On Positive Sasakian Geometry

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Abstract: A Sasakian structure $\mathcal{S}=(\xi,\eta,\Phi,g)$ on a manifold M is called *positive* if its basic first Chern class $c_1(\mathcal{F}_\xi)$ can be represented by a positive (1,1)-form with respect to its transverse holomorphic CR-structure. We prove a theorem that says that every positive Sasakian structure can be deformed to a Sasakian structure whose metric has positive Ricci curvature. This allows us by example to give a completely independent proof of a result of Sha and Yang [SY] that for every nonnegative integer k the 5-manifolds $k\#(S^2\times S^3)$ admits metrics of positive Ricci curvature.

Introduction

Let (M,J) is a compact complex manifold and g a Kähler metric on M, with Kähler form M. Suppose that ρ' is a real, closed $(1,1)$-form on M with $[\rho']=2\pi c_1(M)$. Then there exists a unique Kähler metric g' on M with Kähler form ω', such that $[\omega]=[\omega']\in H^2(M,\mathbb{R})$, and the Ricci form of g' is ρ'. The above statement is the celebrated Calabi Conjecture which was posed by Eugene Calabi in 1954. The conjecture in its full generality was eventually proved by Yau in 1976. In the Fano case when $c_1(M)>0$, i.e., when the first Chern class can be represented by a positive-definite real, closed $(1,1)$-form ρ' on M, the conjecture implies that the Kähler form of M can be represented by a metric of positive Ricci curvature.

There are several reasons one might be interested in a more general Calabi Problem when M is not necessarily a smooth manifold but rather a V-manifold or an orbifold [DK]. In the context of Sasakian manifolds one would naturally consider the Kähler geometry of the associated one-dimensional foliation. In this context one can actually prove a "transverse Yau theorem" and this was done by El Kacimi-Alaoui in 1990 [ElK]. In this note we discuss some consequences of the transverse Yau theorem. In particular, we prove the following which can be viewed as a Sasakian version of the "positive Calabi Conjecture" mentioned above:

Theorem A: *Let $\mathcal{S}=(\xi,\eta,\Phi,g)$ be a positive Sasakian structure on a compact manifold M of dimension $2n+1$. Then M admits a Sasakian structure $\mathcal{S}'=(\xi',\eta',\Phi',g')$ with positive Ricci curvature a-homologous to $\mathcal{S}$.*

Using some of the techniques developed earlier in our studies of circle V-bundles over log del Pezzo surfaces [BG2,BGN1,BGN2] we are able to apply Theorem A to demonstrate

During the preparation of this work the first two authors were partially supported by NSF grant DMS-9970904, and third author by NSF grant DMS-0070190.

THEOREM B: *There exist Sasakian metrics with positive Ricci tensor on $k\#(S^2 \times S^3)$ for every positive integer k. In particular, for each $k > 5$ there is a $2(k-5)$ parameter family of inequivalent Sasakian structures with positive Ricci curvature.*

From the standpoint of Riemannian metrics with positive Ricci curvature, this result is not new as it was first obtained by Sha and Yang [SY] in 1991; however, the Sasakian nature of such metrics is new. Moreover, our method of proof here is completely different.

In [JK] a list of all anticanonically (i.e. orbifold Fano index = 1) quasi-smooth log del Pezzo surfaces embedded in a weighted projective space $\mathbb{P}(\mathbf{w})$ was given, and in [BGN1] it is shown that the largest Picard number occuring is 10. However, as soon as we allow higher Fano index, namely 2, we find:

COROLLARY C: *There exist log del Pezzo surfaces $\subset \mathbb{P}(\mathbf{w})$ with arbitrary Picard number.*

