3-Sasakian Geometry, Nilpotent Orbits, and Exceptional Quotients

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Abstract. Using 3-Sasakian reduction technique we obtain infinite families of new 3-Sasakian manifolds $\mathcal{M}(p_1,p_2,p_3)$ and $\mathcal{M}(p_1,p_2,p_3,p_4)$ in dimension 11 and 15 respectively. The metric cone on $\mathcal{M}(p_1,p_2,p_3)$ is a generalization of the Kronheimer hyperkähler metric on the regular maximal nilpotent orbit of $\mathfrak{sl}(3,\mathbb{C})$ whereas the cone on $\mathcal{M}(p_1,p_2,p_3,p_4)$ generalizes the hyperkähler metric on the 16-dimensional orbit of $\mathfrak{so}(6,\mathbb{C})$. These are first examples of 3-Sasakian metrics which are neither homogeneous nor toric. In addition we consider some further $U(1)$-reductions of $\mathcal{M}(p_1,p_2,p_3)$. These yield examples of non-toric 3-Sasakian orbifold metric in dimensions 7. As a result we obtain explicit families $\mathcal{O}(\Theta)$ of compact self-dual positive scalar curvature Einstein metrics with orbifold singularities and with only one Killing vector field.

Introduction

In 1990 Kronheimer showed that the co-adjoint orbits in the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of both semi-simple and nilpotent elements are hyperkähler [Kr1, Kr2]. In particular these orbits carry Ricci-flat metrics. Later, using Kronheimer's result, Swann showed that the hyperkähler structure on the nilpotent orbits is very special [Sw]. Such orbits admit an action of $\mathbb{H}^*$ with the orbit space being a compact quaternionic Kähler orbifold of positive scalar curvature. Another way of expressing this result is to say that the nilpotent orbits are metric cones $C(\mathcal{S})$ on compact 3-Sasakian orbifolds [BGM1]. The mimimal nilpotent orbit is easily seen to be a metric cone $C(G/K)$, where G/K is a simply-connected 3-Sasakian homogeneous space of [BGM1].

Quaternionic geometry of the regular maximal nilpotent orbit of $\mathfrak{sl}(3,\mathbb{C})$ was investigated by Kobak and Swann in great detail [KS1]. This orbit is 12-dimensional and, in the language of 3-Sasakian geometry, it is a cone $\mathcal{N}=C(\mathcal{S})$ on the 3-Sasakian orbifold $\mathcal{S}=\mathbb{Z}_3\backslash G_2/Sp(1)$. Here $\mathcal{S}$ is simply a $\mathbb{Z}_3$ quotient of the 3-Sasakian homogeneous space associated to the exceptional Lie group G_2, where $\mathbb{Z}_3$ is the center of $SU(3)\subset G_2$. This is a typical example of a bi-quotient which has orbifold singularities and the singular locus can be easily identified with a homogeneous 3-Sasakian 7-manifold $SU(3)/U(1)$.

Furthermore, Kobak and Swann [KS1] show that this particular nilpotent orbit can be obtained as a hyperkähler quotient of another nilpotent orbit by a circle action. In the language of 3-Sasakian geometry it is simply a hyperkähler reduction of the metric cone $C\big(\frac{SO(7)}{SO(3)\times Sp(1)}\big)$ associated to the action of the diagonal $U(1)\subset U(3)\subset SO(7)$. As

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$C\left(\frac{SO(7)}{SO(3)\times Sp(1)}\right)$ can be realized as an $Sp(1)$ reduction of the flat space $\mathbb{H}^7 = C(S^{27})$ [G] we have the following hyperkähler (3-Sasakian [BGM1], quaternionic Kähler [GL]) quotients:

$$
\begin{array}{ccccc}
C(S^{27}) & \overset{Sp(1)}{\Longrightarrow} & C\left(\frac{SO(7)}{SO(3)\times Sp(1)}\right) & \overset{U(1)}{\Longrightarrow} & C(\mathbb{Z}_3\backslash G_2/Sp(1)) \\
\downarrow & & \downarrow & & \downarrow \\
S^{27} & \overset{Sp(1)}{\Longrightarrow} & \frac{SO(7)}{SO(3)\times Sp(1)} & \overset{U(1)}{\Longrightarrow} & \mathbb{Z}_3\backslash G_2/Sp(1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{HP}^6 & \overset{Sp(1)}{\Longrightarrow} & \frac{SO(7)}{SO(4)\times SO(3)} & \overset{U(1)}{\Longrightarrow} & \mathbb{Z}_3\backslash G_2/SO(4).
\end{array}
\tag{0.1}
$$

In a later paper Kobak and Swann show that any classical nilpotent orbit can be obtained as a hyperkähler quotient of a flat space of an appropriate dimension so that the above diagram is only an example [KS2]. The middle horizontal line of Diagram 0.1 is absent from the discussion in [KS1]as the importance of 3-Sasakian geometry in this context was realized later [BGM1].

The starting point of this paper is to revisit Kobak-Swann construction in the context of the associated 3-Sasakian geometry. We are going to examine the quotient construction of the orbifold fibration $\mathbb{Z}_3\backslash G_2/Sp(1) \to \mathbb{Z}_3\backslash G_2/SO(4)$ showing that it admits interesting generalizations. More precisely, the Kobak-Swann quotient can be "deformed" by introducing weights in much the same way toric 3-Sasakian manifolds with $b_2 = 1$ can all be obtained by "deforming" the classical homogeneous example of the fibration $\mathcal{S}(\mathbf{1}) = SU(n)/S(U(n-2)\times U(1)) \to \mathrm{Gr}_2(\mathbb{C}^n)$ [BGM1]. On the other hand our new quotients are quite different from the construction of $\mathcal{S}(\mathbf{p})$ considered in [BGM1] as they cannot be interpreted as bi-quotient. In Section 2 we prove

Theorem A: *Let $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{Z}^3$. For any such non-zero $\mathbf{p}$ one can define an isometric action of $Sp(1)\times U(1)_{\mathbf{p}} \subset Sp(7)$ on S^{27} with the following property: If $0 < p_1 < p_2 < p_3$ are pairwise relatively prime and $\gcd(p_1 \pm p_2, p_1 \pm p_3) = 1$ then the reductions $\mathcal{M}(\mathbf{p}) \to \mathcal{Z}(\mathbf{p}) \to \mathcal{O}(\mathbf{p})$ of $S^{27} \to \mathbb{CP}^{13} \to \mathbb{HP}^6$ give a compact smooth 11-dimensional 3-Sasakian manifold $\mathcal{M}(\mathbf{p})$ together with the (orbifold) leaf spaces of its fundamental foliations $\mathcal{Z}(\mathbf{p})$ and $\mathcal{O}(\mathbf{p})$. Furthermore, the reduced space $\mathcal{M}(1,1,1)$ is a 3-Sasakian orbifold $\mathbb{Z}_3\backslash G_2/Sp(1)$.*

Analysis of the symmetry structure of all the quotients together with the associated foliations gives

Theorem B: The manifold $\mathcal{M}(\mathbf{p})$ of Theorem A is not toric. The corresponding leaf spaces $\mathcal{Z}(\mathbf{p})$ and $\mathcal{O}(\mathbf{p})$ are compact Riemannian orbifolds with inhomogeneous Einstein metrics of positive scalar curvature.

In Section 3 we investigate whether $\mathcal{M}(p_1, p_2, p_3)$ admits further reduction by an isometric circle action. More generally we consider $Sp(1)\times S^1\times S^1$ actions on the 27-sphere and ask if one can get any smooth quotients. Surprisingly, no smooth examples can be found but orbifold quotients exist in profusion. These are interesting, since, due to Theorem B, they are necessarily non-toric (more precisely of cohomogeneity 3). More importantly, they yield new explicit self-dual Einstein metrics of positive scalar curvature with only orbifold singularities and with one-dimensional group of isometries. We get

Theorem C: *Let $\Theta \in \mathcal{M}_{2\times 3}(\mathbb{Z})$ be any integral 2×3 matrix such that each of its three 2×2 minor determinants does not vanish. In addition suppose that the sum of the all minor determinants is nonvanishing, and none of them is equal to the some of the other two. For any such Θ there exists a compact 4-dimensional orbifold $\mathcal{O}(\Theta)$ which admits a self-dual Einstein metric of positive scalar curvature with a one-dimensional group of isometries. Moreover, this metric can be constructed explicitly as a quaternionic Kähler reduction of the real Grassmannian $Gr_4(\mathbb{R}^7)$ by an isometric action of the 2-torus T^2_Θ defined by Θ.*

The first examples of positive self-dual Einstein metrics on orbifolds were obtained in [GL]. Later such metrics were considered by Hitchin [Hi1, Hi2]. Hitchin's examples have large group of isometries. More recently many new orbifold metrics with T^2-symmetry group were constructed in [BGMR]. The examples presented here are perhaps the first self-dual Einstein metrics with only one Killing vector field. We are not aware of any self-dual Einstein metrics which have only discrete isometries.

