Contact Geometry and Einstein Manifolds

My talk is based on joint work with Krzysztof Galicki.

We consider a Riemannian manifold (M, g). On any manifold a linear connection ∇ gives a way of comparing tangent vectors at different points through the notion of parallel transport.

The First Fundamental Theorem of Riemannian geometry says that there exists a unique torsionfree connection that leaves the metric tensor g invariant, i.e. $\nabla g = 0$.

The Riemann curvature tensor R measures the "curvature" of (M, g). It is given by

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$$

where X, Y, Z are vector fields on M.

Two important irreducible components of the Riemann curvature tensor R are: the Ricci curvature

$$\mathrm{Ric}(X,Y) = \mathrm{Tr}\ (Z \mapsto R(X,Z)Y)$$

which is a symmetric tensor field and the scalar curvature

$$s_g = \mathrm{Tr}\ \mathrm{Ric}$$

which is a function on M.

Important Question (R. Thom): Is there a distinquished class or best class of Riemannian manifolds?

Answer: Perhaps the answer is the class of:

Einstein manifolds.

DEFINITION: *A Riemannian manifold (M, g) is called an Einstein manifold if there exists a smooth function λ on M such that*

$$Ric = \lambda g.$$

The Bianchi identities imply that λ is actually a constant, called the *Einstein constant.* So there are 3 cases

1. $\lambda > 0$.
2. $\lambda = 0$.
3. $\lambda < 0$.

There are two reasons why one might believe that the above answer is correct.

1. Einstein manifolds are the critical points of the "Hilbert Action" given by the total scalar curvature

$$S(g) = \int s_g \mu_g$$

where the measure μ_g is normalized to volume one, i.e. $\int \mu_g = 1$.

2. The class of Einstein manifolds is a natural generalization of the class of Riemannian manifolds of constant curvature. For dimension 2 the well known Uniformization Theorem says that every orientable surface admits a metric with curvature either $1, 0$, or -1. These are automatically Einstein. In higher dimensions the condition of constant curvature is too strong, and the Einstein condition is a natural generalization.

Here we shall be interested in the case $\lambda > 0$ in which case we refer to (M, g) as a *positive* Einstein manifold. Then we have the well-known Myers' Theorem which implies any complete positive Einstein manifold (M, g) is compact with finite fundamental group.

Our purpose is the construct explicit examples of positive Einstein manifolds and discuss their topology. In order to do this, we employ two constructions:

1. The method of symmetry reduction combined with a certain additional structure.
2. A construction due to Wang and Ziller of obtaining Einstein metrics on certain torus bundles over Kähler-Einstein manifolds.

In order to get a handle on the geometry we add an additional structure– a type of **Contact Geometry** known as **Sasakian Geometry**. A good reference here is David Blair's book, *Contact Manifolds in Riemannian Geometry.* Although we are interested in Einstein *manifolds* our methods necesitate working with *orbifolds.* In particular, we generalize the Wang-Ziller construction in the case of circle bundles to circle V-bundles over Kähler-Einstein orbifolds.

DEFINITION: *A smooth manifold M is called a* contact manifold *or is said to have a* contact structure *if there exists on M a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Contact manifolds are necessarily odd dimensional and η is called the* characteristic *or* contact *form.*

REMARK: In the complex case it is convenient to generalize this so that η is a 1-form with coefficients in a line bundle or more generally for orbifolds a line V-bundle. In the real case this is a generalization only when M is not orientable.

Definition: *A smooth manifold M is called an almost contact manifold if the structure group reduces to $U(n) \times \mathbb{I}$.*

The following propositions are well-known:

Proposition: *A contact manifold is an almost contact manifold.*

Proposition: *A smooth manifold M is an almost contact manifold if and only if there exist on M a vector field ξ, a 1-form η, and a section Φ of the endomorphism bundle End M such that*

$$\eta(\xi) = 1$$

and

$$\Phi^2 = -\mathbb{I} + \eta \otimes \xi.$$

We need some more definitions:

Let M be an almost contact manifold, then a Riemannian metric g on M is said to be *compatible* with the almost contact structure if

$$g \circ (\Phi \otimes \Phi) = g - \eta \otimes \eta.$$

Such compatible Riemannian metrics always exist.

Definition: *An almost contact manifold (M, η, ξ, Φ) is said to be normal if*

$$[\Phi, \Phi] + 2d\eta \otimes \xi = 0$$

where

$$[\Phi, \Phi](X, Y) = [\Phi X, \Phi Y] + \Phi^2[X, Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y]$$

defines the Nijenhuis bracket.