1. Some Transverse Holomorphic Invariants

Recall that associated with every Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on a smooth manifold M is its characteristic foliation $\mathcal{F}_\xi$, and as described elsewhere (cf. [BGN1]) the transverse geometry of a Sasakian manifold is Kähler. Moreover, there are transverse versions of both the Hodge theory and the Dolbeault theory [ElK] of a Sasakian manifold. In particular if $(\mathcal{S}, \xi, \eta, \Phi, g)$ is a compact Sasakian manifold, then $d\eta$ defines a nontrivial class $[d\eta]_B$ in the basic cohomology group $H^{1,1}_B(\mathcal{F}_\xi) \subset H^2_B(\mathcal{F}_\xi)$. We also define the set $\mathfrak{F}(\xi)$ of all deformed Sasakian structures (ξ, η', Φ', g') on $\mathcal{S}$ that are *homologous* to (ξ, η, Φ, g), that is such that $[d\eta']_B = [d\eta]_B$. More generally, we define the set $\mathfrak{F}(\mathcal{F}_\xi)$ to be the set of all Sasakian structures on M whose characteristic foliation is $\mathcal{F}_\xi$. Clearly, $\mathfrak{F}(\xi) \subset \mathfrak{F}(\mathcal{F}_\xi)$. We shall need

DEFINITION 1.1: *Two Sasakian structures $\mathcal{S} = (\xi, \eta, \Phi, g)$ and $\mathcal{S}' = (\xi', \eta', \Phi', g')$ in $\mathfrak{F}(\mathcal{F}_\xi)$ on a smooth manifold M are said to be a-homologous if there is an $a \in \mathbb{R}^+$ such that $\xi' = a^{-1}\xi$ and $[d\eta']_B = a[d\eta]_B$.*

Next we consider some fundamental invariants of $\mathfrak{F}(\mathcal{F}_\xi)$. Now we have the *basic Betti numbers* and the *basic hodge numbers*

1.2 $$b^B_r(\mathcal{F}_\xi) = \dim\, H^r_B(\mathcal{F}_\xi), \qquad h^{p,q}_B(\mathcal{F}_\xi) = \dim\, H^{p,q}_B(\mathcal{F}_\xi)$$

which satisfy the relations

1.3 $$b^B_r(\mathcal{F}_\xi) = \sum_{p+q=r} h^{p,q}(\mathcal{F}_\xi), \qquad h^{p,q}_B(\mathcal{F}_\xi) = h^{q,p}_B(\mathcal{F}_\xi).$$

Accordingly, we have the *transverse Euler characteristic* of $\mathcal{F}_\xi$ given by

1.4a $$\chi(\mathcal{F}_\xi) = \sum_{p=0}^{2n} (-1)^p \dim\, H^p_B(\mathcal{F}),$$

and the *transverse holomorphic Euler characteristic* defined by

1.4b
$$\chi_{hol}(\mathcal{F}_\xi) = \sum_i (-1)^i \mathrm{dim}\ H_B^{0,q}(\mathcal{F}_\xi).$$

Now the basic cohomology ring $H^*(\mathcal{F}_\xi)$ is invariant under foliated homeomorphisms of M, that is homeomorphisms that preserve the foliation [ElKN], so the basic Betti numbers are also invariant under such homeomorphisms. The basic Hodge numbers, however, are only invariant under homemorphisms that preserve the foliation together with its transverse complex structure. Thus, they are not only invariants of the Sasakian structure, but also of its basic deformation class $\mathfrak{F}(\xi)$. In the case that $\mathcal{S} = (\xi, \eta, \Phi, g)$ is quasi-regular, the basic Betti numbers and basic Hodge numbers coincide with the corresponding Betti numbers and Hodge numbers on the space of leaves $\mathcal{Z} = M/S^1$ which is a compact Kähler orbifold.