In Section 4 we consider the obvious higher-dimensional extension of the problem. Not surprisingly, once again the new examples involve hyperkähler geometry of a nilpotent variety. This time it is the 16-dimensional nilpotent orbit of $\mathfrak{so}(6,\mathbb{C})$ which is the Swann bundle over the quaternionic Kähler orbifold $Gr_4(\mathbb{R}^7)/\mathbb{Z}_2$. In [KS2] it is shown how all classical nilpotent orbits can be obtained as hyperkähler reductions from flat spaces (typically in more than one way). This orbit can be constructed as a $Sp(1)\times U(1)$ reduction of $\mathbb{H}^8$ [KS3] and thus it appears as part of a diagram similar to the one in (0.1). In this context our construction is a systematic study of the general $U(1)$ actions which, at the 3-Sasakian level, produce smooth metrics. We prove the following analogue of the Theorem A:

Theorem D: *Let $\mathbf{p} = (p_1, p_2, p_3, p_4) \in \mathbb{Z}^4$. For any such non-zero $\mathbf{p}$ one can define an action of $Sp(1)\times U(1)_{\mathbf{p}} \subset Sp(8)$ on S^{31} with the following property: If $0 \le p_1 < p_2 < p_3 < p_4$, any triple $p_i < p_j < p_k$ satisfies $\gcd(p_i, p_j, p_k) = 1$, and $\gcd(p_i \pm p_j, p_i \pm p_k) = 1$ then the the reductions $\mathcal{M}(\mathbf{p}) \to \mathcal{Z}(\mathbf{p}) \to \mathcal{O}(\mathbf{p})$ of $S^{31} \to \mathbb{CP}^{15} \to \mathbb{HP}^7$ give a compact smooth 15-dimensional 3-Sasakian manifold $\mathcal{M}(\mathbf{p})$ together with the (orbifold) leaf spaces of its fundamental foliations $\mathcal{Z}(\mathbf{p})$ and $\mathcal{O}(\mathbf{p})$. Furthermore, the reduced space $\mathcal{M}(1,1,1,1)$ is a 3-Sasakian orbifold $\mathbb{Z}_2\backslash\mathrm{Spin}(7)/\mathrm{Spin}(4)$.*

These 15-dimensional quotients are "deformations" of the standard homogeneous 3-Sasakian structure on $SO(7)/SO(3)\times Sp(1)$ which projects to the Wolf space $\mathrm{Gr}_4(\mathbb{R}^7)$ in the quaternionic Kähler base. In higher dimensions the orbifold $\mathrm{Gr}_4(\mathbb{R}^{n+3})/\mathbb{Z}_2$ can equally be obtained as a quaternionic Kähler reduction of $\mathbb{HP}^{n+4}$ by $U(1)\times Sp(1)$. Our construction shows that the $n = 4$ case, from the standpoint of 3-Sasakian geometry, is somewhat exceptional: In higher dimension we can only get orbifold metrics. The orbifold bundles $\mathcal{M}^{15}(\mathbf{p}) \to \mathcal{Z}^{15}(\mathbf{p}) \to \mathcal{O}^{15}(\mathbf{p})$ can be viewed as singular analogues of $SO(7)/SO(3)\times Sp(1) \to SO(7)/SO(3)\times U(2) \to \mathrm{Gr}_4(\mathbb{R}^7)$. This makes use of the well-known isometry between $\mathrm{Gr}_4(\mathbb{R}^7)$ and the space $\mathrm{Spin}(7)/(Sp(1)\times Sp(1)\times Sp(1))/\mathbb{Z}_2$ of the Cayley 4-planes in $\mathbb{R}^8$.

Again, standard analysis of the symmetry structure of all the quotients together with the associated foliations gives

Theorem B$'$: The manifold $\mathcal{M}(\mathbf{p})$ of Theorem D is not toric. The corresponding leaf spaces $\mathcal{Z}(\mathbf{p})$ and $\mathcal{O}(\mathbf{p})$ are compact Riemannian orbifolds with inhomogeneous Einstein metrics of positive scalar curvature.

In Section 5 we give what topological information is available to us. In particular, we show that as long as $\mathbf{p}$ satifies the conditions of Theorems A and D (actually this hypothesis can be weakened) the rational cohomology of the corresponding $\mathcal{M}(\mathbf{p})$ is independent of $\mathbf{p}$. Finally, in section 6 we briefly mention the construction of hypercomplex structures on circle bundles over our new 3-Sasakian manifolds (orbifolds).

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1. Quotient construction of 3-Sasakian structure on $\mathbb{Z}_3\backslash G_2/Sp(1)$

There are two homogeneous Sasakian-Einstein geometries that are naturally associated with the exceptional Lie group G_2 [BG1, BG2]. They both come from the classical Lie group isomorphism between $SO(4) \subset G_2$ and $Sp(1)_- \cdot Sp(1)_+ \subset G_2$. The two $Sp(1)_\pm$ subgroups are very different. One of them has index 1 in G_2 and the other one has index 3. Consequently, the quotients are not of the same homotopy type as can be seen from the exact sequence in homotopy for the fibration

$$Sp(1)_\pm \longrightarrow G_2 \longrightarrow G_2/Sp(1)_\pm .$$

In particular, the two spaces can be distinguished by their third homotopy groups being trivial in one case and $\mathbb{Z}_3$ in the other. One of these quotients, which we shall denote by $G_2/Sp(1)_-$ is diffeomorphic to the real Stiefel manifold $V_{7,2}(\mathbb{R}) = SO(7)/SO(5)$ of 2-frames in $\mathbb{R}^7$ [HL]. As $V_{7,2}(\mathbb{R})$ is 4-connected $\pi_3(G_2/Sp(1)_-) = 0$. The other quotient denoted here by $G_2/Sp(1)_+$ is one of the 11-dimensional 3-Sasakian homogeneous spaces and $\pi_3(G_2/Sp(1)_+) = \mathbb{Z}_3$. $G_2/Sp(1)_+$ fibers as a circle bundle over a generalized flag $\mathcal{Z} = G_2/U(2)_+$, which in turn is well-known to be the twistor space of the exceptional 8-dimensional Wolf space $G_2/SO(4)$. The second homogeneous Sasakian-Einstein manifold is a circle bundle over the complex flag $G_2/U(2)_-$ which can be identified with the complex quadric in the 6-dimensional complex projective space $\mathbb{CP}^6$ or, equivalently, the real Grassmannian $\mathrm{Gr}_2(\mathbb{R}^7) = SO(7)/SO(2) \times SO(5)$ of oriented 2-planes in $\mathbb{R}^7$. Both Sasakian-Einstein metrics are well-known and have been studied in the context of homogeneous Einstein geometries. We have the following diagram of Riemannian submersions:

1.1
$$\begin{array}{ccccc}
 & & G_2 & & \\
 & \swarrow & & \searrow & \\
\frac{G_2}{Sp(1)_+} & & & & \frac{G_2}{Sp(1)_-} \simeq V_{7,2}(\mathbb{R}) \\
\downarrow & & & & \downarrow \\
\mathcal{Z} = \frac{G_2}{U(2)_+} & & & & \frac{G_2}{U(2)_-} \simeq \mathrm{Gr}_2(\mathbb{R}^7) \\
 & \searrow & & \swarrow & \\
 & & \frac{G_2}{SO(4)} & &
\end{array}$$

Salamon and Poon proved that $G_2/SO(4)$ is one of the three possible models of positive quaternionic Kähler manifolds in dimension 8. Later the geometry of $G_2/SO(4)$ was examined by Kobak and Swann [KS1] who proved the following remarkable theorem:

Theorem 1.2: *The quaternionic Kähler manifold* $\mathrm{Gr}_4(\mathbb{R}^7)$ *admits an action of* $U(1)$ *such that the quaternionic Kähler quotient is a compact quaternionic Kähler orbifold* $\mathcal{O} = \mathcal{O}_r \cup \mathbb{CP}(2) = G_2/(SO(4) \times \mathbb{Z}_3)$.

We will first re-examine the Kobak-Swann construction from the point of view of the 3-Sasakian geometry of a certain $SO(3)$ V-bundle over $\mathcal{O}$ (or, equivalently, the hyperkähler geometry of the regular nilpotent orbit of $\mathfrak{sl}(3,\mathbb{C})$). In particular, we have the following:

Theorem 1.3: *The 3-Sasakian homogeneous manifold* $SO(7)/SO(3) \times Sp(1)$ *admits an action of* $U(1)$ *such that the 3-Sasakian quotient is a compact 3-Sasakian orbifold* $\mathcal{M} = \mathcal{M}_r \cup SU(3)/U(1) = \mathbb{Z}_3\backslash G_2/Sp(1)_+$.

Theorem 1.3 is a straightforward translation of Theorem 1.2 into the language of 3-Sasakian geometry and we could leave it at that. However, we will outline a constructive proof of this result as our description of the corresponding quotient slightly differs from the one given in [KS1].

One can think of the homogeneous 3-Sasakian manifold $SO(7)/SO(3) \times Sp(1)$ as the 3-Sasakian reduction $S^{4n-1}/\!\!/\!\!/ Sp(1)$ as follows [G, BGM1]: Let $\mathbf{u} = (u_1, ..., u_7) \in S^{27}$. Consider the $Sp(1)$ action given by multiplication by unit quaternion $\lambda \in Sp(1)$ on the left that is

1.4 $$\varphi_\lambda(\mathbf{u}) = \lambda\mathbf{u}.$$

In the $\{i, j, k\}$ basis the 3-Sasakian moment maps for this action read:

1.5 $$\mu_i(\mathbf{u}) = \sum_{\alpha=1}^{7} \overline{u}_\alpha i u_\alpha, \qquad \mu_j(\mathbf{u}) = \sum_{\alpha=1}^{7} \overline{u}_\alpha j u_\alpha, \qquad \mu_k(\mathbf{u}) = \sum_{\alpha=1}^{7} \overline{u}_\alpha k u_\alpha.$$

Then, the common zero-locus of the moment maps

1.6 $$N = \{\mathbf{u} \in S^{4n-1} : \quad \mu_i(\mathbf{u}) = \mu_j(\mathbf{u}) = \mu_k(\mathbf{u}) = 0\}$$

is the Stiefel manifold $N \simeq SO(7)/SO(3) = V_{7,4}(\mathbb{R})$ of the orthonormal 4-frames in $\mathbb{R}^7$ and the corresponding 3-Sasakian quotient $\mathcal{S} = N/Sp(1)$ is the Konishi's $\mathbb{RP}^3$-bundle over the real Grassmannian of oriented 4-planes in $\mathbb{R}^7$. We can combine Theorem 1.3 with this description to get

Corollary 1.7: *The 3-Sasakian sphere* S^{27} *admits an action of* $U(1) \times Sp(1)$ *such that the 3-Sasakian quotient is a compact 3-Sasakian orbifold* $\mathcal{M} = \mathcal{M}_r \cup SU(3)/U(1) = \mathbb{Z}_3\backslash G_2/Sp(1)_+$.