Definition: *A normal contact manifold with a compatible metric (M, g, η, ξ, Φ) is called a Sasakian manifold.*

Sasakian geometry was defined by Sasaki and was further developed by the Japaneese school during the 1960's and 1970's. There are many equivalent ways to characterize Sasakian manifolds and each has its own advantage. We have

THEOREM: *On a Riemannian manifold* $(\mathcal{S}, g)$ *the following conditions are equivalent:*

(i) *There exists a Killing vector field* ξ *of unit length on* $\mathcal{S}$ *so that the tensor field* Φ *of type* $(1,1)$*, defined by* $\Phi(X) = \nabla_X \xi$*, satisfies the condition*

$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi$$

for any pair of vector fields X *and* Y *on* $\mathcal{S}$.

(ii) *There exists a Killing vector field* ξ *of unit length on* $\mathcal{S}$ *so that the Riemann curvature satisfies the condition*

$$R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi,$$

for any pair of vector fields X *and* Y *on* $\mathcal{S}$.

(iii) *There exists a Killing vector field* ξ *of unit length on* $\mathcal{S}$ *so that the sectional curvature of every section containing* ξ *equals one.*

(iv) *The holonomy group of the Riemannian cone on* $\mathcal{S}$,

$$(C(\mathcal{S}), \bar{g}) = (\mathbb{R}_+ \times \mathcal{S},\ dr^2 + r^2 g)$$

reduces to a subgroup of $U(n+1)$*, where* $\dim \mathcal{S} = 2n+1$*, that is,* $(C(\mathcal{S}), \bar{g})$ *is Kähler.*

(v) $(\mathcal{S}, g)$ *is Sasakian.*

DEFINITION: *A Sasakian manifold* $(\mathcal{S}, g)$ *is called* Sasakian-Einstein *if* g *is also Einstein.*

Proposition(B-Galicki): *A Riemannian manifold of dimension $2n+1$ is Sasakian-Einstein if and only if the restricted holonomy group of the Riemannian cone is contained in $SU(n+1)$.*

On a Sasakian manifold the Ricci tensor satisfies

$$Ric(X,\xi) = 2n\eta(X).$$

It follows immediately that an Sasakian-Einstein manifold necessarily has positive scalar curvature equal to $2n(2n+1)$ and that any complete Sasakian-Einstein manifold is compact with finite fundamental group.

Let $\mathcal{S}$ be a Sasakian manifold. Since ξ is nowhere vanishing it defines canonically a 1-dimensional foliation $\mathcal{F}$ on $\mathcal{S}$. We are interested in the case that all the leaves of $\mathcal{F}$ are compact. The assumption that all leaves are compact is equivalent to the assumption that $\mathcal{F}$ is *quasi-regular*, i.e. each point $p \in \mathcal{S}$ has a cubical neighborhood U such that any leaf $\mathcal{L}$ of $\mathcal{F}$ intersects a transversal through p at most a finite number of times $N(p)$. Furthermore, $\mathcal{S}$ is called *regular* if $N(p) = 1$ for all $p \in \mathcal{S}$. In this case, the foliation $\mathcal{F}$ is simple, and defines a global submersion. In the quasi-regular case it is well-known that ξ generates a locally free circle action on $\mathcal{S}$, and that the space of leaves is a compact orbifold (or V-manifold). We shall denote the space of leaves of the foliation $\mathcal{F}$ on $\mathcal{S}$ by $\mathcal{Z}$. Then the natural projection $\pi : \mathcal{S} \longrightarrow \mathcal{Z}$ is an orbifold submersion and a Siefert fibration. Actually, much more is true as we show below.

Recall that an orbifold or V-manifold (Satake) is very similar to a manifold except that the local Euclidean condition is replaced by neighborhoods U that are homeomorphic to $\mathbb{R}^n/\Gamma$ where Γ is a finite group that can always be taken to be a subgroup of $O(n)$. Furthermore, Γ can be taken to be the identity on a dense open set. Then (U,Γ) together with the homeomorphism is called a *local uniformizing system.* Much of the standard differential geometry on manifolds carries over to the orbifold category.

STRUCTURE THEOREM (B-Galicki): *Let $(\mathcal{S},g)$ be a complete quasi-regular Sasakian-Einstein manifold of dimension $2n+1$. Then*

(i) The leaf space $\mathcal{Z}$ is a compact complex orbifold with a Kähler-Einstein metric h of positive scalar curvature $4n(n+1)$ in such a way that $\pi:(\mathcal{S},g)\longrightarrow(\mathcal{Z},h)$ is an orbifold Riemannian submersion. The fibres of π are totally geodesic submanifolds of $\mathcal{S}$ diffeomorphic to S^1.

(ii) $\mathcal{Z}$ is also a simply connected normal projective algebraic variety. In fact, it is a $\mathbb{Q}$-factorial Fano variety. Hence, it is uniruled with Kodaira dimension $\kappa(\mathcal{Z})=-\infty$.

Recall that a *Fano variety* means that the anti-canonical sheaf K^{-1} is ample, $\mathbb{Q}$-factorial means that for every Weil divisor D on $\mathcal{Z}$ there is an integer m such that mD is a Cartier divisor.

