On Sasakian-Einstein Geometry

Charles P. Boyer   Krzysztof Galicki

An Introduction

In 1960 Sasaki [Sas] introduced a type of metric-contact structure which can be thought of as the odd-dimensional version of Kähler geometry. This geometry became known as Sasakian geometry, and although it has been studied fairly extensively ever since it has never gained quite the reputation of its older sister – Kählerian geometry. Nevertheless, it has appeared in an increasing number of different contexts from quaternionic geometry to mathematical physics.

In this article we shall focus our attention on a special class of Sasakian manifolds (and orbifolds) which have the property that the metric g is Einstein, that is $\mathrm{Ric}_g = \lambda g$. Such spaces are called Sasakian-Einstein. Perhaps one reason that the study of Sasakian-Einstein manifolds is so attractive is that they have generic holonomy [BG2]. However, the reason that they are so tractable is that they are closely related to a reduced holonomy. In fact, the most geometric definition of a Sasakian structure is: a smooth manifold $(\mathcal{S}, g)$ is Sasakian if the metric cone $(C(\mathcal{S}), \bar{g} = dr^2 + r^2 g)$ is Kähler, *i.e.,* its holonomy group reduces to a subgroup of $U(\frac{m+1}{2})$, where $m = \dim(\mathcal{S}) = 2n+1$. Moreover, $(\mathcal{S}, g)$ is Sasakian-Einstein if and only if its metric cone $(C(\mathcal{S}), \bar{g})$ is Kähler and Ricci-flat, *i.e.,* its restricted holonomy group reduces to a subgroup of $SU(n+1)$ (Calabi-Yau geometry). In particular, the Sasakian-Einstein geometry properly contains the so-called 3-Sasakian spaces for which the metric cone is not just Calabi-Yau but hyperkähler. The 3-Sasakian spaces are intimately related to quaternionic Kähler geometry and from this point of view they have been investigated in a series of recent articles [BGM1-6, BGMR, BG1-2, GS]. The simplest example of a compact simply connected Sasakian-Einstein manifold is furnished by the unit round sphere S^{2n+1} whose associated metric cone $\mathbb{C}^{n+1} \setminus \{0\}$ is flat. One of the first examples for which the cone $C(\mathcal{S})$ is not flat was constructed by Tanno [Tan]. Applying certain embedding techniques Tanno showed that $\mathcal{S} = S^2 \times S^3$ supports a homogeneous Sasakian-Einstein structure.

Sasakian geometry has associated with it a characteristic vector field [Bl]. This vector field is non-vanishing and thus generates a 1-dimensional foliation – the characteristic foliation on the Sasakian manifold $\mathcal{S}$. If we make an additional assumption that the leaves of this foliation are compact then the space of leaves will be a Kähler orbifold. This is both at once a generalization and a specialization of the well-known Boothby-Wang fibration [Bl, Hat]. It is a specialization since we are dealing with Sasakian and not the more general contact geometry. It is a generalization in that we make no regularity assumption on the foliation, but only assume that the leaves are compact. We refer to this as *quasi-regularity*, and it is this condition that places us within the category of orbifolds.

In the context of Einstein geometry we are dealing with the orbifold version of a result of Kobayashi [Kob2] which says that the total space of a principal circle bundle over a Kähler-Einstein manifold of positive scalar curvature admits an Einstein metric of positive scalar curvature. Recently a generalization of the Kobayashi's construction was brought to fruition in a paper by Wang and Ziller [WZ], where the authors construct Einstein metrics on circle bundles (and higher dimensional torus bundles) over products of positive scalar

---

During the preparation of this work the authors were supported by NSF grant DMS-9423752.

curvature Kähler-Einstein manifolds. We contend that the correct setting for Wang and Ziller's result in the case of circle bundles is a commutative topological monoid structure $(\mathcal{R},\star)$ on the set $\mathcal{R}$ of all *regular* Sasakian-Einstein manifolds. (We refer to "$\star$" as the "join".) In general the Einstein metrics obtained via the Wang-Ziller bundle construction are not Sasakian-Einstein. However, over each base $M = M_1\times\cdots\times M_n$ one finds a unique simply connected circle bundle $\mathcal{S}$ which is Sasakian-Einstein and as an element of $\mathcal{R}$ it is a product of n factors $\mathcal{S}_1\star\cdots\star\mathcal{S}_n$. Furthermore, such $\mathcal{S}$ has an n-dimensional lattice $L(\mathcal{S}_1,\ldots,\mathcal{S}_n)$ of compact Einstein manifolds canonically associated to it. The points on this lattice give all the other circle bundles in the Wang-Ziller construction.