We now turn to the explicit description of the $U(1)$ quotient. Consider the following subgroups of the group of 3-Sasakian isometries of the 27-sphere:

1.8 $$Sp(7) \supset SO(7) \supset 1 \times SO(6) \supset 1 \times U(3),$$

where $U(1) \subset U(3)$ is the central subgroup. Explicitly, we shall write $f : [0, 2\pi) \to SO(7)$

1.9 $$f(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A(t) & 0 & 0 \\ 0 & 0 & A(t) & 0 \\ 0 & 0 & 0 & A(t) \end{pmatrix} \in SO(7),$$

where

1.10 $$A(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

are the real rotations in $\mathbb{R}^2$. The homomorphism $f(t)$ yields a circle action on S^{27} or, equivalently, (after performing the "$Sp(1)$-reduction" first) on the homogeneous 3-Sasakian manifold $SO(7)/SO(3)\times Sp(1)$ via left multiplication $f(t)\mathbf{u}$ and the associated 3-Sasakian moment map can be written as

1.11 $$\nu(\mathbf{u}) = \sum_{\alpha=1,2,3} (\overline{u}_{2\alpha}u_{2\alpha+1} - \overline{u}_{2\alpha+1}u_{2\alpha}).$$

Note here that $\nu(\mathbf{u})$ does not depend on the u_1 quaternionic coordinate.

Definition 1.12: *Let us define the zero level set of this new moment map intersected with N, that is*

1.12 $$N_\nu \equiv N \cap \nu^{-1}(0)$$

First we observe, following Kobak and Swann [KS1] that

Lemma 1.13: *The manifold N_ν can be identified with $U(1)\cdot G_2 = (S^1\times G_2)/\mathbb{Z}_3$ where $U(1)\cap G_2 = \mathbb{Z}_3$.*

Proof: The argument is similar here to the one used by Kobak and Swann in [KS1] and it is based on the Proposition 1.10 of [HL]. First, using the basis $\{i,j,k\}$ of unit imaginary quaternions, we write $u_\alpha = u^0_\alpha + iu^1_\alpha + ju^2_\alpha + ku^3_\alpha$ and introduce the 4×7 real matrix

1.14 $$\mathbb{A} = \begin{pmatrix} u^0_1 & u^0_2 & u^0_3 & u^0_4 & u^0_5 & u^0_6 & u^0_7 \\ u^1_1 & u^1_2 & u^1_3 & u^1_4 & u^1_5 & u^1_6 & u^1_7 \\ u^2_1 & u^2_2 & u^2_3 & u^2_4 & u^2_5 & u^2_6 & u^2_7 \\ u^3_1 & u^3_2 & u^3_3 & u^3_4 & u^3_5 & u^3_6 & u^3_7 \end{pmatrix} \equiv \begin{pmatrix} f^0 \\ f^1 \\ f^2 \\ f^3 \end{pmatrix},$$

where to make the connection with the notation in [KS1] we also think of the rows of $\mathbb{A}$ as purely imaginary octonions $\mathrm{Im}(\mathbb{O})$. In the standard basis of $\mathrm{Im}(\mathbb{O})$ we write $f^a = u^a_1 i + u^a_2 j + u^a_3 k + u^a_4 e + u^a_5 ie + u^a_6 je + u^a_7 ke$. Let $\phi(a,b,c) = <ab, c>$ denote the 3-form defining the associative calibration [HL] on $\mathrm{Im}(\mathbb{O})$ where $<\cdot,\cdot>$ denotes the standard Euclidean inner product. Then writing $\nu = \nu_1 i + \nu_2 j + \nu_3 k$ a straightforward computation shows that for $a = 1,2,3$ and ϵ^{abc} the totally antisymmetric tensor satisfying $\epsilon^{123} = 1$

1.15 $$\nu_a = 2\phi(f^0 f^a + \epsilon^{abc} f^b f^c, i),$$

where the summation convention on repeated indices is used. Now, $\mathbf{u}\in N$ if and only if the rows $\{f^0,f^1,f^2,f^3\}$ of $\mathbb{A}$ form an orthonormal frame in $\mathbb{R}^7 \simeq \mathrm{Im}(\mathbb{O})$, and one can identify G_2 with a special kind of oriented orthonormal 4-frame, namely those which are co-associative. This means that the 3-plane that is orthogonal to the 4-plane defined by the frame $\{f^0,f^1,f^2,f^3\}$ is spanned by an associative subalgebra of $\mathrm{Im}(\mathbb{O})$. Then one shows that these special 4-frames satisfy the $U(1)$-moment map equation $\nu(\mathbf{u}) = 0$ and, hence, $U(1)\cdot G_2 \subset N\cap\nu^{-1}(0)$. As, $U(1)\cap G_2 = \mathbb{Z}_3$ it is enough to show that by acting with $U(1)$ one gets the whole $N \cap \nu^{-1}(0)$. The argument is similar to the one presented in [KS1]. (See [KS1] Lemma 5.1 and the discussion that follows.) ∎

Now, Theorems 1.2 and 1.3 and Corollary 1.7 all follow from the above lemma as we get the quotient

1.16 $$\mathcal{M} = \frac{N_\nu}{U(1) \times Sp(1)} \simeq \frac{U(1) \cdot G_2}{U(1) \times Sp(1)} \simeq \mathbb{Z}_3 \backslash G_2 / Sp(1).$$

REMARK 1.17: The $U(1) \times Sp(1)$ action on the level set N_ν is not locally free. If we divide by $Sp(1)$ first and consider the $U(1)$ action on the orbit space $N_\nu / Sp(1)$ this circle action is quasi-free. This means that there are only two kinds orbits: regular orbits with the trivial isotropy group and singular orbits (points) where the isotropy group is the whole $U(1)$. In such cases the quotient space is often an orbifold (or even a smooth manifold). The stratification of the Theorem 2 is precisely with respect to the orbit types as will be seen in the next section.

2. Generalizations of the Kobak-Swann Quotient

In this section we will consider the simplest possible family of quotients which generalize the construction of Kobak and Swann via an introduction of weights. Instead of considering the central $U(1) \subset U(3)$ in 1.9 we can consider and arbitrary circle subgroup of the maximal torus $U(1) \subset T^3 \subset U(3)$. Again, to be more specific, we have the following inclusions:

2.1 $$Sp(7) \supset SO(7) \supset 1 \times SO(6) \supset 1 \times SO(2) \times SO(2) \times SO(2).$$

We can consider arbitrary circle subgroups of the last 3-torus. Let $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{Z}^3$ and define the following homomorphism

2.2 $$f_{\mathbf{p}}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A(p_1 t) & 0 & 0 \\ 0 & 0 & A(p_2 t) & 0 \\ 0 & 0 & 0 & A(p_3 t) \end{pmatrix} \in SO(7),$$

where

2.3 $$A(p_i t) = \begin{pmatrix} \cos(p_i t) & \sin(p_i t) \\ -\sin(p_i t) & \cos(p_i t) \end{pmatrix} \in SO(2), \qquad i = 1, 2, 3$$

are 2-dimensional real rotations. Note that for $p_1 = p_2 = p_3 = 1$ we recover the example of the previous section. The homomorphism $f_{\mathbf{p}}(t)$ yields a circle action on the homogeneous 3-Sasakian manifold $SO(7)/SO(3) \times Sp(1)$ via left multiplication $f_{\mathbf{p}}(t)\mathbf{u}$ and the moment map can now be written as

2.4 $$\nu_{\mathbf{p}}(u_1, ..., u_7) = \sum_{\alpha=1,2,3} p_\alpha (\overline{u}_{2\alpha} u_{2\alpha+1} - \overline{u}_{2\alpha+1} u_{2\alpha}).$$

Observe that without loss of generality we can assume all weights to be non-negative as p_i can be changed to $-p_i$ by renaming the quaternions in the associated pair (u_{2i}, u_{2i+1}). We begin analysis of this quotient by considering the level set of the moment map

DEFINITION 2.5: *Let $N_\nu(\mathbf{p}) \subset S^{27}$ be the level set of the 3-Sasakian moment map of the $Sp(1)\times U(1)_{\mathbf{p}}$-action, i.e., $N_\nu(\mathbf{p}) \equiv N \cap \{\nu_{\mathbf{p}}^{-1}(0)\}$.*

We want to consider a stratification of the level set $N_\nu(\mathbf{p})$ that will allow us to analyze the quotient space

2.6
$$\mathcal{M}(\mathbf{p}) = \frac{N_\nu(\mathbf{p}) \equiv V_{7,4}(\mathbb{R}) \cap \{\nu_{\mathbf{p}}^{-1}(0)\}}{Sp(1)\times U(1)_{\mathbf{p}}}.$$

Since $N_\nu(\mathbf{p})$ is a submanifold of the Stiefel manifold $V_{7,4}(\mathbb{R})$ at most 3 quaternionic coordinates can vanish on $N_\nu(\mathbf{p})$. So setting various quaternionic coordinates equal to zero determines a stratification of $N_\nu(\mathbf{p})$ in which the strata of minimal dimension play an important role. We call these strata *vertices* although, as we shall see, they each have two connected components.

LEMMA 2.7: *Let $0 < p_1 < p_2 < p_3$. At a vertex neither u_1 nor any of the three pairs of quaternions (u_{2i}, u_{2i+1}), $i = 1,2,3$ can vanish. Thus, there are precisely eight vertices and they are all diffeomorphic to $O(4)$.*

PROOF: Every vertex has precisely 3 quaternionic coordinates vanishing, so the Stiefel manifold becomes $V_{4,4}(\mathbb{R}) = O(4)$. Let V be a vertex. Then $V = V_{ijkl}$ can be represented by a matrix of the form

2.8
$$\mathbb{B} = \begin{pmatrix} u_i^0 & u_i^1 & u_i^2 & u_i^3 \\ u_j^0 & u_j^1 & u_j^2 & u_j^3 \\ u_k^0 & u_k^1 & u_k^2 & u_k^3 \\ u_l^0 & u_l^1 & u_l^2 & u_l^3 \end{pmatrix}, \qquad \mathbb{B}^t\mathbb{B} = \mathbb{B}\mathbb{B}^t = \frac{1}{4}\mathbb{I}_4,$$

where $1 \le i < j < k < l \le 7$. Here we have written a quaternionic coordinate as $u_i = u_i^0 + iu_i^1 + ju_i^2 + ku_i^3$. Suppose $i > 1$, *i.e.*, $u_1 = 0$ on on V_{ijkl}. Then there are two possibilities: (1) the quadruple $i < j < k < l$ contains only one quaternionic pair $(2\alpha, 2\alpha+1)$ or (2) it contains two such pairs. Let us examine the first possibility. Without loss of generality we can take $i = 2, j = 3, k = 4, l = 6$. With this choice the vanishing of the $U(1)$ moment map 2.4 now becomes

$$\mathrm{Im}(\bar{u}_2 u_3) = 0.$$

One can easily check that orthogonality of the basis forces

$$\mathrm{Re}(\bar{u}_2 u_3) = u_2^1u_3^1 + u_2^2u_3^2 + u_2^3u_3^3 + u_2^4u_3^4 = 0.$$