Since an orbifold fibration is not a fibration in the usual sense, the usual techniques in topology for fibrations do not apply directly. However, Haefliger [Hae] has defined orbifold homology, cohomology, and homotopy groups which do have an analogue in the standard theory. Let X be an orbifold of dimension n, and let P denote the bundle of orthonormal frames on X. It is a smooth manifold on which the orthogonal group $O(n)$ acts locally freely with the quotient X. Let $EO(n)\longrightarrow BO(n)$ denote the universal $O(n)$ bundle. Consider the diagonal action of $O(n)$ on $EO(n)\times P$ and denote the quotient by BX. Now there is a natural projection $p:BX\longrightarrow X$ with generic fiber the contractible space $EO(n)$.

Haefliger defines the orbifold cohomology, homology, and homotopy groups by

$$H^i_{orb}(X,\mathbb{Z}) = H^i(BX,\mathbb{Z}), \quad H_i^{orb}(X,\mathbb{Z}) = H_i(BX,\mathbb{Z}),$$

$$\pi_i^{orb}(X) = \pi_i(BX).$$

This definition of π_1^{orb} is equivalent to Thurston's better known definition in terms of orbifold deck transformations, and when X is a smooth manifold these orbifold groups coincide with the usual groups. Using the Leray spectral sequence one easily sees that the map $p : BX \longrightarrow X$ induces an isomorphism $H^i_{orb}(\mathcal{S},\mathbb{Z})\otimes\mathbb{Q} \simeq H^i(\mathcal{S},\mathbb{Z})\otimes\mathbb{Q}$.

Definition: *Let $c_1(\mathcal{L})$ denote the first Chern class of a line V-bundle $\mathcal{L}$. The index of a Fano orbifold $\mathcal{Z}$ is the largest positive integer m such that $\frac{c_1(K_{\mathcal{Z}}^{-1})}{m}$ is an element of $H^2_{orb}(\mathcal{Z},\mathbb{Z})$. The index of $\mathcal{Z}$ is denoted by Ind$(\mathcal{Z})$.*

Definition: *The index of a complete Sasakian-Einstein manifold $\mathcal{S}$ is defined to be the index of $\mathcal{Z}$ if $\mathcal{S}$ is quasi-regular, and 0 if $\mathcal{S}$ is not quasi-regular. The index is denoted by Ind$(\mathcal{S})$.*

Ind$(\mathcal{S})$ is an invariant of the Sasakian-Einstein structure, and we have:

Theorem(B-Galicki): *The index Ind$(\mathcal{S})$ of an Sasakian-Einstein manifold $\mathcal{S}$ of dimension $2n+1$ satisfies the inequality Ind$(\mathcal{S}) \leq 2n+2$ and equals $2n+2$ if and only if the universal cover $\tilde{\mathcal{S}}$ of $\mathcal{S}$ is the sphere S^{2n+1} with its standard Sasakian structure.*

We now have:

INVERSION THEOREM (B-Galicki): *Let $\mathcal{Z}$ be a compact Fano orbifold of complex dimension n with $\pi_1^{orb}(\mathcal{Z})=0$ and a Kähler-Einstein metric h of scalar curvature $4n(n+1)$. Let $\mathcal{S}$ denote the total space of the S^1 V-bundle on $\mathcal{Z}$ whose first Chern class is $\frac{c_1(\mathcal{Z})}{\mathrm{Ind}(\mathcal{Z})}$. Then $(\mathcal{S},g)$ is a Sasakian-Einstein orbifold with $g=\pi^*h+\eta\otimes\eta$ and where η is the connection 1-form in the V-bundle $\pi:\mathcal{S}\longrightarrow\mathcal{Z}$ whose curvature is the Kähler 2-form on $\mathcal{Z}$. Furthermore, if the local uniformizing groups Γ_i inject into the group S^1 of the bundle $\mathcal{S}$ is a simply connected smooth manifold.*

In the regular case when $\mathcal{Z}$ is a smooth manifold this theorem is due to Baum, Friedrich, Grunewald, and Kath. Using these results we can easily determine the homogeneous Sasakian-Einstein manifolds:

THEOREM (B-Galicki): *Let $\mathcal{S}$ be a quasi-regular homogeneous Sasakian-Einstein manifold. Then $\mathcal{S}$ is an S^1-bundle over a generalized flag manifold G/P. Conversely, given any generalized flag manifold G/P there is a circle bundle $\pi:\mathcal{S}\longrightarrow G/P$ whose total space $\mathcal{S}$ is a homogeneous Sasakian-Einstein manifold.*

An important special case is what is known as a *3-Sasakian structure.*

DEFINITION: *Let $(\mathcal{S},g)$ be a Riemannian manifold of real dimension m. We say that $(\mathcal{S},g)$ is 3-Sasakian if the holonomy group of the Riemannian cone on $\mathcal{S}$ reduces to a subgroup of $Sp(\frac{m+1}{4})$. In particular, $m=4n+3, n\geq 1$ and $(C(\mathcal{S}),\bar{g})$ is hyperkähler.*

3-Sasakian manifolds are manifolds with a certain metric quaternionic contact structure. They were first defined independently by Kuo and Udriste around 1970 using tensorial methods. Essentially there is a triple of Sasakian structures which satisfy certain quaternionic-like relations.

