: *Let $(\mathcal{S},g)$ be a simply-connected 3-Sasakian 7-manifold. Can g be of maximal cohomogeneity 4?*

We are not aware of any such examples. All toric examples are of cohomogeneity 0,1,2 and some new construction of [B-G 2] gives 3-Sasakian 7-manifolds of cohomogeneity 3.

Concerning topology and Problem 8.2, we can ask the following questions:

QUESTION 8.6: *Let $(\mathcal{S},g)$ be a compact simply-connected 3-Sasakian 7-manifold. Can $\mathcal{S}$ be topologically a product?*

If so then $\mathcal{S}$ must be $S^2\times S^5$. In the Sasakian-Einstein case it is known that such a splitting can occur. The simplest example is $S^2\times S^3$ which has a Sasakian-Einstein structure [Tan 4]. Of course, the above problem and questions have versions in higher dimension. More generally,

QUESTIONS 8.7: *Other than the vanishing of the odd Betti numbers up to the middle dimension and the finiteness of the fundamental group, what more can be said about the topology of a compact 3-Sasakian manifold? For instance, is $H_2(\mathcal{S},\mathbb{Z})$ always torsion free? Are there further restrictions on the fundamental group?*

Specifically in higher dimensions we ask:

QUESTION 8.8: *Are there 3-Sasakian manifolds $\mathcal{S}$ of dimension* 19 *or greater with* $b_2(\mathcal{S}) > 1$?

From a differentiable topological viewpoint we can ask:

QUESTIONS 8.9: *Let $(\mathcal{S}, g)$, $(\mathcal{S}', g')$ be two compact simply-connected 3-Sasakian 7-manifolds which are not 3-Sasakian equivalent. Can $\mathcal{S}$ be diffeomorphic (homeomorphic) to $\mathcal{S}'$? In particular, is there a non-standard 3-Sasakian structure on S^7? Can one have 3-Sasakian structures on exotic 7-spheres?*

As pointed out in the Remark 7.4.11, we expect the positive answer to the first question. But the problem of existence of other 3-Sasakian structures on S^7 or exotic spheres lacks even the slightest hint, one way or the other. In general, due to the local rigidity, the moduli space of inequivalent 3-Sasakian structures must be discrete. So we have

QUESTION 8.10: *Is the moduli space always finite, or can a 3-Sasakian manifold admit infinitely many inequivalent 3-Sasakian structures?*

Concerning related geometries, we have

PROBLEM 8.11: *Classify all compact simply-connected proper G_2-manifolds.*

This appears to be more involved than Problem 8.2 because of Theorem 5.2.9. On the other hand, perhaps the G_2-structure can be investigated without reference to the 3-Sasakian geometry. It could happen that Problem 8.10 might admit a simpler solution and become the right approach to Problem 8.2. Maybe even one could try to classify all weak holonomy G_2-manifolds. At the moment we do not even know if the converse of the Theorem 5.2.9 is true, that is if a proper G_2-manifold always admits a metric which is 3-Sasakian. This is unlikely though and one could start by looking for possible proper G_2-manifolds with $b_3 \neq 0$.

Last but not least, we turn to the regular case. All regular 3-Sasakian manifolds in dimension 7 and 11 are known as explained in Section 4.5. Any classification in higher dimensions would translate into the classification of positive quaternionic Kähler manifolds. Below we give the 3-Sasakian version of the conjecture that all compact positive quaternionic Kähler spaces are symmetric:

CONJECTURE 8.12: *Let $(\mathcal{S}, g)$ be a compact regular 3-Sasakian manifold of dimension $4n+3$. Then $\mathcal{S}$ is homogeneous.*

This is simply theorem 4.4.5 without $n < 3$ in the hypothesis. One might hope that 3-Sasakian geometry would provide some new input in the regular case. So far we have mostly used results about positive quaternionic Kähler manifolds to describe properties of regular 3-Sasakian manifolds. But Section 4 does give some indication that 3-Sasakian geometry can be used, at the very least, to give new proves of known theorems.