But this implies $\bar{u}_2 u_3 = 0$ which forces either u_2 or u_3 to vanish, giving a contradiction. Now assume that the second case, that is that $i < j < k < l$ consists of two quaternionic pairs $(2\alpha, 2\alpha+1)$. Again, when we take $i = 2, j = 3, k = 4, l = 5$, the orthogonality of the vectors in the associated $\mathbb{B}$-matrix forces

2.9
$$\begin{aligned} \mathrm{Re}(\bar{u}_2 u_3) &= u_2^1u_3^1 + u_2^2u_3^2 + u_2^3u_3^3 + u_2^4u_3^4 = 0, \\ \mathrm{Re}(\bar{u}_4 u_5) &= u_4^1u_5^1 + u_4^2u_5^2 + u_4^3u_5^3 + u_4^4u_5^4 = 0. \end{aligned}$$

But orthogonality also implies

$$|u_2|^2 = |u_3|^2 = |u_4|^2 = |u_5|^2 = \frac{1}{4}.$$

Then we have

$$|\mathrm{Im}(\bar{u}_2u_3)|^2 = |\mathrm{Im}(\bar{u}_4u_5)|^2 = \frac{1}{16},$$

and this contradicts the $U(1)$ moment map equation

$$2p_1\mathrm{Im}(\bar{u}_2u_3) + 2p_2\mathrm{Im}(\bar{u}_4u_5) = 0$$

if $p_1 \neq p_2$. Repeating the argument for other choices of $1 < i < j < k < l$ requires $p_1 < p_2 < p_3$ of the hypothesis. Thus, at a vertex we must have $u_1 \neq 0$. This proves the "neither" part of the statement.

Now assume that $i = 1 < j < k < l$ and suppose either $j < k$ or $k < l$ is a quaternionic pair. We can take $j = 2, k = 3$ and let l be arbitrary. Then the orthogonality of the corresponding $\mathbb{B}$-matrix again forces

$$\mathrm{Re}(\bar{u}_2u_3) = 0.$$

But again the $U(1)$ moment map constraint is

$$\mathrm{Im}(\bar{u}_2u_3) = 0$$

giving a contradiction. This proves the "nor" part of the lemma. It is now clear that the vertices must automatically satisfy the $U(1)$ moment map constraint $\nu_{\mathbf{p}}(\mathbf{u}) = 0$; hence, they are all diffeomorphic to $O(4)$. Moreover, a simple counting shows that there are precisely eight vertices. ∎

Our analysis suggests the importance of the following strata:

2.10
$$\begin{aligned}
S_0 &= \{\mathbf{u} \in N_\nu(\mathbf{p}) \mid u_1 = 0\},\\
S_1 &= \{\mathbf{u} \in N_\nu(\mathbf{p}) \mid u_1 \neq 0\},\\
S_2 &= \{\mathbf{u} \in N_\nu(\mathbf{p}) \mid \text{some quaternionic pair } (u_{2i}, u_{2i+1}) \text{ vanishes}\},\\
S_3 &= \{\mathbf{u} \in N_\nu(\mathbf{p}) \mid \text{no quaternionic pair } (u_{2i}, u_{2i+1}) \text{ vanishes}\}.
\end{aligned}$$

Then Lemma 2.7 easily implies that

COROLLARY 2.11: *Let $0 < p_1 < p_2 < p_3$. Then*

(i) $S_0 \cup S_1 = S_2 \cup S_3 = N_\nu(\mathbf{p})$.

(ii) $S_0 \cap S_1 = S_2 \cap S_3 = \emptyset$.

(iii) $S_0 \cap S_2 = \emptyset$.

(iv) $S_2 \subset S_1$.

(v) $S_0 \subset S_3$.

Notice that (iii) fails if $p_i = p_j$ for some $i \neq j$. In particular it fails for the level set N_ν of 1.12 in the previous section, and this is the reason that the quotient $\mathcal{M}$ of 1.16 is not smooth. We now are ready to give necessary conditions to guarantee a smooth quotient.

LEMMA 2.12: *Let $\mathbf{p} = (p_1, p_2, p_3) \in (\mathbb{Z}_+)^3$ be pairwise relatively prime. Then the isotropy group of the $Sp(1) \times U(1)_{\mathbf{p}}$ action at every point of S_1 is the identity.*

PROOF: The action of $Sp(1) \times U(1)_{\mathbf{p}}$ on $\mathbb{H}^7$ is the diagonal action of $Sp(1)$ by quaternionic multiplication by a unit quaternion λ on the left, and the matrix multiplication $\mathbf{u} \mapsto f_{\mathbf{p}}(t)\mathbf{u}$

for the $U(1)_{\mathbf{p}}$ action. These two actions clearly commute. Since $u_1 \neq 0$ we immediately get that $\lambda = 1$. Consider the set where a quaternionic pair $(u_6, u_7) = (0, 0)$. Then the fixed point equation becomes

2.13 $$A(p_1 t) = A(p_2 t) = \mathbb{I}_2,$$

which has only the trivial solution provided that $\gcd(p_1, p_2) = 1$. Setting the other two quaternionic pairs to be zero gives $\gcd(p_1, p_3) = \gcd(p_2, p_3) = 1$. As onne cannot set more than one quaternionic pair equal to $(0, 0)$ the lemma is proved. ∎

LEMMA 2.14: *Let* $\mathbf{p} = (p_1, p_2, p_3) \in (\mathbb{Z}_+)^3$ *satisfy the four conditions* $\gcd(p_1 \pm p_2, p_1 \pm p_3) = 1$. *Then the isotropy group of the* $Sp(1) \times U(1)_{\mathbf{p}}$ *action at every point of* S_0 *is the identity.*

PROOF: To determine the conditions for fixed points of the action we consider the following equations

$$A(p_i t) \begin{pmatrix} u_{2i} \\ u_{2i+1} \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \begin{pmatrix} u_{2i} \\ u_{2i+1} \end{pmatrix} = \lambda \begin{pmatrix} u_{2i} \\ u_{2i+1}, \end{pmatrix} \qquad i = 1, 2, 3$$

for $\lambda \in Sp(1)$ and $t \in [0, 2\pi)$. For each $i = 1, 2, 3$ this reads

$$a_i u_{2i} + b_i u_{2i+1} = \lambda u_{2i},$$

$$-b_i u_{2i} + a_i u_{2i+1} = \lambda u_{2i+1}.$$

For each $i = 1, 2, 3$ we multiply the first equation from the right by $\overline{u}_{2i}$ and the second by $\overline{u}_{2i+1}$ to get

$$a_i |u_{2i}|^2 + b_i u_{2i+1} \overline{u}_{2i} = \lambda |u_{2i}|^2,$$

2.15 $$-b_i u_{2i} \overline{u}_{2i+1} + a_i |u_{2i+1}|^2 = \lambda |u_{2i+1}|^2.$$

By adding these two equations we get

$$a_i(|u_{2i}|^2 + |u_{2i+1}|^2) + b_i(u_{2i+1}\overline{u}_{2i} - u_{2i}\overline{u}_{2i+1}) = \lambda(|u_{2i}|^2 + |u_{2i+1}|^2), \quad i = 1, 2, 3.$$

By (iii) of Lemma 2.10 the term multiplying λ on the right hand side of this equation never vanishes. This gives for each $i = 1, 2, 3$

2.16 $$\begin{aligned} \mathrm{Re}(\lambda) &= a_i, \\ \mathrm{Im}(\lambda) &= b_i \frac{u_{2i+1}\overline{u}_{2i} - u_{2i}\overline{u}_{2i+1}}{|u_{2i}|^2 + |u_{2i+1}|^2}. \end{aligned}$$

The first of these equations gives

2.17 $$a_1 = a_2 = a_3,$$

and combining this with $a_i^2 + b_i^2 = 1$ implies

2.18 $$b_1 = \pm b_2 = \pm b_3.$$

Let us write $\tau = e^{it}$. Then 2.17 and 2.18 give

2.19 $$\tau^{p_1} = \tau^{\pm p_2}, \qquad \tau^{p_1} = \tau^{\pm p_3}.$$

These have only trivial solutions if and only if $\gcd(p_1 \pm p_2, p_1 \pm p_3) = 1$. ∎

It is convenient to make the following:

DEFINITION 2.20: *Let* $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{Z}^3$*. We say that the weight vector* $\mathbf{p}$ *is admissible if* $0 < p_1 < p_2 < p_3$, $\gcd(p_i, p_j) = 1$ *for all* $i < j$*, and* $\gcd(p_1 \pm p_2, p_1 \pm p_3) = 1$.

It now follows immediately from Lemmas 2.12, 2.14 and Definition 2.20 that

THEOREM 2.21: *The* $Sp(1) \times U(1)_{\mathbf{p}}$ *action on* $N_\nu(\mathbf{p})$ *is free if and only if* $\mathbf{p} \in \mathbb{Z}^3_+$ *is admissible.*

Note that there are infinitely many admissible weight vectors. For example we can take $\mathbf{p} = (2k-1, 2k, 2k+1)$, where $k \in \mathbb{Z}_+$. Thus, there are infinite families of smooth quotients $\mathcal{M}(\mathbf{p})$ and infinite families of the associated triples $\mathcal{M}(\mathbf{p}) \to \mathcal{Z}(\mathbf{p}) \to \mathcal{O}(\mathbf{p})$ with their (orbifold) Einstein metrics.

Theorem A now follows from the Theorem 2.21 and various theorems concerning 3-Sasakian (complex contact, quaternionic Kähler) reductions [BGM1, GL]. The last statement of Theorem A follows from Corollary 1.7.

We briefly return to the $\mathbf{p} = (1,1,1)$ case of the previous section. As already observed we get singularities here because the corresponding action is not even locally free on the level set of the moment map. But it is easy to see that it is quasi-free. That is there are only two types if isotropy subgroups in the circle: the identity and the whole group. In such a situation Dancer and Swann [DS] observed that 3-Sasakian quotients, stratify as the union of 3-Sasakian manifolds [DS]. This is true in this case in particular, but as we saw in the previous section the two strata nicely fit together and one gets a compact 3-Sasakian orbifold. Let us explicitly describe the singular part $\mathcal{M}_1 \subset \mathcal{M}(1,1,1)$.