Shortly thereafter, Kashiwada proved that every 3-Sasakian manifold is Einstein with positive scalar curvature. Other important properties of 3-Sasakian manifolds were obtained early to mid 1970's by Ishihara, Konishi, Sasaki, Tachibana, Tanno, and Yu. Nevertheless, the only examples known until fairly recently were spheres and their quotients. Recently the authors together with B. Mann and E. Rees discovered many infinite sequences of 3-Sasakian manifolds with highly non-trivial topology.

Proposition: *Let $(\mathcal{S}, g)$ be a Riemannian manifold Then $\mathcal{S}$ is 3-Sasakian if and only if it admits three characteristic vector fields $\{\xi^1, \xi^2, \xi^3\}$ of 3 different Sasakian structures such that $g(\xi^a, \xi^b) = \delta_{ab}$ and $[\xi^a, \xi^b] = 2\epsilon_{abc}\xi^c$.*

From the point of view of the holonomy group of the cone we see that since $Sp(n+1) \subset SU(2n+2)$ a 3-Sasakian manifold is automatically Sasakian-Einstein. A 3-Sasakian manifold necessarily has relatively high index.

Theorem (B-Galicki): *The index $Ind(\mathcal{S})$ of a 3-Sasakian manifold of dimension $4n+3$ is either $n+1$ or $2n+2$ and equals $2n+2$ if and only if the universal cover $\tilde{\mathcal{S}}$ of $\mathcal{S}$ is a sphere S^{4n+3}.*

It should be mentioned that there are Sasakian-Einstein manifolds of dimension $4n+3$ and index $n+1$ that are not 3-Sasakian, for example, the Sasakian-Einstein circle bundle over $\mathbb{CP}^n \times Q_{n+1}(\mathbb{C})$ where $Q_{n+1}(\mathbb{C})$ is the complex quadric of dimension $n+1$.

The Quotient Construction

Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold with a nontrivial group $I_0(\mathcal{S}, g)$ of automorphisms of the 3-Sasakian structure. Now $I_0(\mathcal{S}, g)$ extends to a group of hyperkähler automorphisms $I_0(\mathcal{S}, \bar{g})$ on the cone $C(\mathcal{S})$ by defining each element to act trivially on $\mathbb{R}^+$. Recall that the hyperkähler quotient of Hitchin, Karlhede, Linström, and Roček shows that any subgroup $G \subset I_0(M, \bar{g})$ gives rise to a hyperkähler moment map $\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$, where $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}^*$ is its dual. Thus, we can define a 3-Sasakian moment map

$$\mu_{\mathcal{S}} : \mathcal{S} \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

simply by restriction $\mu_{\mathcal{S}} = \mu | \mathcal{S}$. Such a moment map takes the simple form

$$< \mu^a_{\mathcal{S}}, \tau > = \frac{1}{2}\eta^a(X^\tau).$$

Then we have

THEOREM (B-G-M): *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold with a connected compact Lie group G acting on $\mathcal{S}$ smoothly and properly by 3-Sasakian isometries. Let $\mu_{\mathcal{S}}$ be the corresponding 3-Sasakian moment map and assume both that 0 is a regular value of $\mu_{\mathcal{S}}$ and that G acts freely on the submanifold $\mu^{-1}_{\mathcal{S}}(0)$. Furthermore, let $\iota : \mu^{-1}_{\mathcal{S}}(0) \longrightarrow \mathcal{S}$ and $\pi : \mu^{-1}_{\mathcal{S}}(0) \longrightarrow \mu^{-1}_{\mathcal{S}}(0)/G$ denote the corresponding embedding and submersion. Then $(\mathcal{S}/\!/\!/G = \mu^{-1}_{\mathcal{S}}(0)/G, \check{g})$ is a smooth 3-Sasakian manifold of dimension $4(n - \dim \mathfrak{g}) + 3$ with metric $\check{g}$ and Sasakian vector fields $\check{\xi}^a$ determined uniquely by the two conditions $\iota^* g = \pi^* \check{g}$ and $\pi_*(\xi^a | \mu^{-1}_{\mathcal{S}}(0)) = \check{\xi}^a$.*