The relationship between the basic cohomology $H^r_B(\mathcal{F}_\xi)$ and the ordinary cohomology $H^r(M,\mathbb{R})$ is given by the exact sequence [Ton]

1.5
$$\cdots \longrightarrow H^p_B(\mathcal{F}_\xi) \xrightarrow{\iota_*} H^p(M,\mathbb{R}) \xrightarrow{j_p} H^{p-1}_B(\mathcal{F}_\xi) \xrightarrow{\delta} H^{p+1}_B(\mathcal{F}_\xi) \longrightarrow \cdots$$

where δ is the connecting homomorphism given by $\delta[\alpha]_B = [d\eta \wedge \alpha]_B = [d\eta]_B \cup [\alpha]_B$, and j_p is the composition of the map induced by $\xi\rfloor$ with the well known isomorphism $H^r(M,\mathbb{R}) \approx H^r(M,\mathbb{R})^{S^1}$ where $H^r(M,\mathbb{R})^{S^1}$ is the S^1-invariant cohomology defined from the S^1-invariant r-forms $\Omega^r(M)^{S^1}$. Another important transverse invariant of the class $\mathfrak{F}(\xi)$ is the basic first Chern class $c_1(\mathcal{F}_\xi) \in H^2(\mathcal{F}_\xi)$ of the foliation $\mathcal{F}_\xi$. The image $\iota_*(c_1(\mathcal{F}_\xi))$ is the real first Chern class of the normal bundle $\nu(\mathcal{F}_\xi)$ to the foliation with its induced transverse complex structure.

If the dimension of $\mathcal{S}$ is $4n+1$ we can also define the *basic Hirzebruch signature* $\tau_B(\mathcal{F}_\xi)$ to be the signature of the bilinear form defined by the cup product on the middle basic cohomology group $H^{2n}_B(\mathcal{F}_\xi)$. Using transverse Lefschetz theory [ElK] one can obtain the usual formula:

1.6
$$\tau_B(\mathcal{F}_\xi) = \sum_{p,q} (-1)^p h^{p,q}_B(\mathcal{F}_\xi) = \sum_{p\equiv q(2)} (-1)^p h^{p,q}_B(\mathcal{F}_\xi).$$

As is usual in complex geometry we introduce the *geometric genus* $p_g(\mathcal{F}_\xi) = h^{0,n}_B(\mathcal{F}_\xi)$, the *arithmetic genus* $p_a(\mathcal{F}_\xi) = (-1)^n(\chi_{hol}(\mathcal{F}_\xi) - 1)$, and the irregularity $q = q(\mathcal{S}) = h^{0,1}_B$. We remark that $q = \frac{1}{2}b^B_1(\mathcal{F}_\xi) = \frac{1}{2}b_1(M)$ is actually a topological invariant by (5) of Proposition 1.9 of [BGN1]. In the case $n = 2$, things simplify much more. All the basic Hodge numbers are given in terms of the three invariants $q, p_g(\mathcal{F}_\xi)$ and $h^{1,1}_B(\mathcal{F}_\xi)$, and we have the relations

1.7
$$\chi_B(\mathcal{F}_\xi) = 2 + 2p_g(\mathcal{F}_\xi) - 4q + h^{1,1}_B(\mathcal{F}_\xi), \qquad \tau_B(\mathcal{F}_\xi) = 2 + 2p_g(\mathcal{F}_\xi) - h^{1,1}_B(\mathcal{F}_\xi)$$
$$\chi_B(\mathcal{F}_\xi) + \tau_B(\mathcal{F}_\xi) = 4\chi_{hol}(\mathcal{F}_\xi).$$

The fact that q is actually a topological invariant has some nice consequences for $n = 2$.