Note that if $p_1 = p_2 = p_3 = 1$ then $a_1 = a_2 = a_3 = a$ and $b_1 = b_2 = b_3 = b$ and we can add equations (2.15) to get

2.22 $$a + b\rho = \lambda,$$

where now

2.23 $$\rho = \sum_{i=1,2,3} (u_{2i+1}\overline{u}_{2i} - u_{2i}\overline{u}_{2i+1}).$$

When $u_1 \neq 0$ one does not get any fixed points of the action. But when $u_1 = 0$ for any imaginary unit ρ there is a $U(1) \subset Sp(1) \times U(1)$ subgroup

$$(\cos t + \rho \sin t, A(t)) \in Sp(1) \times U(1),$$

which acts trivially on the following set

2.24 $$\{\mathbf{u} \in N_\nu \mid u_{2i+1} = \rho u_{2i}\}.$$

In this case all 4 moment map equations reduce to the same one and it reads

$$\sum_{i=1,2,3} \overline{u}_{2i}\rho u_{2i} = 0.$$

For any fixed ρ we can recognize this set as the complex Stiefel manifold $U(3)/U(1)$ and it follows that the quotient $SU(3)/U(1) = \mathcal{S}(1,1,1)$ is the singular stratum $\mathcal{M}_1$.

The geometry of the smooth families $\mathcal{M}(\mathbf{p})$ is rather interesting. First we observe that these spaces cannot be toric. This can be seen in several different ways, for example by careful analysis of the associated foliations. One can also generalize the analysis of [BGM2] to show that the only isometries of the level set of the moment map $N_\nu(\mathbf{p}) \subset S^{27} \subset \mathbb{R}^{28}$ can come from the restriction of the isometries of the Euclidean space $\mathbb{R}^{28}$. From this we conclude

THEOREM 2.25: *Let* $\mathbf{p}$ *be admissible so that* $\mathcal{M}(\mathbf{p})$ *is a smooth compact 3-Sasakian 11-manifold. Then the Lie algebra* $\mathrm{Isom}^0(\mathcal{M}(\mathbf{p}), g(\mathbf{p}))$ *of the group of 3-Sasakian isometries of* $\mathcal{M}(\mathbf{p})$ *is isomorphic to* $\mathbb{R}^2 \oplus \mathfrak{sp}(1)$*. In particular, all such quotients are non-toric.*

This proves part of Theorem C of the introduction which relates to the 11-dimensional quotients.

Finally, observe that $\mathcal{M}(\mathbf{p})$ contains a special 7-manifold which is embedded as a 3-Sasakian submanifold. Define

2.26
$$S_0(\mathbf{p}) = \frac{N_\nu(\mathbf{p}) \cap \{u_1 = 0\}}{Sp(1) \times U(1)_{\mathbf{p}}}.$$

One can see that $S_0(\mathbf{p})$ is a submanifold and, as it is itself a 3-Sasakian reduction, it must be a 3-Sasakian submanifold. One can even identify this space. Observe that the classical group isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ implies that the $Sp(1)$ quotient yields the homogeneous 3-Sasakian 11-dimensional manifold $\mathcal{S}(1,1,1,1)$. Hence, $S_0(\mathbf{p})$ is either a $U(1)_{\mathbf{p}}$-reduction of $\mathcal{S}(1,1,1,1)$ or, equivalently, a T^2-reduction of S^{15}. Hence, there exists an admissible integer weight matrix $\Omega \in \mathcal{A}_{2\times 4}(\mathbb{Z})$ (see [BGMR]) such that $S_0(\mathbf{p}) \simeq \mathcal{S}(\Omega)$. Hence $S_0(\mathbf{p})$ is toric with second Betti number equal to 2.

3. Further Reductions of $\mathcal{M}(p_1, p_2, p_3)$ by a Circle

In this section we will examine reductions of $\mathcal{M}(p_1, p_2, p_3)$ by isometric circle actions. More generally we shall consider an arbitrary 2-torus subgroup of the maximal torus $T^2 \subset T^3 \subset SO(7)$. Let

3.1
$$\Theta = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix} \in \mathcal{M}_{2\times 3}(\mathbb{Z})$$

be an arbitrary integral 2×3 matrix and define the homomorphism $f_\Theta : T^2 \to SO(7)$

3.2
$$f_\Theta(t,s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A(p_1 t + q_1 s) & 0 & 0 \\ 0 & 0 & A(p_2 t + q_2 s) & 0 \\ 0 & 0 & 0 & A(p_3 t + q_3 s) \end{pmatrix} \in SO(7),$$

where A is the $SO(2)$ rotations as in 2.3. Note that if $\mathbf{p} = (p_1, p_2, p_3)$ were admissible then we would be considering arbitrary isometric circle actions on the quotient $\mathcal{M}(\mathbf{p})$; however, so far we assume nothing about Θ. The homomorphism $f_\Theta(t,s)$ yields a 2-torus action on the homogeneous 3-Sasakian manifold $SO(7)/SO(3)\times Sp(1)$ via left multiplication $f_\Theta(t,s)\mathbf{u}$ and the moment map can now be written as

3.3 $$\nu_\Theta(u_1,...,u_7)=\begin{pmatrix}\sum_{\alpha=1,2,3}p_\alpha(\overline{u}_{2\alpha}u_{2\alpha+1}-\overline{u}_{2\alpha+1}u_{2\alpha})\\ \sum_{\alpha=1,2,3}q_\alpha(\overline{u}_{2\alpha}u_{2\alpha+1}-\overline{u}_{2\alpha+1}u_{2\alpha})\end{pmatrix}\in\mathbb{R}^2\otimes\mathfrak{sp}(1).$$

We begin our analysis of this quotient by considering the level set of the moment map.

Definition 3.4: *Let $N_\nu(\Theta)\subset S^{27}$ denote the level set of the 3-Sasakian moment map of the $Sp(1)\times T^2_\Theta$-action, i.e., $N_\nu(\Theta)\equiv N\cap\{\nu_\Theta^{-1}(\mathbf{0})\}$ and let*

3.5 $$\mathcal{M}(\Theta)=N_\nu(\Theta)/Sp(1)\times T^2_\Theta.$$

We want to determine for which $\Theta\in\mathcal{M}_{2\times3}(\mathbb{Z})$ the 7-dimensional quotient $\mathcal{M}(\Theta)$ is an orbifold and, if possible which weight matrices yield smooth quotients. Let us define

3.6 $$\Delta_{ij}=\Delta_{ij}(\Theta)=\det\begin{pmatrix}p_i & p_j\\ q_i & q_j\end{pmatrix},\qquad 1\le i<j\le3,$$

the three minor determinants of Θ.

Lemma 3.7: *The action of $Sp(1)\times T^2_\Theta$ on $N_\nu(\Theta)\cap\{u_1\neq0\}$ is*

(i) locally free if and only if $\Delta_{ij}(\Theta)\neq0,\quad\forall\ 1\le i<j\le3$,

(ii) free if and only if $|\Delta_{ij}(\Theta)|=1,\quad\forall\ 1\le i<j\le3$.

Proof: Since $u_1\neq0$ we must have $\lambda=1$ and, hence, it is enough to consider the T^2_Θ-action. Now, suppose that a quaternionic pair, say (u_6,u_7) vanishes. Then the fixed point equation reads:

3.8 $$A(p_it+q_is)\begin{pmatrix}u_{2i}\\ u_{2i+1}\end{pmatrix}=\begin{pmatrix}u_{2i}\\ u_{2i+1}\end{pmatrix},\qquad i=1,2$$

for $t,s\in[0,2\pi)$, or, equivalently,

3.9 $$A(p_it+q_is)=\mathbb{I}_2,\qquad i=1,2.$$

Let $\tau=e^{it}$ and $\rho=e^{is}$. Then we can rewrite 3.9 as

3.10 $$\tau^{p_i}\rho^{q_i}=1,\qquad i=1,2.$$

This has only discrete solutions provided $\Delta_{12}(\Theta)\neq0$. Furthermore, the isotropy group at all such points will be trivial provided $\Delta_{12}(\Theta)=\pm1$. This proves the lemma. ∎

Note that the second condition is already very restrictive as it says that any 2×2 submatrix of Θ must be an element of $PSL(2,\mathbb{Z})$. However, there are many matrices which satisfy both conditions, for example

$$\Theta_1=\begin{pmatrix}1&0&1\\0&1&1\end{pmatrix},\qquad\Theta_2=\begin{pmatrix}9&2&7\\40&9&31\end{pmatrix}.$$

It remains to analyze the fixed point equations on $N_\nu(\Theta) \cap \{u_1 = 0\}$. We now prove

LEMMA 3.11: *The action of $Sp(1) \times T^2_\Theta$ on $N_\nu(\Theta)$ is locally free if and only if $\Delta_{ij}(\Theta) \neq 0, \ \ \forall\, 1 \leq i < j \leq 3$ and*

3.12
$$\Box^{\mp}_{\mp} = \det \begin{pmatrix} p_1 \mp p_2 & q_1 \mp q_2 \\ p_1 \mp p_3 & q_1 \mp q_3 \end{pmatrix} \neq 0$$

Furthermore, there is no weight matrix $\Theta \in \mathcal{M}_{2\times 3}(\mathbb{Z})$ for which the actions is free.

PROOF: First let us clarify that what we mean in 3.12 is that four determinants $\Box^{+}_{+}$, $\Box^{-}_{+}$, $\Box^{+}_{-}$, and $\Box^{-}_{-}$ must vanish (in any row we can choose either upper or lower signs). Since the $\Delta_{ij}(\Theta) \neq 0$, by the previous Lemma, we know that the action is locally free on the $u_1 \neq 0$ part. Hence, it is enough to consider $u_1 = 0$. Here the analysis is similar to the one presented in Lemma 2.14 and it is entirely based on the fact that no quaternionic pair can vanish. As a result we get the following analogue of the fixed point equations 2.19:

3.13
$$\tau^{p_1}\rho^{q_1} = \left(\tau^{p_2}\rho^{q_2}\right)^{\pm 1}, \qquad \tau^{p_1}\rho^{q_1} = \left(\tau^{p_3}\rho^{q_3}\right)^{\pm 1}.$$

We can rewrite these as

3.14
$$\tau^{p_1 \mp p_2} = \rho^{-q_1 \pm q_2}, \qquad \tau^{p_1 \mp p_3} = \rho^{-q_1 \pm q_3}.$$

These are four systems of two equations in (τ, ρ) variables. We want all four of them to have at most discrete solutions. This requires that the four determinants

3.15
$$\det \begin{pmatrix} p_1 \mp p_2 & -q_1 \pm q_2 \\ p_1 \mp p_3 & -q_1 \pm q_3 \end{pmatrix} = -\det \begin{pmatrix} p_1 \mp p_2 & q_1 \mp q_2 \\ p_1 \mp p_3 & q_1 \mp q_3 \end{pmatrix}$$

do not vanish and gives 3.12.