3-Sasakian Toric Manifolds

We can use the 3-Sasakian reduction theorem to obtain new 3-Sasakian manifolds by toral reductions from the 3-Sasakian sphere S^{4n+3}. On S^{4n+3} the group $I_0(\mathcal{S},g)$ of 3-Sasakian automorphisms is $Sp(n+1)$. We choose a k-dimensional subtorus T^k in the maximal torus T^n. Every quaternionic representation of a k-torus T^k on $\mathbb{H}^{n+1}$ can be described via a homomorphism $f_\Omega : T^k \longrightarrow T^{n+1}$

$$f_\Omega(\tau_1,\ldots,\tau_k) = \begin{pmatrix} \prod_{i=1}^k \tau_i^{a_1^i} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \prod_{i=1}^k \tau_i^{a_{n+1}^i} \end{pmatrix},$$

where $(\tau_1,..,\tau_k) \in S^1 \times \cdots \times S^1 = T^k$ are the complex coordinates on T^k, and $a_j^i \in \mathbb{Z}$ are the coefficients of a $k \times (n+1)$ integral *weight* matrix $\Omega = (a_\alpha^l)_{\alpha=1,\ldots,n+1}^{l=1,\ldots,k} \in \mathcal{M}_{k,n+1}(\mathbb{Z})$.

Let $\{e_l\}_{l=1}^k$ denote the standard basis for $\mathfrak{t}_k^* \simeq \mathbb{R}^k$. Then the 3-Sasakian moment map $\mu_\Omega : S^{4n+3} \longrightarrow \mathfrak{t}_k^* \otimes \mathbb{R}^3$ of the k-torus action defined by $\varphi_{(\tau_1,\ldots,\tau_k)}(\mathbf{u}) = f_\Omega(\tau_1,\ldots,\tau_k)\mathbf{u}$, is given by $\mu_\Omega = \sum_l \mu_\Omega^l e_l$ with

$$\mu_\Omega^l(\mathbf{u}) = \sum_\alpha \bar{u}_\alpha i a_\alpha^l u_\alpha.$$

Let us denote further denote the triple $(T^k, f_\Omega, \varphi_{(\tau_1,\ldots,\tau_k)})$ by $T^k(\Omega)$. We want to study the reduced space

$$\mathcal{S}(\Omega) = S^{4n+3} /\!\!/\!\!/ T^k(\Omega) = \mu_\Omega^{-1}(0)/T^k(\Omega)$$

.

Consider the $\binom{n}{k}$ minor determinants

$$\Delta_{\alpha_1\ldots\alpha_k} = \det \begin{pmatrix} a^1_{\alpha_1} & \ldots & a^1_{\alpha_k} \\ \vdots & & \vdots \\ a^k_{\alpha_1} & \ldots & a^k_{\alpha_k} \end{pmatrix}$$

obtained by deleting $n+1-k$ columns of Ω.

DEFINITION: *Let $\Omega \in \mathcal{M}_{k,n+1}(\mathbb{Z})$ be the weight matrix.*

(i) If $\Delta_{\alpha_1\cdots\cdots\alpha_k} \neq 0, \forall \quad 1 \leq \alpha_1 < \cdots < \alpha_k \leq n+1$, then we say that Ω is non-degenerate.

Suppose Ω is non-degenerate and let g be the kth determinantal divisor, i.e., the gcd of all the k by k minor determinants $\Delta_{\alpha_1\cdots\cdots\alpha_k}$. Then Ω is said to be admissible if in addition we have

(ii) $\gcd(\Delta_{\alpha_2\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\alpha_k}) = g$ for all sequences of length (k+1) such that $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n+1$.

THEOREM (B-G-M): *Let $\mathcal{S}(\Omega)$ be a 3-Sasakian toral quotient space. Then*

(i) if Ω is non-degenerate, $\mathcal{S}(\Omega)$ is an orbifold.

(ii) If Ω is degenerate, then either $\mathcal{S}(\Omega)$ is a singular stratified manifold which is not an orbifold or it is an orbifold obtained by reduction of a lower dimensional sphere $S^{4n-4r-1}$ by a torus $T^{k-r}(\Omega')$ or a finite quotient of such, where $1 \leq r \leq k$ and Ω' is non-degenerate. (When $r = k$ the quotient is the sphere $S^{4n-4k-1}$).

(iii) Assuming that Ω is non-degenerate $\mathcal{S}(\Omega)$ is a smooth manifold if and only if Ω is admissible.

When can we find admissible Ω?

Generally this question is non-trivial. Admissible Ω clearly exist for $k = 1$ for any n, i.e. any dimension by simply choosing the elements of Ω to be pairwise relatively prime. They also exist in dimension 7 for any k as the following Ω shows:

$$\Omega = \begin{pmatrix} 1 & 0 & \dots & 0 & a_1 & b_1 \\ 0 & 1 & \dots & 0 & a_2 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_k & b_k \end{pmatrix}.$$

We simply choose (a_i, b_i) to be relatively prime, and if for some pair i, j $a_i = \pm a_j$ or $b_i = \pm b_j$ then we must have $b_i \neq \pm b_j$ or $a_i \neq \pm a_j$, respectively.