REMARK 8.13: *Sasakian-Einstein, 3-Sasakian, and proper G_2-manifolds in the AdS/CFT Correspondence.*

Very recently Sasakian-Einstein geometry has emerged quite naturally in the conformal field theory and string theory. In particular, Klebanov and Witten [K-W] cnosidered $\mathcal{S} = S^3 \times S^2$ in the context of superconformal field theory dual to the string theory on $AdS_5 \times \mathcal{S}$. Their article originates in an influential result of Maldacena [Mal] who noticed that large N limit of certain conformal field theories in d dimensions can be described in terms of supergravity (and string theory) on a product of $(d+1)$-dimensional anti-de-Sitter AdS_{d+1} space with a compact manifold M. The idea was later examined by Witten who proposed a precise correspondence between conformal field theory observables and those of supergravity [Wi]. It turns out, and this observation has recently been made by Figueroa [Fi], that M necessarily has real Killing spinors and the number of them determines the number of supersymmetries preserved. Depending on the dimension and the amount of supersymmetry, the following geometries are possible: spherical in any dimension, Sasakian-Einstein in dimension $2k+1$, 3-Sasakian in dimension $4k+3$, 7-manifolds with weak G_2-holonomy, and 6-dimensional nearly Kähler manifolds [A-F-H-B]. The case when $\dim(M) = 5, 7$ seems to be of particular interest. For other developements concerning Sasakian and 3-Sasakian manifolds in supersymmetric field theories see [M-P, O-T, G, G-R, C^2-D-F^2-T].

Appendix: Fundamentals of Orbifolds

The notion of orbifold was introduced under the name V-manifold by Satake [Sat 1] in 1956, and subsequently he developed Riemannian geometry on V-manifolds [Sat 2] ending with a proof of the Gauss-Bonnet theorem for V-manifolds. Contemporaneously, Baily introduced complex V-manifolds and generalized both the Hodge decomposition theorem [Bai 1], and Kodaira's projective embedding theorem [Bai 2] to V-manifolds. Somewhat later in the late 1970's and early 1980's Kawasaki generalized various index theorems [Kaw 1-3] to the category of V-manifolds. It was about this time that Thurston [Thu] rediscovered the concept of V-manifold, under the name of orbifold, in his study of the geometry of 3-manifolds, and defined the orbifold fundamental group π_1^{orb}. By now orbifold has become the accepted term for these objects and we shall follow suit. However, we do use the name V-bundle for fibre bundles in this category.

Orbifolds arise naturally as spaces of leaves of Riemannian foliations with compact leaves, and we are particularly interested in this point of view. Conversely, every orbifold can be realized in this way. In fact, given an orbifold $\mathcal{O}$, we can construct on it the V-bundle of orthonormal frames whose total space P is a smooth manifold with a locally free action of the orthogonal group $O(n)$ such that $\mathcal{O} = P/O(n)$. Thus, every orbifold can be realized as the quotient space by a locally free action of a Lie group. We are not certain of the history of this connection, but it was surely well understood by Haefliger [Hae] in 1982 who developed the basic techniques for studying the topology of orbifolds.

Definition A.1: *A smooth orbifold (or V-manifold) is a second countable Hausdorff space X together with a family $\{U_i\}_{\subset \in I}$ of open sets that satisfy:*

i) *$\{U_i\}_{\subset \in I}$ is an open cover of X that is closed under finite intersections.*

ii) *For each $i \in I$ a local uniformizing system consisting of a triple $\{\tilde{U}_i, \Gamma_i, \varphi_i\}$, where $\tilde{U}_i$ is connected open subset of $\mathbb{R}^n$ containing the origin, Γ_i is a finite group of diffeomorphisms acting effectively and properly on $\tilde{U}_i$, and $\varphi_i : \tilde{U}_i \longrightarrow U_i$ is a continuous map onto U_i such that $\varphi_i \circ \gamma = \varphi_i$ for all $\gamma \in \Gamma_i$ and the induced natural map of $\tilde{U}_i/\Gamma_i$ onto U_i is a homeomorphism. The finite group Γ_i is called a local uniformizing group.*

iii) *Given $\tilde{x}_i \in \tilde{U}_i$ and $\tilde{x}_j \in \tilde{U}_j$ such that $\varphi_i(\tilde{x}_i) = \varphi_j(\tilde{x}_j)$, there is a diffeomorphism $g_{ji} : \tilde{V}_i \longrightarrow \tilde{V}_j$ from a neighborhood $\tilde{V}_i \subset \tilde{U}_i$ of $\tilde{x}_i$ onto a neighborhood $\tilde{V}_j \subset \tilde{U}_j$ of $\tilde{x}_j$ such that $\varphi_i = \varphi_j \circ g_{ji}$.*

Remarks A.2: 1) We can always take the finite subgroups Γ_i to be subgroups of the orthogonal group $O(n)$ and in the orientable case $SO(n)$. 2) Condition iii) implies that for each $\gamma_i \in \Gamma_i$ there exists a unique $\gamma_j \in \Gamma_j$ such that $g_{ji} \circ \gamma_i = \gamma_j \circ g_{ji}$. 3) One can define the notion of equivalence of families of open sets, any such family of open sets is contained in a unique maximal family satisfying the required properties. 4) The standard notions of smooth maps between orbifolds, and isomorphism classes of orbifolds, etc. can then be given in an analogous manner to manifolds (see [Sat 1-2, Bai 1-2]). We leave this to the reader to fill in. Notice that in particular a diffeomorphism between orbifolds gives a homeomorphism of the underlying topological spaces. Similarly, a *complex orbifold* can be defined by making the obvious changes.