The fact that orbifold singularities are always present requires more subtle analysis. To have smooth quotients we must assume

3.16
$$\Delta_{ij}(\Theta) = \pm 1, \qquad \forall\, 1 \leq i < j \leq 3,$$

one one hand, and

3.17
$$(\Box^{\mp}_{\mp}) = \begin{vmatrix} p_1 \mp p_2 & q_1 \mp q_2 \\ p_1 \mp p_3 & q_1 \mp q_3 \end{vmatrix} = \pm 1$$

on the other. A simple computation relates all of these four determinants to the 3 minor determinants $\Delta_{ij}(\Theta)$ and we get

$$\Box^{-}_{-} = \det \begin{pmatrix} p_1 - p_2 & q_1 - q_2 \\ p_1 - p_3 & q_1 - q_3 \end{pmatrix} = \Delta_{12}(\Theta) + \Delta_{23}(\Theta) - \Delta_{13}(\Theta),$$

$$\Box^{-}_{+} = \det \begin{pmatrix} p_1 - p_2 & q_1 - q_2 \\ p_1 + p_3 & q_1 + q_3 \end{pmatrix} = \Delta_{12}(\Theta) - \Delta_{23}(\Theta) + \Delta_{13}(\Theta),$$

$$\square^+_- = \det\begin{pmatrix} p_1+p_2 & q_1+q_2 \\ p_1-p_3 & q_1-q_3\end{pmatrix} = -\Delta_{12}(\Theta)-\Delta_{23}(\Theta)-\Delta_{13}(\Theta),$$

3.18 $$\square^+_+ = \det\begin{pmatrix} p_1+p_2 & q_1+q_2 \\ p_1+p_3 & q_1+q_3\end{pmatrix} = -\Delta_{12}(\Theta)+\Delta_{23}(\Theta)+\Delta_{13}(\Theta).$$

Now, because of 3.16, all three minor determinants must be ±1. This gives 8 possible combinations of the values of $\Delta_{ij}(\Theta)$. It is trivial to check that for any one out of these eight at least two determinants $\square^{\mp}_{\mp}$ will be equal to ±3 (the other six all being equal to ±1). Hence, even if we choose Θ so that 3.16 holds, the quotient will necessarily have orbifold singularities of type $\mathbb{Z}_3$. This concludes the proof of Lemma 3.11 ∎

Using the calculation in the proof of the above lemma we restate condition 3.12 to get

Theorem 3.19: *The action of $Sp(1)\times T^2_\Theta$ on $N_\nu(\Theta)$ is locally free if and only if*

(1) all their determinants $\Delta_{12}(\Theta),\Delta_{23}(\Theta),\Delta_{13}(\Theta)$ do not vanish, and

(2) their sum does not vanish, and

(3) none of the determinants is equal to the sum of the other two.

In such a case the quotient $\mathcal{M}(\Theta)$ is a compact 7-dimensional 3-Sasakian orbifold. Furthermore, there is no weight matrix Θ for which $\mathcal{M}(\Theta)$ is a smooth manifold.

Now, Theorem C of the introduction follows from Theorem 3.19, and the fact that the fundamental 3-dimensional foliation $\mathcal{M}(\Theta)\to\mathcal{O}(\Theta)$, in the case $\mathcal{M}(\Theta)$ is a compact orbifold, yields a compact self-dual Einstein orbifold with a positive scalar curvature (orbifold) metric as the space of leaves. The fact that this metric has only one Killing vector field follows from an appropriate generalization of Theorem B.

Remark 3.20: Note that both $\mathcal{M}(\Theta)$ and $\mathcal{O}(\Theta)$ depend only on the three minor determinants $\Delta_{12}(\Theta),\Delta_{23}(\Theta),\Delta_{13}(\Theta)$ rather that on Θ itself. Different weight matrices can certainly lead to equivalent quotients. One could compute the self-dual Einstein metrics $g(\Theta)$ on $\mathcal{O}(\Theta)$ explicitly. Locally we can change variables so that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A(p_1t+q_1s) & 0 & 0 \\ 0 & 0 & A(p_2t+q_2s) & 0 \\ 0 & 0 & 0 & A(p_3t+q_3s)\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A(\lambda_1) & 0 & 0 \\ 0 & 0 & A(\lambda_2) & 0 \\ 0 & 0 & 0 & A(a\lambda_1+b\lambda_2)\end{pmatrix},$$

where $a=-\frac{\Delta_{23}(\Theta)}{\Delta_{12}(\Theta)}$ and $b=\frac{\Delta_{13}(\Theta)}{\Delta_{12}(\Theta)}$ are now non-zero rational numbers. Such a choice simplifies both the 2-torus action as well as the moment map equations. The self-dual Einstein quotient metric in question (up to scale) will depend on these two parameters $g(\Theta)=g(a,b)$.

4. 15-Dimensional Examples

At first, it may appear that there are obvious higher dimensional analogues of our construction. However, a simple parity argument shows that such actions do not yield 3-Sasakian metrics other than in dimensions 7, 11, and 15. One can have orbifold metrics in any 3-Sasakian dimension. On the other hand the existence of these smooth quotient in dimension 11 and 15 is closely related to the geometry of G_2 and Spin(7).

We consider two separate cases of $U(1)_{\mathbf{p}} \subset SO(2k+1)$ and $U(1)_{\mathbf{p}} \subset SO(2k)$. In the $SO(2k+1)$ case it suffices to take $k=4$. Then $\mathbf{p}=(p_1,p_2,p_3,p_4)$ and the new $U(1)_{\mathbf{p}}$ action is defined by adding an $SO(2)$ matrix $A(p_4t)$ and let it act by rotating the additional quaternionic coordinates (u_8,u_9). To get a smooth quotient any triple $p_i<p_j<p_k$ would have to be admissible according to the definition 2.31, as is easily seen from the analysis given in Section 2. However, this is clearly impossible as admissibility implies that each triple contains two odd and one even number.

In the $SO(2k)$ case we have already seen in the previous section that $k=3$ leads to the toric 7-manifolds $\mathcal{S}(\Omega)$ with $b_2=2$. Now we will show that, in fact, $k=4$ leads to smooth quotients. Let $\mathbf{p}=(p_1,p_2,p_3,p_4)\in\mathbb{Z}^4$ and define the following homomorphism

4.1
$$f_{\mathbf{p}}(t)=\begin{pmatrix} A(p_1t) & 0 & 0 & 0\\ 0 & A(p_2t) & 0 & 0\\ 0 & 0 & A(p_3t) & 0\\ & 0 & 0 & A(p_4t)\end{pmatrix}\in SO(8),$$

where

4.2
$$A(p_it)=\begin{pmatrix}\cos(p_it) & \sin(p_it)\\ -\sin(p_it) & \cos(p_it)\end{pmatrix}\in SO(2),\qquad i=1,2,3,4$$

are 2-dimensional real rotations. As before we can choose all weights to be non-negative. Further note that at most one of the weights can vanish. We prove the following

Lemma 4.3: *Let $(p_1,p_2,p_3,p_4)\in\mathbb{Z}^4$. Then the action of $U(1)_{\mathbf{p}}\times Sp(1)$ on $N_\nu(\mathbf{p})$ is free if and only if $0\le p_1<p_2<p_3<p_4$, $\gcd(p_i,p_j,p_k)=1$ and $\gcd(p_i\pm p_j,p_i\pm p_k)=1$ for any triple in $\mathbf{p}$.*

Proof: First, let us assume that all of the weights are non-negative. Consider a triple, say (p_1,p_2,p_3). Set $u_7=u_8=0$. Then the analysis of the previous section shows that the three must be distinct and that we must have $\gcd(p_1\pm p_2,p_1\pm p_2)=1$. However, we no longer need the three weights to be pairwise relatively prime as one cannot set two of the quaternionic pairs $(u_{2i-1},u_{2i}), i=1,2,3,4$ equal to $(0,0)$ at the same time. One such quaternionic pair can vanish; hence, we need $\gcd(p_1,p_2,p_3)=1$ to get a free action. The analysis in the case when $p_1=0$ is similar. Then one sees that the triple (p_2,p_3,p_4) has to be admissible in the sense of the Definition 2.31. But that is what Lemma 2.9 says in this case. ∎

It is immediately clear that one cannot extend this construction for $k>4$ without admitting orbifold singularities in the quotient.

Theorem B follows from the above lemma except for its last statement. When $\mathbf{p}=(1,1,1,1)$ the quotient is singular. It easy to see that the action is quasi-regular just as in the 11-dimensional case. The fact that the two strata fit together giving the 3-Sasakian orbifold $\mathbb{Z}_2\backslash\mathrm{Spin}(7)/\mathrm{Spin}(4)$ can be seen by first identifying the zero level set $N_\nu(1,1,1,1)$ with the $U(1)\cdot\big(\mathrm{Spin}(7)/Sp(1)\big)$ and observing that $U(1)\cap\mathrm{Spin}(7)\simeq U(1)\cap SU(4)\simeq\mathbb{Z}_2$ [BGOP].

Actually, $\mathbf{p}=(1,1,1,1)$ is not quite the reduction considered by Kobak and Swann [KS3]. Instead they use $\mathbf{p}=(0,0,0,1)$ for the $U(1)$-action. The latter is easily seen to give $\mathrm{Gr}_4(\mathbb{R}^7)/\mathbb{Z}_2$ as the quaternionic Kähler quotient. This is a simple consequence of the fact that a "zero momentum" hyperkähler reduction of $\mathbb{H}^2$ by any circle action is isometric to

$\mathbb{H}/\Gamma$ ($\Gamma = \mathbb{Z}_2$ for the standard action). The identification of the cases $\mathbf{p} = (1,1,1,1)$ and $\mathbf{p} = (0,0,0,1)$ owes to the isomorphism between $\mathfrak{so}(6)$ and $\mathfrak{su}(4)$.