PROPOSITION 1.8: *Let $\mathcal{S} = (\xi, \eta, \Phi, g)$ be a Sasakian structure on a 5-manifold M with $b_1(M) = 0$. Then we have*

$$b_2(\mathcal{F}_\xi) = 1 + b_2(M), \qquad \chi_B(\mathcal{F}_\xi) = 3 + b_2(M) \geq 3, \qquad \chi_{hol}(\mathcal{F}_\xi) = 1 + p_g(\mathcal{F}_\xi) \geq 1.$$

PROOF: Since $b_1(M) = 0$ Proposition 1.9 of [BGN1] and the exact cohomology sequence 1.5 imply

1.9 $$H^2_B(\mathcal{F}_\xi) \approx \mathbb{R} \oplus H^2(M, \mathbb{R}),$$

and the results easily follow from this and equations 1.7. ∎

COROLLARY 1.10: *On a Sasakian 5-manifold M with $b_1(M) = 0$ we must have $b_2(M) \geq 2p_g(\mathcal{F}_\xi)$. In particular, every Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on S^5 or $S^2 \times S^3$ satisfies $p_g(\mathcal{F}_\xi) = 0$ and $\chi_{hol}(\mathcal{F}_\xi) = 1$.*

PROOF: This follows from Proposition 1.8 since by Proposition 1.9 of [BGN1] we must have $h^{1,1}_B(\mathcal{F}_\xi) \geq 1$. ∎

The reader no doubt notices that a classification of Sasakian structures on compact 5-manifolds involves a "transverse Enriques-Kodaira classification" paralleling complex surface theory. Such a classification is currently under study.

2. Positive Sasakian Geometry

DEFINITION 2.1: *A Sasakian manifold M is said to be positive if its basic first Chern class $c_1(\mathcal{F}_\xi)$ can be represented by a basic positive definite $(1,1)$-form.*

A basic ingredient in studying positive Sasakian geometry is the "transverse Yau Theorem" of El Kacimi-Alaoui [ElK]:

THEOREM 2.2 [ElK]: *If $c_1(\mathcal{F}_\xi)$ is represented by a real basic $(1,1)$ form ρ^T, then it is the Ricci curvature form of a unique transverse Kähler form ω^T in the same basic cohomology class as $d\eta$.*

In [BGN1] this theorem was reformulated in terms of Sasakian geometry in a convenient way, namely

THEOREM 2.3 [BGN1]: *Let (M, ξ, η, Φ, g) be a Sasakian manifold whose basic first Chern class $c_1(\mathcal{F}_\xi)$ is represented by the real basic $(1,1)$ form ρ, then there is a unique Sasakian structure $(\xi, \eta_1, \Phi_1, g_1) \in \mathfrak{F}(\xi)$ homologous to (ξ, η, Φ, g) such that $\rho_{g_1} = \rho - 2d\eta_1$ is the Ricci form of g_1, and $\eta_1 = \eta + \zeta_1$, with $\zeta_1 = \frac{1}{2}d^c\phi$. The metric g_1 and endomorphism Φ_1 are then given by*

$$\Phi_1 = \Phi - \xi \otimes \zeta_1 \circ \Phi, \qquad g_1 = d\eta_1 \circ (id \otimes \Phi_1) + \eta_1 \otimes \eta_1.$$

One key ingredient still missing in the study of transverse holomorphic geometry is the transverse analogue of Dolbeault's Theorem. The reason is that the sheaves in the basic Dolbeault complex are not fine, since they are constant along the leaves of the foliation. We can, however, obtain some partial results.

PROPOSITION 2.4: *If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a complete (i.e. g is complete) and positive Sasakian structure on a manifold M of dimension $2n+1$, then M is compact and $q = 0$. If in addition $\mathcal{S} = (\xi, \eta, \Phi, g)$ is quasi-regular, then $h^{p,0}(\mathcal{F}_\xi) = 0$ for all $p > 0$, and so $\chi_{hol}(\mathcal{F}_\xi) = 1$, $p_g(\mathcal{F}_\xi) = p_a(\mathcal{F}_\xi) = 0$. Moreover, any two quasi-regular positive Sasakian structures on a 5-manifold must have the same basic Hodge number $h^{1,1}_B$, the same basic Euler characteristic χ_B, and the same basic Hirzebruch signature τ_B.*