An alternative definition of orbifold given by Haefliger [Hae] can be obtained as follows: Let $G_{\Gamma,g}$ denote the groupoid of germs of diffeomorphisms generated by the germs of elements in Γ_i and the germs of the diffeomorphisms g_{ji} described above. Let $\tilde{U} = \sqcup_i \tilde{U}_i$ denote the disjoint union of the $\tilde{U}_i$. Then $x, y \in \tilde{U}$ are equivalent if there is a germ $\gamma \in G_{\Gamma,g}$ such that $y = \gamma(x)$. The quotient space $X = \tilde{U}/G_{\Gamma,g}$ defines an orbifold (actually an isomorphism class of orbifolds). In the case that an orbifold X is given as the space of leaves of a foliation $\mathcal{F}$ on a smooth manifold, the groupoid $G_{\Gamma,g}$ is the transverse holonomy groupoid of $\mathcal{F}$.

The following result relating to foliations, which is given in Molino, is fundamental to our work:

THEOREM A.3: [Mol: Proposition 3.7] *Let $(M, \mathcal{F}, g)$ be a Riemannian foliation of codimension q with compact leaves and bundle-like metric g. Then the space of leaves $M/\mathcal{F}$ admits the structure of a q-dimensional orbifold such that the natural projection $\pi : M \longrightarrow M/\mathcal{F}$ is an orbifold submersion.*

Let X be an orbifold and choose a local uniformizing system $\{U, \Gamma, \varphi\}$. Let $x \in X$ be any point, and let $p \in \varphi^{-1}(x)$, then up to conjugacy the isotropy subgroup $\Gamma_p \subset \Gamma$ depends only on x, and accordingly we shall denote this isotropy subgroup by Γ_x. A point of X whose isotropy subgroups $\Gamma_x \neq \mathrm{id}$ is called a *singular* point. Those points with $\Gamma_x = \mathrm{id}$ are called *regular* points. The subset of regular points is an open dense subset of X. The isotropy groups give a natural stratification of X by saying that two points lie in the same stratum if their isotropy subgroups are conjugate. Thus, the dense open subset of regular points forms the principal stratum. In the case that X is the space of leaves of a foliation, the isotropy subgroup Γ_x is precisely the leaf holonomy group of the leaf x. An orbifold X is a smooth manifold or in the complex analytic category a complex manifold if and only if $\Gamma_x = \mathrm{id}$ for all $x \in X$. In this case we can take $\Gamma = \mathrm{id}$ and $\varphi = \mathrm{id}$, and the definition of an orbifold reduces to the usual definition of a smooth manifold.

Many of the usual differential geometric concepts that hold for smooth or complex analytic manifolds also hold in the orbifold category, in particular the important notion of a fiber bundle.

DEFINITION A.4: *A V-bundle over an orbifold X consists of a bundle $B_{\tilde{U}}$ over $\tilde{U}$ for each local uniformizing system $\{\tilde{U}_i, \Gamma_i, \varphi_i\}$ with Lie group G and fiber F (independent of $\tilde{U}_i$) together with a homomorphism $h_{\tilde{U}_i} : \Gamma_i \longrightarrow G$ satisfying:*

i) *If b lies in the fiber over $\tilde{x}_i \in \tilde{U}_i$ then for each $\gamma \in \Gamma_i$, $bh_{\tilde{U}_i}(\gamma)$ lies in the fiber over $\gamma^{-1}\tilde{x}_i$.*

ii) *If $g_{ji} : \tilde{U}_i \longrightarrow \tilde{U}_j$ is a diffeomorphism onto an open set, then there is a bundle map $g^*_{ij} : B_{U_j}|g_{ji}(\tilde{U}_i) \longrightarrow B_{\tilde{U}_i}$ satisfying the condition that if $\gamma \in \Gamma_i$, and $\gamma' \in \Gamma_j$ is the unique element such that $g_{ji} \circ \gamma = \gamma' \circ g_{ji}$, then $h_{\tilde{U}_i}(\gamma) \circ g^*_{ji} = g^*_{ji} \circ h_{\tilde{U}_j}(\gamma')$, and if $g_{kj} : \tilde{U}_j \longrightarrow \tilde{U}_k$ is another such diffeomorphism then $(g_{kj} \circ g_{ji})^* = g^*_{ji} \circ g^*_{kj}$.*

If the fiber F is a vector space and G acts on F as linear transformations of F, then the V-bundle is called a vector V-bundle. Similarly, if F is the Lie group G with its right action, then the V-bundle is called a principal V-bundle.