The fact that these quotients cannot be toric follows from the same type of argument as the one used for 11-dimensional quotients. The Lie algebra of the of the group of the 3-Sasakian isometries is $\mathbb{R}^3 \oplus \mathfrak{sp}(1)$.

The smooth 15-dimensional manifolds $\mathcal{M}(\mathbf{p})$ for $p_1 = 0$ and $p_1 > 0$ are geometrically different. In the first case $\mathcal{M}(\mathbf{p})$ contains two copies of the 11-dimensional 3-Sasakian manifold $\mathcal{M}(p_2,p_3,p_4)$ which intersect in the 7-dimensional toric 3-Sasakian submanifold $\mathcal{S}(\Omega(p_2,p_3,p_4))$. When $p_1 > 0$ then $\mathcal{M}(\mathbf{p})$ does not have any obvious 11-dimensional 3-Sasakian submanifolds. However, we do get 4 disjoint toric 7-dimensional 3-Sasakian submanifolds $\mathcal{S}(\Omega(p_1,\ldots,\hat{p}_i,\ldots,p_4))$ by setting one of the quaternionic pairs $(u_{2i-1},u_{2i}) = (0,0)$.

We can also consider non-zero momentum $\xi \in \mathfrak{sp}(1)$-deformations of the hyperkähler metric on the cones $C(\mathcal{M}(\mathbf{p}))$. (Up to scale one can set ξ to be any imaginary quaternion, so we really have just one parameter family). In some sense they are all deformations of Kronheimer's hyperkähler metrics on the two nilpotent orbits in $\mathfrak{sl}(3,\mathbb{C})$ and $\mathfrak{so}(6,\mathbb{C})$. Unfortunately such hyperkähler quotients are rarely free from orbifold singularities. In fact, in the 12-dimensional case we never get any complete metrics. In 16 dimensions we do get a complete metric only when $\mathbf{p} = (1,1,1,1)$ (or equivalently $\mathbf{p} = (0,0,0,1)$). The metric is $SU(4)$-invariant and it gives the Kronheimer metric on the 16-dimensional nilpotent orbit of $\mathfrak{so}(6,\mathbb{C})$ in the $\xi \to 0$ scaling limit. More generally, cohomogeneity 2 metrics were studied by Kobak and Swann [KS4].

As each classical nilpotent orbit is a hyperkähler reduction of some quaternionic vector space [KS2] it is tempting to undertake a more systematic study of the following problem: Which of the nilpotent orbits can give rise to compact 3-Sasakian manifolds? Certainly, any time a quotient involves some $U(1)$-factor one can introduce weights. However, as demonstrated here, requiring smoothness often puts very severe restrictions on the weights. We plan to address some of these questions in a future work [BGOP].

Remark 4.4: Note that all of the quotients considered in this paper are examples of toric reductions of the 3-Sasakian homogeneous space associated to the real Grassmanian $Gr_4(\mathbb{R}^n)$. This space is $SO(n)/SO(n-4)\times Sp(1)$ and it has $SO(n)$ as the group of isometries preserving the 3-Sasakian structure. Consider the maximal torus $T^l \subset SO(n)$. Then the relevant question is: Which subgroups $T^m \subset T^l$ yield smooth 3-Sasakian quotients? Sections 2 and 3 give the complete analysis of the $n = 2k+1 = 7$ case. In Section 2 $m = 1$ and in Section 3 $m = 2$ which exhaust all interesting possibilities. This section considers $n = 2k = 8$ case with $m = 1$. One can see that $m > 1$ does not yield any smooth quotient but we leave the analysis of this to a future work where we plan to give a complete answer in the most general case of arbitrary (m,n) [BGP].

5. Comments on the Topology of $\mathcal{M}(\mathbf{p})$ and Related Spaces

In this section we denote by $\mathcal{M}(\mathbf{p})$ either one of the 11 or 15 dimensional 3-Sasakian manifolds discussed in sections 2 and 4 with $\mathbf{p}$ admissible. Actually as discussed below $\mathcal{M}(\mathbf{p})$ will denote a component of the manifolds discussed previously. It would be interesting to know the topology of our quotients, most importantly $\pi_1(\mathcal{M}(\mathbf{p}))$ and $H_2(\mathcal{M}(\mathbf{p}),\mathbb{Z})$. For this one needs to understand the topology of the level set of the moment map $N_\nu(\mathbf{p})$. Of course, we do know that $\pi_1(\mathcal{M}(\mathbf{p}))$ is finite and that the odd Betti numbers of $\mathcal{M}(\mathbf{p})$ vanish up to the middle dimension [GS]. However, beyond this not much explicit topological information can be obtained. For example, so far we have been unable to determine whether $\mathcal{M}(\mathbf{p})$ and $N_\nu(\mathbf{p})$ are even connected. This presents no real problem as we shall always mean by these spaces connected components such that $N_\nu(\mathbf{p})$ is a $S^1\times Sp(1)$ bundle over $\mathcal{M}(\mathbf{p})$. Generally, the determination of the topology of an intersection of real quadrics such as $N_\nu(\mathbf{p})$ is quite complicated. The analysis in previous work [BGM1] and [BGMR] relied heavily on very specialized information. In the former case the level sets in question were diffeomorphic to certain Stiefel manifolds whose topology is completely understood, whereas in the later case there was a large symmetry group whose quotient was a two dimensional space whose topology could be analysized. In the present case we meet with no such good fortune. Nevertheless, we are able to obtain a small amount of general information about the topology of $N_\nu(\mathbf{p})$. Our first result is that for $\mathbf{p}$ and $\mathbf{p}'$ admissible the level sets $N_\nu(\mathbf{p})$ and $N_\nu(\mathbf{p}')$ are diffeomorphic. This does not hold generally as the case of $N_\nu(1,1,1)$ shows. In this case the Jacobian matrix drops rank making the quotient singular.

Lemma 5.1: *For $\mathbf{p}$ and $\mathbf{p}'$ admissible, the level sets $N_\nu(\mathbf{p})$ and $N_\nu(\mathbf{p}')$ are diffeomorphic.*

Proof: For $\mathbf{p}$ admissible there are no fixed points of the $U(2)$ action so by the 3-Sasakian version of a well known result in symplectic geometry, zero is a regular value of the $U(2)$ moment map

$$\tilde{\nu}_{\mathbf{p}}:S^{27}\longrightarrow\mathbb{R}^{12}=\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3$$

defined by $\tilde{\nu}_{\mathbf{p}}=(\nu_{\mathbf{p}},\mu_i,\mu_j,\mu_k)$. Now the level sets $N_\nu(\mathbf{p})$ are defined for all $\mathbf{p}\in\mathbb{R}^3$. Moreover, for any non-zero $\rho\in\mathbb{R}$ we see that level sets of $\tilde{\nu}_{\rho\mathbf{p}}$ and $\tilde{\nu}_{\mathbf{p}}$ coincide, that is $N_\nu(\rho\mathbf{p})=N_\nu(\mathbf{p})$. Thus, by scaling we can choose ρ such that $\tilde{\nu}_{\rho\mathbf{p}'}$ is in an ϵ-neighborhood of $\tilde{\nu}_{\rho\mathbf{p}}$ in $C^\infty(S^{27},\mathbb{R}^{12})$ with the C^∞ compact-open topology. Since zero is a regular value of both $\tilde{\nu}_{\rho\mathbf{p}}$ and $\tilde{\nu}_{\rho\mathbf{p}'}$, it follows from a well known theorem (cf. [BCR], 14.1.1) that $N_\nu(\mathbf{p})$ and $N_\nu(\mathbf{p}')$ are diffeotopic, that is there is a one parameter family of diffeomorphisms $\phi_t:S^{27}\longrightarrow S^{27}$ parameterized by the unit interval such that ϕ_0 is the identity and ϕ_1 takes $N_\nu(\mathbf{p}')$ diffeomorphically to $N_\nu(\mathbf{p})$. ∎

This immediately implies that the homotopy groups as well as the cohomology rings of $N_\nu(\mathbf{p})$ and $N_\nu(\mathbf{p}')$ are isomorphic.

Now once and for all we shall choose a component of $N_\nu(\mathbf{p})$ and the corresponding component of $\mathcal{M}(\mathbf{p})$ so that $N_\nu(\mathbf{p})$ is an $S^1\times Sp(1)$ bundle over $\mathcal{M}(\mathbf{p})$ with both base space and total space connected. We also denote by $\mathcal{H}(\mathbf{p})$ the circle bundle over $\mathcal{M}(\mathbf{p})$ that coincides with the quotient of the corresponding component of $N_\nu(\mathbf{p})$ by the $Sp(1)$ action. We now study the latter as a fibration, namely $S^3\longrightarrow N_\nu(\mathbf{p})\xrightarrow{\pi}\mathcal{H}(\mathbf{p})$. Note that since ϕ_1 in Lemma 5.1 is not necessarily a bundle map, we cannot claim that the manifolds $\mathcal{H}(\mathbf{p})$ and $\mathcal{H}(\mathbf{p}')$ are diffeomorphic. Nevertheless, we can obtain some useful information about their cohomology and homotopy groups. The manifolds $\mathcal{H}(\mathbf{p})$ are of interest in their own right since as discussed briefly in the next section they admit hypercomplex structures.