PROOF: As mentioned above $q = \frac{1}{2}b_1(M)$ follows from Proposition 1.9 of [BGN1]. Since $c_1(\mathcal{F}_\xi) > 0$ Theorem 2.3 implies there is a Sasakian structure $(\xi, \eta_1, \Phi_1, g_1) \in \mathfrak{F}(\xi)$ homologous to (ξ, η, Φ, g) whose Ricci curvature is positive and represents $c_1(\mathcal{F}_\xi)$. It then follows from Hasegawa and Seino's [HS] application of Myers' Theorem to Sasakian geometry that g_1 is complete, and hence, M is compact with finite fundamental group. This implies $q = 0$.

To prove the second part we note that any nontrivial element of $H^{p,0}_B(\mathcal{F}_\xi)$ is represented by a basic $(p,0)$-form in the kernel of $\bar{\partial}$, i.e. by a transversely holomorphic p-form α. Since $\mathcal{S} = (\xi, \eta, \Phi, g)$ is quasi-regular, M is the total space of an orbifold S^1 V-bundle over a compact Kähler orbifold $\mathcal{Z}$. Moreover, since α is basic, it descends to a nontrivial element $\check{\alpha} \in H^{p,0}(\mathcal{Z})$. Now since the Sasakian structure $\mathcal{S}$ is positive so is the Kähler structure on $\mathcal{Z}$, i.e. $c_1(\mathcal{Z}) > 0$, and this implies $h^{p,0}(\mathcal{Z}) = 0$ by the Kodaira-Baily vanishing Theorem. ∎

PROOF OF THEOREM A: $c_1(\mathcal{F}_\xi)$ can be represented by a positive definite basic $(1,1)$ form ρ, so by Theorem 2.3 there is a Sasakian structure $\mathcal{S}_1 = (\xi, \eta_1, \Phi_1, g_1) \in \mathfrak{F}(\xi)$ homologous to (ξ, η, Φ, g) such that $\rho_{g_1} = \rho - 2d\eta_1$ is the Ricci form of g_1. Let g_1^T denote the transverse Kähler metric of this Sasakian structure. Then the Ricci curvatures of g_1^T and g_1 are related by

$$\text{2.3} \qquad \mathrm{Ric}_{g_1}|_{\mathcal{D}_1 \times \mathcal{D}_1} = \mathrm{Ric}_{g_1^T} - 2g_1^T.$$

Next for any real number $a > 0$ we can perform a homothetic deformation [YK] of the Sasakian structure by defining

$$\text{2.4} \qquad g_2^T = \frac{1}{a} g_1^T, \quad \eta_2 = \frac{1}{a}\eta_1, \quad \xi_2 = a\xi, \quad \Phi_2 = \Phi_1,$$

in which case $\mathcal{S}_2 = (\xi_2, \eta_2, \Phi_2, g_2)$ is a Sasakian structure where $g_2 = g_2^T + \eta_2 \otimes \eta_2$. Notice also that $\mathcal{S}_1$ and $\mathcal{S}_2$ both have the same contact subbundles with the same underlying transverse complex structures, and the same characteristic foliations. Now since the Ricci tensor is invariant under homothety we find

$$\text{2.5} \qquad \mathrm{Ric}_{g_2}|_{\mathcal{D}_2 \times \mathcal{D}_2} = \mathrm{Ric}_{g_2^T} - 2g_2^T = \mathrm{Ric}_{g_1^T} - \frac{2}{a} g_1^T.$$

But since $Ric_{g_1^T} > 0$ and M is compact there exists $a_0 \in \mathbb{R}^+$ such that for all $a > a_0$ we have $\mathrm{Ric}_{g_2}|_{\mathcal{D}_2 \times \mathcal{D}_2} > 0$. But also for any Sasakian metric we have $\mathrm{Ric}_{g_2}(X, \xi_2) = 2n\eta_2(X)$ which implies that the Ricci curvature of g_2 is positive. But clearly $\mathcal{S}_2$ is a-homologous to $\mathcal{S}$ which proves the result. ∎