The *total space* of a V-bundle over X is an orbifold E with local uniformizing systems $\{B_{\tilde{U}_i}, \Gamma^*_i, \varphi^*_i\}$. By choosing the local uniformizing neighborhoods of X small enough, we can always take $B_{\tilde{U}_i}$ to be the product $\tilde{U}_i \times F$ which we shall heretofore assume. There is an action of the local uniformizing group Γ_i on $\tilde{U}_i \times F$ given by sending $(\tilde{x}_i, b) \in \tilde{U}_i \times F$ to $(\gamma^{-1}\tilde{x}_i, bh_{\tilde{U}_i}(\gamma))$, so the local uniformizing groups Γ_i^* can be taken to be subgroups of Γ_i. We are particularly interested in the case of a principal bundle. In the case the fibre is the Lie group G, so the image $h_{\tilde{U}_i}(\Gamma_i^*)$ acts freely on F. Thus the total space P of a principal V-bundle will be smooth if and only if $h_{\tilde{U}_i}$ is injective for all i.

Remarks A.5: We shall often denote a V-bundle by the standard notation $\pi : P \longrightarrow X$ and think of this as an "orbifold fibration". It must be understood, however, that an orbifold fibration is not a fibration in the usual sense. Shortly, we shall show that it is a fibration rationally. Again the standard notions of smooth maps between V-bundles, and isomorphism classes of V-bundles can be given in the usual manner. We let this description to the reader. An *absolute* V-bundle resembles a bundle in the ordinary sense, and corresponds to being able to take $h_{\tilde{U}} = \mathrm{id}$, for all local uniformizing neighborhoods $\tilde{U}$. In particular, the trivial V-bundle $X \times F$ is absolute. Another important notion introduced by Kawasaki [Kaw 2] is that of proper. A V-bundle E is said to be *proper* if the local uniformizing groups Γ_i^* of E act effectively on X when viewed as subgroups of the local uniformizing groups Γ_i on X. Any V-bundle with smooth total space is clearly proper. The Kawasaki index theorems such as his Riemann-Roch Theorem used in section 2.2 require the V-bundles to be proper.

Since an orbifold fibration is not a fibration in the usual sense, the usual techniques in topology for fibrations do not apply directly. However, Haefliger [Hae] has defined orbifold homology, cohomology, and homotopy groups which do have an analogue in the standard theory. Let X be an orbifold of dimension n and let P denote the bundle of orthonormal frames on X. It is a smooth manifold on which the orthogonal group $O(n)$ acts locally freely with the quotient X. Let $EO(n) \longrightarrow BO(n)$ denote the universal $O(n)$ bundle. Consider the diagonal action of $O(n)$ on $EO(n) \times P$ and denote the quotient by BX. Now there is a natural projection $p : BX \longrightarrow X$ with generic fiber the contractible space $EO(n)$, and Haefliger defines the orbifold cohomology, homology, and homotopy groups by

$$A.6 \qquad H^i_{orb}(X, \mathbb{Z}) = H^i(BX, \mathbb{Z}), \quad H_i^{orb}(X, \mathbb{Z}) = H_i(BX, \mathbb{Z}), \quad \pi_i^{orb}(X) = \pi_i(BX).$$

This definition of π_1^{orb} is equivalent to Thurston's better known definition [Thu] in terms of orbifold deck transformations, and when X is a smooth manifold these orbifold groups coincide with the usual groups. Moreover, we have

Proposition A.7 [Hae]: *The map $p : BX \longrightarrow X$ induces an isomorphism $H^i_{orb}(\mathcal{S}, \mathbb{Z}) \otimes \mathbb{Q} \simeq H^i(\mathcal{S}, \mathbb{Z}) \otimes \mathbb{Q}$.*