However, there are mod 2 obstructions to smoothness which show somewhat surprisingly:

THEOREM (B-G-M): *Let Ω be admissible and $k > 1$. Then dim $\mathcal{S}(\Omega) \leq 15$. Furthermore, if $k > 4$, then dim $\mathcal{S}(\Omega) = 7$.*

This says that there are **smooth** toral quotients $\mathcal{S}(\Omega)$ only in the cases indicated, and we shall see shortly that k is precisely the second Betti number $b_2(\mathcal{S}(\Omega))$. Moreover, a theorem of R. Bielawski says that any 3-Sasakian manifold $(\mathcal{S}, g)$ whose automorphism group $I_0(\mathcal{S}, g)$ contains a torus T^{n+1-k} is homeomorphic to a $\mathcal{S}(\Omega)$.

Furthermore, we have an existence theorem:

THEOREM (B-G-M-Rees,B-G-M): *There exist smooth toric quotients $\mathcal{S}(\Omega)$ in dimension 7 with any second Betti number $b_2(\mathcal{S}(\Omega)) = k$, and in dimensions* 11 *and* 15 *with second Betti numbers $b_2(\mathcal{S}(\Omega)) = 2, 3,$ and* 4.

The Topology of $\mathcal{S}(\Omega)$

The full isometry group of $\mathcal{S}(\Omega)$ is $Tn + 1 - k \times Sp(1)$ and in the case dim $\mathcal{S}(\Omega) = 7$ one can obtain a stratification of $\mathcal{S}$ by analysing the quotient $Q(\Omega) = \mathcal{S}(\Omega)/(T^2 \times Sp(1)$. For general dimension Bielawski has obtained the rational homology of $\mathcal{S}(\Omega)$ by viewing $\mathcal{S}(\Omega)$ as a boundary at infinity of a hyperkähler orbifold that is the limit of certain toric hyperkähler orbifolds studied by Bielawski and Dancer.

The orbit space $Q(\Omega)$ can be stratified as follows:

$$Q(\Omega) = Q_0(\Omega) \sqcup Q_1(\Omega) \sqcup Q_2(\Omega)$$

and:

LEMMA (B-G-M-R): *Let dim $\mathcal{S}(\Omega) = 7$. Then*

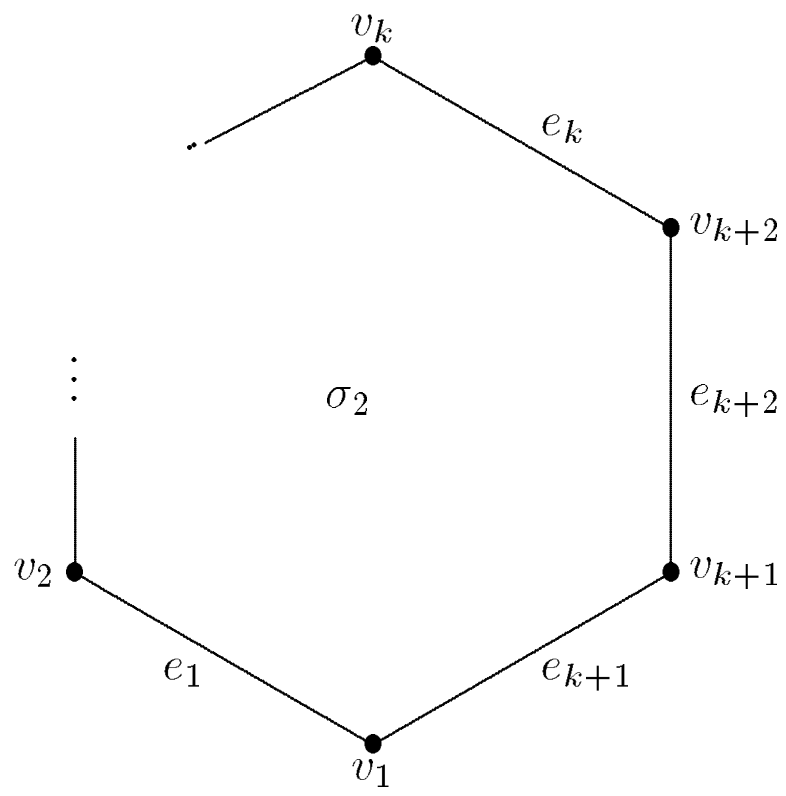
(i) The orbit space $Q(\Omega)$ is homeomorphic to the closed disc $\bar{D}^2$, and the subset of singular orbits $Q_1(\Omega) \sqcup Q_0(\Omega)$ is homeomorphic to the boundary $\partial\bar{D}^2 \simeq S^1$.