Consider the following commutative diagram of Gysin sequences with $\mathbb{Z}$ coefficients:

5.2

$$\begin{tikzcd}
H^{r+3}(N_\nu(\mathbf{p})) \arrow[r] \arrow[d] & H^r(\mathcal{H}(\mathbf{p})) \arrow[r, "\cup\chi"] & H^{r+4}(\mathcal{H}(\mathbf{p})) \arrow[r, "\pi^*"] & H^{r+4}(N_\nu(\mathbf{p})) \arrow[r] \arrow[d] & H^{r+1}(\mathcal{H}(\mathbf{p})) \\
H^{r+3}(N_\nu(\mathbf{p}')) \arrow[r] & H^r(\mathcal{H}(\mathbf{p}')) \arrow[r, "\cup\chi"] & H^{r+4}(\mathcal{H}(\mathbf{p}')) \arrow[r, "\pi^*"] & H^{r+4}(N_\nu(\mathbf{p}')) \arrow[r] & H^{r+1}(\mathcal{H}(\mathbf{p}')),
\end{tikzcd}$$

where the two vertical arrows are isomorphisms by Lemma 5.1, and $\cup\chi$ denotes cupping by the Euler class of the bundle. The idea is to construct, as best as possible, the missing vertical arrows and relate the cohomology of $\mathcal{H}(\mathbf{p})$ and $\mathcal{H}(\mathbf{p}')$ by the Five Lemma. First we notice that setting $r=-3$ and -2 and using Lemma 5.1 gives isomorphisms

$$H^1(\mathcal{H}(\mathbf{p}),\mathbb{Z})\approx H^1(N_\nu(\mathbf{p}),\mathbb{Z})\approx H^1(N_\nu(\mathbf{p}'),\mathbb{Z})\approx H^1(\mathcal{H}(\mathbf{p}'),\mathbb{Z})$$

$$H^2(\mathcal{H}(\mathbf{p}),\mathbb{Z})\approx H^2(N_\nu(\mathbf{p}),\mathbb{Z})\approx H^2(N_\nu(\mathbf{p}'),\mathbb{Z})\approx H^2(\mathcal{H}(\mathbf{p}'),\mathbb{Z})$$

Next by setting $r=-1$ we have

5.3
$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "="] & H^3(\mathcal{H}(\mathbf{p})) \arrow[r, "\pi^*"] & H^3(N_\nu(\mathbf{p})) \arrow[r] \arrow[d, "\phi^*"] & H^0(\mathcal{H}(\mathbf{p})) \arrow[d, "\approx"] \\
0 \arrow[r] & H^3(\mathcal{H}(\mathbf{p}')) \arrow[r, "(\pi')^*"] & H^3(N_\nu(\mathbf{p}')) \arrow[r] & H^0(\mathcal{H}(\mathbf{p}'))
\end{tikzcd}$$

Since both π^* and $(\pi')^*$ are injective, ϕ^* is an isomorphism, and the diagram is exact and commutative, we can define the missing vertical arrow by $\psi=((\pi')^*)^{-1}\circ\phi^*\circ\pi^*$ and it is an isomorphism. We thus have

$$H^3(\mathcal{H}(\mathbf{p}),\mathbb{Z})\approx H^3(\mathcal{H}(\mathbf{p}'),\mathbb{Z}).$$

Now generally we cannot construct the missing vertical maps; however, we can construct them if the groups are free. We thus change to rational coefficients $\mathbb{Q}$.

Consider now diagram 5.2 for $r=0$. We can now fill in the second and last columns with vertical arrows that are isomorphisms. Now considering the diagram with rational coefficients, we can split the middle groups, for all admissable $\mathbf{p}$, as

$$H^4(\mathcal{H}(\mathbf{p}),\mathbb{Q})\approx \mathrm{im}(\cup\chi)\oplus\mathrm{coker}(\cup\chi).$$

Choosing bases for these groups we can define the middle vertical map simply by sending a basis element in $\mathrm{im}(\cup\chi)\subset H^4(\mathcal{H}(\mathbf{p}),\mathbb{Q})$ to a basis element in $\mathrm{im}(\cup\chi)\subset H^4(\mathcal{H}(\mathbf{p}'),\mathbb{Q})$, and a basis element in $\mathrm{coker}(\cup\chi)\subset H^4(\mathcal{H}(\mathbf{p}),\mathbb{Q})$ to a basis element in $\mathrm{coker}(\cup\chi)\subset H^4(\mathcal{H}(\mathbf{p}'),\mathbb{Q})$. That these subspaces have the same dimension making this possible follows from exactness and commutativity of the diagram. It then follows from the Five Lemma that this middle arrow is an isomorphism. This argument is general and using a simple induction we arrive at

THEOREM 5.4: *For admissible* $\mathbf{p}$ *and* $\mathbf{p}'$ *we have*

(i) $H^r(\mathcal{H}(\mathbf{p}),\mathbb{Z})\approx H^r(\mathcal{H}(\mathbf{p}'),\mathbb{Z})$ *for* $r=0,1,2,3$.

(ii) $b_r(\mathcal{H}(\mathbf{p}))=b_r(\mathcal{H}(\mathbf{p}'))$ *for all* r.

In particular, $\mathcal{H}(\mathbf{p})$ *and* $\mathcal{H}(\mathbf{p}')$ *have isomorphic rational cohomology groups.*

Here $b_r(M)$ denotes the rth Betti number of M. Similar arguments can be used for the homotopy groups, and combining these results with well known facts about circle bundles over 3-Sasakian manifolds [BGM3] we obtain

PROPOSITION 5.5: *For* $\mathbf{p}$ *and* $\mathbf{p}'$ *admissible, we have isomorphisms:*

(i) $\pi_i(\mathcal{H}(\mathbf{p}))\approx\pi_i(N_\nu(\mathbf{p}))\approx\pi_i(\mathcal{H}(\mathbf{p}'))$ *for* $i=0,1,2$.

(ii) $H^1(\mathcal{H}(\mathbf{p}),\mathbb{Z})\approx H^1(N_\nu(\mathbf{p}),\mathbb{Z})\approx H^1(\mathcal{H}(\mathbf{p}'),\mathbb{Z})\approx 0$ *or* $\mathbb{Z}$.

(iii) $H^2(\mathcal{H}(\mathbf{p}),\mathbb{Z})\approx H^2(N_\nu(\mathbf{p}),\mathbb{Z})$.

Next we consider our manifolds of primary interest, namely $\mathcal{M}(\mathbf{p})$. First, it is easy to see the following relations:

PROPOSITION 5.6: *For admissible* $\mathbf{p}$ *we have:*

(i) $\pi_i(\mathcal{H}(\mathbf{p}))\approx\pi_i(\mathcal{M}(\mathbf{p}))$ *for all* $i>2$.

(ii) $b_2(N_\nu(\mathbf{p}))=\begin{cases} b_2(\mathcal{M}(\mathbf{p}))-1 & \text{if } b_1(N_\nu(\mathbf{p}))=0;\\ b_2(\mathcal{M}(\mathbf{p})) & \text{if } b_1(N_\nu(\mathbf{p}))=1.\end{cases}$

Next we have the analogue of Theorem 5.4 for our 3-Sasakian manifolds $\mathcal{M}(\mathbf{p})$.

THEOREM 5.7: *For admissible* $\mathbf{p}$ *and* $\mathbf{p}'$ *we have*

(i) $H^r(\mathcal{M}(\mathbf{p}),\mathbb{Z})\approx H^r(\mathcal{M}(\mathbf{p}'),\mathbb{Z})$ *for* $r=0,1,2,3$.

(ii) $b_r(\mathcal{M}(\mathbf{p}))=b_r(\mathcal{M}(\mathbf{p}'))$ *for all* r.

In particular, $\mathcal{M}(\mathbf{p})$ *and* $\mathcal{M}(\mathbf{p}')$ *have isomorphic rational cohomology groups.*

PROOF: The proof of this Theorem is analagous to the proof of Theorem 5.4 with diagram 5.2 replaced by the following commutative diagram of Gysin sequences:

$$\begin{tikzcd} H^{r+1}(\mathcal{H}(\mathbf{p})) \arrow[r] \arrow[d] & H^r(\mathcal{M}(\mathbf{p})) \arrow[r, "\cup\chi"] & H^{r+2}(\mathcal{M}(\mathbf{p})) \arrow[r] & H^{r+2}(\mathcal{H}(\mathbf{p})) \arrow[r] \arrow[d] & H^{r+1}(\mathcal{M}(\mathbf{p})) \\ H^{r+1}(\mathcal{H}(\mathbf{p}')) \arrow[r] & H^r(\mathcal{M}(\mathbf{p}')) \arrow[r, "\cup\chi"] & H^{r+2}(\mathcal{M}(\mathbf{p}')) \arrow[r] & H^{r+2}(\mathcal{H}(\mathbf{p}')) \arrow[r] & H^{r+1}(\mathcal{M}(\mathbf{p}')), \end{tikzcd}$$

where we have used Theorem 5.4 and its proof to construct the isomorphisms indicated by the vertical arrows. ∎

REMARK 5.8: Actually we can weaken the hypothesis that $\mathbf{p}$ be admissible by noting any of the results of this section concerning rational cohomology hold in the cases when $\mathcal{M}(\mathbf{p})$ is an orbifold obtained as the quotient by a locally free action. It follows from the analysis in section 2, that the action is locally free precisely when the components of $\mathbf{p}$ are all distinct, and in this case the level sets $\mathcal{H}(\mathbf{p})$ are smooth manifolds; hence, Theorem 5.4 and Proposition 5.5 hold in this case as well. It is interesting to note that in the case that $\mathcal{M}(\mathbf{p})$ is an orbifold, but not a smooth manifold, the smooth manifold $\mathcal{H}(\mathbf{p})$ cannot be the trivial V-bundle. The above remarks apply equally as well to the 7-dimensional orbifolds $\mathcal{M}(\Theta)$ constructed in section 3 with the condition for a locally free action being that all the minor determinants $\Delta_{ij}(\Theta)$ are nonvanishing.

Finally we briefly discuss the two singular cases

$$\mathcal{M}(1,1,1)\simeq\mathbb{Z}_3\backslash G_2/Sp(1)\quad\text{and}\quad\mathcal{M}(1,1,1,1)\simeq\mathbb{Z}_2\backslash\mathrm{Spin}(7)/\mathrm{Spin}(4).$$

Since these are biquotients of Lie groups the topology is more accessible. In particular, their rational cohomology is that of the corresponding 3-Sasakian homogeneous space, $G_2/Sp(1)$ and $\mathrm{Spin}(7)/\mathrm{Spin}(4)$, respectively, which is well known [GS,BG2]. Thus, $\mathcal{M}(1,1,1)$ has the rational cohomology of S^{11}, whereas $\mathcal{M}(1,1,1,1)$ has the rational cohomology of $S^4\times S^{11}$. In both cases b_2 vanishes, and we do not expect this in the non-singular cases.

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