Now with this in hand for the orbifold category, the circle V-bundles over $\mathcal{S}$ are classified [H-S] by $H^2_{orb}(\mathcal{S}, \mathbb{Z})$. Of course, rationally there is no difference by Proposition A.7. The rational Gysin sequence for orbifold sphere bundles whose generic fibres are spheres also holds. Haefliger's theory also applies to the following situation. Let G be a compact Lie group acting locally freely on an orbifold Y with quotient X. This gives rise to a fibration $EO(n) \times G \longrightarrow BY \longrightarrow BX$, which induces the long exact homotopy sequence

$$A.8 \qquad \cdots \longrightarrow \pi_i(G) \longrightarrow \pi_i^{orb}(Y) \longrightarrow \pi_i^{orb}(X) \longrightarrow \pi_{i-1}(G) \longrightarrow \cdots.$$

This was used by Haefliger and Salem [H-S] in their study of torus actions on orbifolds.

We are particularly interested in the case of circle V-bundles. Using the exponential exact sequence one sees as the usual case that $H^2_{orb}(X, \mathbb{Z})$ classifies equivalence classes of circle V-bundles over an orbifold X. Furthermore, in [H-S] it is shown that $H^2(X,\mathbb{Z})$ classifies circle V-bundles up to local equivalence. This gives a monomorphism $H^2(X,\mathbb{Z})\longrightarrow H^2_{orb}(X,\mathbb{Z})$ which is an isomorphism rationally.

In [B-G 1] we introduced the set $\mathrm{Pic}^{orb}(X)$ of equivalence classes holomorphic line V-bundles over a complex orbifold X and one easily sees [B-G 1]:

LEMMA A.9: *$Pic^{orb}(X)$ forms an Abelian group. Furthermore, there is a monomorphism $Pic(X)\longrightarrow Pic^{orb}(X)$ which is an isomorphism rationally.*

The notion of sections of bundles works just as well in the orbifold category.

DEFINITION A.10: *Let E be a V-bundle over an orbifold X. Then a section σ of E over the open set $V\subset X$ is a section σ_U of the bundle B_U for each local uniformizing system $\{U,\Gamma,\varphi\}\in\mathcal{F}_V$ such that for any $x\in U$ we have*

(i) For each $\gamma\in\Gamma$ $\sigma_U(\gamma^{-1}x)=h_U(\gamma)\sigma_U(x)$.

(ii) If $\lambda:\{U,\Gamma,\varphi\}\longrightarrow\{U',\Gamma',\varphi'\}$ is an injection, then $\lambda^*\sigma_{U'}(\lambda(x))=\sigma_U(x)$.

If each of the local sections σ_U is continuous, smooth, holomorphic, etc., we say that σ is continuous, smooth, holomorphic, etc., respectively. Given local sections σ_U of a vector V-bundle we can always construct Γ-invariant local sections by "averaging over the group", i.e., we define $\sigma^I_U=\frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\sigma_U\circ\gamma$. A similar procedure holds for product structures. For example, if L is a holomorphic line V-bundle on X, and if σ is a holomorphic section, we can construct local invariant sections σ^I_U of $L^{|\Gamma|}$ by taking products, viz., $\sigma^I_U=\frac{1}{|\Gamma|}\prod_{\gamma\in\Gamma}\sigma_U\circ\gamma$.

The standard notions of tangent bundle, cotangent bundle, and all the associated tensor bundles all have V-bundle analogues [Bai 1, Sat 1-2]. In particular, if V is an open subset of $\varphi(U)$ then the integral of an n-form (measurable) σ is defined by $\int_V\sigma=\frac{1}{|\Gamma|}\int_{\varphi^{-1}(V)}\sigma_U$. All of the standard integration techniques, such as Stokes' theorem, hold on V-manifolds.

Riemannian metrics also exist by the standard partition of unity argument, and we shall always work with Γ-invariant metrics. Moreover, all the standard differential geometric objects involving curvature and metric concepts, such as the Ricci tensor, Hodge star operator, etc., hold equally well. On a complex orbifold there is a Γ-invariant tensor field J of type $(1,1)$ which describes the complex structure on the tangent V-bundle TX. The almost complex structure J gives rise in the usual way to the V-bundles $A^{r,s}$ of differential forms of type (r,s). The standard concepts of Hermitian and Kähler metrics hold equally well on V-manifolds, and all the special identities involving Kähler, Einstein, or Kähler-Einstein geometry hold. In particular, the standard Weizenböck formulas hold.

Finally, there is associated to every compact orbifold X an integer m_0 called the *order* of X and defined to be the least common multiple of the orders of the local uniformizing groups.

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September 1998

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