We shall make use of

PROPOSITION 2.6: *Any simply connected compact positive Sasakian manifold M is spin.*

PROOF: M is spin if and only if its second Stiefel-Whitney class $w_2(M)$ vanishes. But if $\mathcal{D}$ is the contact subbundle of a Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on M, we have a natural splitting

$$TM = \mathcal{D} \oplus L_\xi$$

where L_ξ is the trivial real line bundle generated by ξ. Thus,

$$w_2(M) = w_2(TM) = w_2(\mathcal{D})$$

which is the mod 2 reduction of $c_1(\mathcal{D}) \in H^2(M, \mathbb{Z})$.

Now suppose that $\mathcal{S}$ is a positive Sasakian structure so that $c_1(\mathcal{F}_\xi) > 0$. Choose a transverse Kähler metric g'_T whose Kähler form $\omega'_T \in c_1(\mathcal{F}_\xi)$. Then both ω'_T and its transverse Ricci form ρ'_T represent $c_1(\mathcal{F}_\xi)$. This transverse Kähler structure defines a positive Sasakian structure $\mathcal{S}' = (\xi, \eta', \Phi', g')$ on M such that $g' = g'_T + \eta' \otimes \eta'$, and ω'_T represents $c_1(\mathcal{F}_\xi)$. The exact sequence 1.11 of [BGN1] becomes

2.7
$$\begin{array}{ccccccc} & & & & H^2(M,\mathbb{Z}) & & \\ & & & & \downarrow & & \\ 0 \longrightarrow \mathbb{R} & \xrightarrow{\delta} & H^2_B(\mathcal{F}_\xi) & \xrightarrow{\iota_*} & H^2(M,\mathbb{R}) & \longrightarrow & \end{array}$$

where $\delta(c) = c[\omega'_T]_B$. But since $c_1(\mathcal{F}_\xi)$ is represented by ω'_T we have $c_1(\mathcal{D}') = \iota_* c_1(\mathcal{F}_\xi) = \iota_*\delta(1) = 0$ in $H^2(M, \mathbb{R})$. But since M is simply connected $c_1(\mathcal{D}') = 0$, so this implies that $w_2(M) = w_2(\mathcal{D}') = 0$. ∎

Actually we have proven more.

PROPOSITION 2.8: *For every positive Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on M the first Chern class $c_1(\mathcal{D}) \in H^2(M, \mathbb{Z})$ is torsion.*

So on a Sasakian manifold the free part of the first Chern class of the contact bundle $\mathcal{D}$ is an obstruction to positivity, and in particular to the existence of a compatible Sasakian-Einstein metric. However, the authors know of no example of a simply connected non-spin manifold admitting a Sasakian structure, or more generally a Sasakian manifold with the free part of $c_1(\mathcal{D})$ nontrivial. It should be mentioned that Sasakian manifolds are known to admit $\mathrm{Spin}^{\mathbb{C}}$-structures [Mor]. Now combining our results with Smale's [Sm] well-known classification of simply connected spin 5-manifolds, we arrive at

THEOREM 2.9: *Let M^5 be a complete positive Sasakian 5-manifold. Then the universal cover $\tilde{M}^5$ is diffeomorphic to one of the following:*

$$M^5(k,\alpha_i) = S^5\#k\#(S^2\times S^3)\#M^5_{\alpha_1}\#\cdots\#M^5_{\alpha_r}$$

for some nonnegative integers k,r where M^5_α is a compact simply connected 5-manifold with $H_2(M^5_\alpha,\mathbb{Z})\approx\mathbb{Z}_\alpha\oplus\mathbb{Z}_\alpha$, and α_i are positive integers satisfying $\alpha_1|\alpha_2|\cdots|\alpha_r$.