(ii) $Q_2(\Omega)$ is homeomorphic to the open disc D^2.

(iii) $Q_1(\Omega)$ is homeomorphic to the disjoint union of $k + 2$ copies of the open unit interval.

(iv) $Q_0(\Omega)$ is a set of $k + 2$ points.

Hence, the quotient $Q(\Omega)$ can be depicted by the polygon:



The orbit space $Q(\Omega)$

The topology of $\mathcal{S}(\Omega)$ can then be analysed by using a Leray spectral sequence on the induced stratification of $\mathcal{S}(\Omega)$. This gives

THEOREM (B-G-M-R,B-G-M): *Let $\mathcal{S}(\Omega)$ be a smooth 3-Sasakian toric quotient of dimension 7. Then $\mathcal{S}(\Omega)$ is simply connected and $H_2(\mathcal{S}(\Omega),\mathbb{Z})\simeq\mathbb{Z}^k$ where $k=b_2(\mathcal{S}(\Omega))$. Furthermore, the rational homology is given by*

$$b_i(\mathcal{S}(\Omega))=\begin{cases}1 & \textit{if } i=0,7;\\ 0 & \textit{if } i=1,3,4,6;\\ k & \textit{if } i=2,5.\end{cases}$$

There is certainly torsion in $H_3(\mathcal{S}(\Omega),\mathbb{Z})$, but we have only been able to determine this in the case $b_2(\mathcal{S}(\Omega)) = k = 1$. In this case we have determined the complete cohomology ring in all dimensions. Here, we have $\Omega = \mathbf{p} = (p_1, \cdots, p_n)$ with the p_i's relatively prime. Then

THEOREM (B-G-M): *Let $\mathcal{S}(\mathbf{p})$ be a 3-Sasakian manifold with $b_2 = k = 1$ obtained by a circle reduction from S^{4n+3}. Then, as rings,*

$$H^*\big(\mathcal{S}(\mathbf{p}),\mathbb{Z}\big) \;\cong\; \left(\frac{\mathbb{Z}[e_2]}{[e_2^{n+1}=0]} \otimes E[f_{2n+1}]\right)/\mathcal{R}(\mathbf{p}).$$

Here e_2 and f_{2n+1} are generators with the subscripts indicating the cohomological dimension of each generator. Furthermore, the relations $\mathcal{R}(\mathbf{p})$ are generated by $\sigma_n(\mathbf{p})e_2^n = 0$ and $f_{2n+1}e_2^n = 0$, where $\sigma_n(\mathbf{p}) = \sum_{j=1}^{n+1} p_1 \cdots \hat{p_j} \cdots p_{n+1}$ is the $(n)^{th}$ elementary symmetric polynomial in the entries of $\mathbf{p}$.

Notice this theorem shows that $H^{2n}(\mathcal{S}(\mathbf{p});\mathbb{Z}) \;=\; \mathbb{Z}_{\sigma_n(\mathbf{p})}$.

In more general dimension there is:

THEOREM (Bielawski): *Let $\mathcal{S}(\Omega)$ is a toric 3-Sasakian orbifold of dimension $4(n-k)+3$. Then we have*

$$b_{2i}(\mathcal{S}(\Omega)) = \begin{pmatrix} k+i-1 \\ k \end{pmatrix}, \qquad i < n+1-k.$$

The Wang-Ziller Construction

Here we apply the Wang-Ziller construction in the case of circle bundles over positive Kähler-Einstein orbifolds.

THEOREM: *Let* $(\mathcal{Z}_i, h_i)$, $i = 1, \cdots m$ *be Kähler-Einstein Fano orbifolds of complex dimension* n_i *with* $\pi_1^{orb}(\mathcal{Z}_i) = 0$. *There is a Kähler-Einstein metric* h *on the product* $\mathcal{Z} = \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_m$ *with scalar curvature* $4n(n+1)$ *where* $n = n_1 + \cdots + n_m$. *Let* $\mathcal{S}$ *be the total space of the* S^1 *V-bundle on* $\mathcal{Z}$ *whose first Chern class is* $\frac{c_1(\mathcal{Z})}{\mathrm{Ind}(\mathcal{Z})}$ *where* $\mathit{Ind}(\mathcal{Z}) = \gcd\{\mathit{Ind}(\mathcal{Z}_i)\}$. *Then* $\mathcal{S}$ *has a natural Sasakian-Einstein structure. Furthermore, there is an Einstein metric on any circle V-bundle over* $\mathcal{Z}$ *whose first Chern class is positive.*

We have the following:

COROLLARY (B-Galicki): *Let* $(\mathcal{S}_i, g_i)$ $i = 1, ..., m$ *be compact simply connected Sasakian-Einstein manifolds. Then the product* $\mathcal{S}_1 \times \cdots \times \mathcal{S}_m$ *has a locally free, isometric* T^m*-action. Choose any homomorphism* $f : T^l \to T^m$ *so that* T^l *acts on* $\mathcal{S}_1 \times \cdots \times \mathcal{S}_m$ *as a subgroup of* T^m. *If this action is free then the quotient* $P(f) = (\mathcal{S}_1 \times \cdots \times \mathcal{S}_m)/f(T^l)$ *is a compact manifold admitting an Einstein metric of positive scalar curvature.*

EXAMPLE: Let $\mathcal{S}_1 = \mathcal{S}(\Omega)$ where Ω is the k by $k+2$ integral weight matrix defined previously, and $\mathcal{S}_2 = S^{2n+1}$, $n \geq 1$. Let $(\tau, \rho) \in S^1 \times S^1$ be the product of the Sasakian circle actions by the characteristic vector fields with quotient spaces $\mathcal{S}(\Omega)/S^1_\tau = \mathcal{Z}(\Omega), S^{2n+1}/S^1_\rho = \mathbb{CP}^n$. Consider the homomorphism $f_{m,l} : S^1 \to T^2$ given by $f_{m,l}(\sigma) = (\sigma^m, \sigma^l)$. The homomorphism define a circle action on the product $\mathcal{S}_1 \times \mathcal{S}_2$, let us denote this action by S^1_{ml}. Define the quotient

$$M(\Omega;n;m,l) = \frac{\mathcal{S}(\Omega)\times S^{2n+1}}{S^1_{ml}}.$$

In general, the bundle $M(\Omega;n;m,l)$ is going to be an orbifold but one can choose infinitely many pairs (m,l) so that it is smooth. The condition $(m,l)=1$ is necessary for $M(\Omega;n;m,l)$ to be smooth but not sufficient. Necessary and sufficient condition involve the weight matrix Ω and are complicated. We get infinitely many pairs but observing that m can be chosen arbitrary and $l > m\mathrm{index}(\mathcal{Z}(\Omega))$, with l prime. Also any pair $(m,1)$ yields a smooth quotient. With appropriate normalizations of both $\mathcal{S}(\Omega)$ and S^{2n+1} and $(m,l)=(1,1)$ we get $M(\Omega;1,1)$ as a Sasakian-Einstein manifold.

Using standard spectral sequence arguments we find the rational cohomology:

$$H^q(M(\Omega;n;m,l)) \approx \begin{cases} H^q(\mathcal{Z}(\Omega)\times S^{2n+1}) & \text{if } n>2; \\ H^q(\mathcal{S}(\Omega)\times \mathbb{CP}^1) & \text{if } n=1; \\ \begin{cases} \mathbb{Q} & \text{if } q=0,11; \\ \mathbb{Q}^{k+1} & \text{if } q=2,4,7,9; \\ 0 & \text{if otherwise}; \end{cases} & \text{if } n=2. \end{cases}$$

Notice that the rational cohomology of these spaces depends only on Ω through its size k and is independent of the integers m and l. Moreover, when $n=2$ it does not have the rational cohomology of a product.

Some Consequences and an Open Problem

We give a list of some of the consequences that follow from our work together with other known results.

COROLLARY 1: *The are infinite families of simply connected Sasakian-Einstein manifolds in all odd dimensions* ≥ 7 *with arbitrary second Betti number.*

A remarkable theorem of Gromov says that if there is a negative number κ and a Riemannian metric with bounded diameter such that all sectional curvatures are $\geq \kappa$, then the total Betti number of the manifold must be bounded. This together with our results give

COROLLARY 2: *For any negative real number* κ *there are infinitely many Sasakian-Einstein manifolds in all odd dimensions* ≥ 7, *which do not admit metrics whose sectional curvatures are all greater than or equal to* κ.

A well-known result of Smale says that every simply connected 5-manifold with free second homology group is diffeomorphic to $S^5\#_k S^2 \times S^3$ for some k. All regular simply connected Sasakian-Einstein manifolds of dimension 5 are circle bundles over del Pezzo surfaces with a positive Kähler-Einstein metric. All such del Pezzo surfaces are $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ blown up at k points in general position with $0 \leq k \leq 8$. By Matsushima's well-known theorem the cases $k = 1$ and 2 cannot admit a Kähler-Einstein metric, and by results of Tian and Yau the cases $k = 3, \cdots, 8$ do admit Kähler-Einstein metrics. We know of no non-regular Sasakian-Einstein manifolds of dimension 5 that are not quotients of regular Sasakian-Einstein manifolds by a finite group.

OPEN PROBLEM: Classify all simply connected Sasakian-Einstein manifolds of dimension 5.

The study of this problems entails the study of del Pezzo orbifolds admitting a Kähler-Einstein metric.