The problem of existence of positive Sasakian structures on the above 5-manifolds is still open in general. However, in the next section we prove existence in the case $r=0$ for all $k>0$. We should mention that for $r=0$ and $0\leq k\leq 9$ the manifolds are known to have Sasakian-Einstein structures [BG2, BGN1, BGN2] which are automatically positive. We should also mention that generally there is an a priori obstruction for M^{2n+1} to admit a Sasakian structure, namely the top Stiefel-Whitney class w_{2n+1}. However, it follows from Smale's Theorem 1.3 [Sm] that $w_5(M^5(k,\alpha_i))=0$; hence, all $M^5(k,\alpha_i)$ are candidates for admitting Sasakian structures. These 5-manifolds with α_i nontrivial can never be realized as hypersurfaces in a well-formed weighted projective space since a result of [BG2] says that such hypersurfaces necessarily have no torsion in $H_2(M,\mathbb{Z})$. Nevertheless, other methods are available to show that $M^5(k,\alpha_i)$ can admit Sasakian structures. Such examples with $k=0$ and $r>0$ are given in [BL].

3. Positive Sasakian Structures on $k\#(S^2\times S^3)$

We consider weighted homogeneous polynomials f of degree d in four complex variables z_0,z_1,z_2,z_3 which describe hypersurfaces in $\mathbb{C}^4$ with only an isolated singularity at the origin. Here we are interested in the case that f has degree $d=k+1$ with weights $\mathbf{w}=(1,1,1,k)$, $k\geq 2$ and Fano index $I=2$. Explicitly f is given by a general polynomial of the form

3.1 $$f(z_0,z_1,z_2,z_3)=g_{(k+1)}(z_0,z_1,z_2)+g_{(1)}(z_0,z_1,z_2)z_3$$

where $g_{(k+1)}(z_0,z_1,z_2)$ is a homogeneous polynomial of degree $(k+1)$ which is not divisible by $g_{(1)}(z_0,z_1,z_2)\neq 0$. This guarantees that f has only an isolated singularity at the origin. The zero locus of f cuts out a cone $C_f\subset\mathbb{C}^4$ and the link L_f is defined by intersecting with the unit sphere S^7, viz.

3.2 $$L_f=C_f\cap S^7.$$

The link L_f is a smooth simply connected 5-manifold which for $k\geq 5$ depends on $2(k-5)$ effective parameters. To see this, note that the group $\mathfrak{G}(1,1,1,k)$ of complex automorphisms of the projective space $\mathbb{CP}(1,1,1,k)$ is obtained by the projectivisation of the following group of transformations [BGN1]

$$\varphi_{\mathbb{A},\lambda,\phi}\begin{pmatrix}z_0\\z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}\mathbb{A}\begin{pmatrix}z_0\\z_1\\z_2\end{pmatrix}\\ \lambda z_3+\phi_{(k)}(z_0,z_1,z_2)\end{pmatrix},$$

where $\mathbb{A} \in GL(3,\mathbb{C})$, $\lambda \in \mathbb{C}^*$, and $\phi_{(k)}(z_0,z_1,z_2)$ is an arbitrary homogeneous polynomial of degree k.

To prove Theorem B of the introduction we give several lemmas.

LEMMA 3.2: *The 5-manifold L_f is spin.*

PROOF: We show that L_f has natural positive Sasakian structures. The result will then follow by Proposition 2.6. It is shown in [BG2] that links of isolated hypersurface singularities have natural Sasakian structures $\mathcal{S}$ whose characteristic foliation $\mathcal{F}_\xi$ is determined by the weights. Let $\mathcal{Z}_f$ denote the space of leaves of $\mathcal{F}_\xi$ which is embedded in the weighted projective space $\mathbb{CP}(1,1,1,k)$. The Sasakian structures will be positive if the basic first Chern class $c_1(\mathcal{F}_\xi)$ is positive. But since $c_1(\mathcal{F}_\xi)$ is the pullback of the first Chern class $c_1(\mathcal{Z}_f)$ of $\mathcal{Z}_f$ by the natural projection, it suffices to show that $c_1(\mathcal{Z}_f)$ is positive. But

$$c_1(\mathcal{Z}_f) \simeq c_1(K^{-1}_{\mathcal{Z}_f}) \simeq c_1(\mathcal{O}(2)_{\mathcal{Z}_f})$$

by 3.6 of [BGN1] which is clearly positive. ∎

LEMMA 3.3: $b_2(L_f) = k$.

PROOF: The procedure for computing the Betti numbers of links is given by Milnor and Orlik [MO]. Associate to any monic polynomial f with roots $\alpha_1,\ldots,\alpha_k \in \mathbb{C}^*$ its divisor

3.4 $$\mathrm{div}\ f = <\alpha_1> + \cdots + <\alpha_k>$$

as an element of the integral ring $\mathbb{Z}[\mathbb{C}^*]$ and let $\Lambda_n = \mathrm{div}(t^n - 1)$. The 'rational weights' used in [MO] are just $\frac{d}{w_i}$, and are written in irreducible form, $\frac{d}{w_i} = \frac{u_i}{v_i}$. The divisor of the characteristic polynomial $\Delta(t)$ is then determined by

3.5. $$\mathrm{div}\ \Delta(t) \ = \prod_i (\frac{\Lambda_{u_i}}{v_i} - 1) \ = 1 + \sum a_j\Lambda_j,$$

where $a_j \in \mathbb{Z}$ and the second equality is obtained by using the relations $\Lambda_a\Lambda_b = \gcd(a,b)\Lambda_{lcm(a,b)}$. The second Betti number of the link is then given by

3.6 $$b_2(L_f) = 1 + \sum_j a_j.$$

In our case we have

$$\mathrm{div}\ \Delta(t) \ = (\frac{\Lambda_{k+1}}{k} - 1)(\Lambda_{k+1} - 1)^3 = (\frac{\Lambda_{k+1}}{k} - 1)(\Lambda_{k+1} - 1)((k-1)\Lambda_{k+1} + 1) = (k-1)\Lambda_{k+1} + 1$$

implying $b_2(L_f) = k$. ∎

To prove Theorem B of the introduction it suffices to prove

LEMMA 3.7: *L_f is diffeomorphic to $k\#(S^2\times S^3)$.*

PROOF: This will follow from a Theorem of Smale [Sm] and Lemmas 3.2 and 3.3 if we can show that $H_2(L_f,\mathbb{Z})$ has no torsion. But since the gcd of any three weights is one this follows from Lemma 5.8 of [BG2]. ∎

Finally, we mention the failure of the sufficient conditions described in [JK,BGN1] for $\mathcal{Z}_f$ to admit a Kähler-Einstein metric, and hence, for L_f to admit a Sasakian-Einstein metric. Indeed, it follows from Lemma 5.1 of [BGN1] that for any $k\geq 2$, $(\mathcal{Z}_f,\frac{2+\epsilon}{3}D)$ cannot be Kawamata log terminal for every effective $\mathbb{Q}$-divisor that is numerically equivalent to $K^{-1}_{\mathcal{Z}_f}$. Furthermore, the Hitchin-Thorpe inequality ($c_1^2\geq 0$) which prohibits smooth compact complex surfaces with $q=p_g=0$ and $b_2\geq 9$ from admitting any Einstein metrics whatsoever, is ineffective for hypersurfaces in the weighted projective space $\mathbb{P}(\mathbf{w})$ since any such hypersurface $\mathcal{Z}_f$ of degree d satisfies

$$c_1^2(\mathcal{Z}_f)=\frac{d(|\mathbf{w}|-d)^2}{w_0w_1w_2w_3}\geq 0.$$

In our case we have $c_1^2=4(1+\frac{1}{k})$.

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March 2001