It is not hard to realize that the regularity assumption on S_i's is much too restrictive even if one is solely interested in the smooth category of compact Einstein manifolds. Indeed, our contribution is not merely a novel look at Wang and Ziller's bundle construction, nor is it the recognition of the central role played by Sasakian-Einstein geometry; it is, however, mainly the non-trivial generalization of the monoid $(\mathcal{R},\star)$ of regular Sasakian-Einstein manifolds to a monoid structure on the set $\mathcal{SE}$ of compact Sasakian-Einstein orbifolds. A crucial point is that even though the subset of quasi-regular Sasakian-Einstein manifolds in $\mathcal{SE}$ is not closed under the join operation, one can analyze the conditions necessary for the join to be smooth. A key ingredient in this construction is Haefliger's description [Hae] of the classifying space of an orbifold which allows him to define "orbifold cohomology". In turn this allows us to generalize the notion of the index of a smooth Fano variety to Fano orbifolds; hence, we can construct a V-bundle with a Sasakian-Einstein structure whose total space is simply connected, and the index is associated to the Sasakian structure. Then given a pair of quasi-regular Sasakian-Einstein manifolds, or more generally orbifolds, we can define their relative indices by dividing each index by the gcd of their indices. It also makes sense to define the order of a quasi-regular Sasakian manifold (orbifold) to be the order as an orbifold of the space of leaves of the characteristic foliation. Our main result is:

Theorem A: *Let $\mathcal{S}_1,\mathcal{S}_2$ be two simply connected quasi-regular Sasakian-Einstein orbifolds of dimensions $2n_1+1$ and $2n_2+1$, respectively. Let $\mathcal{S}_i$ have orders m_i and relative indices l_i. Then there exists a multiplication called the join and denoted by $\star$ such that $\mathcal{S}_1\star\mathcal{S}_2$ is a simply connected quasi-regular Sasakian-Einstein orbifold of dimension $2(n_1+n_2)+1$. If both $\mathcal{S}_1$ and $\mathcal{S}_2$ are smooth Sasakian-Einstein manifolds then $\mathcal{S}_1\star\mathcal{S}_2$ is a smooth manifold if and only if $\gcd(m_1l_2,m_2l_1)=1$. Furthermore, to each such pair there is a two-dimensional lattice of Einstein orbifolds each having the same rational cohomology as $\mathcal{S}_1\star\mathcal{S}_2$.*

Similarly, one can determine necessary conditions for a lattice point on $L(\mathcal{S}_1,\mathcal{S}_2)$ to be a smooth Einstein manifold. As a consequence we obtain many new Einstein and Sasakian-Einstein manifolds. Using a simple and elegant spectral sequence argument employed by Wang and Ziller we are able to compute the cohomology rings of many examples of the joins of Sasakian-Einstein manifolds. This will allow us to generalize some of our previous results [BGMR, BGM2]. For example, we have

Corollary B: *In every odd dimension greater than 5, there are infinitely many distinct homotopy types of simply connected compact Sasakian-Einstein manifolds having the same rational cohomology groups. In particular, in each such dimension, there are infinitely many Sasakian-Einstein manifolds with arbitrarily small injectivity radii.*

Corollary C: *In every odd dimension greater than 5 there are simply connected compact Sasakian-Einstein manifolds with any second Betti number. In particular in each such dimension, there are infinitely many simply connected Sasakian-Einstein manifolds with the property that given any negative real number κ there exist no metrics on $\mathcal{S}$ whose sectional curvatures are all greater than or equal to κ.*

Corollary D: *In each dimension of the form $4n+3$ with $n\geq 1$ there are compact Sasakian-Einstein manifolds that do not admit a 3-Sasakian structure, and for $n>2$ there* *are such manifolds having arbitrary second Betti number.*

COROLLARY E: *In every odd dimension greater than 3 there are compact simply connected manifolds that admit continuous families of Sasakian-Einstein structures.*

The second statement in Corollary B follows by a well-known result of Anderson [An] while the second statement of Corollary C follows from a famous result of Gromov [Gro].

Concluding this introduction it seems worth mentioning that very recently Sasakian-Einstein geometry has emerged in the context of dualities of certain supersymmetric conformal field theories. The whole story begins with an important conjecture of of Maldacena [Mal] who noticed that the large N limit of certain conformal field theories in d dimensions can be described in terms of supergravity (and string theory) on a product of $(d+1)$-dimensional anti-de-Sitter AdS_{d+1} space with a compact manifold M. The idea was later examined by Witten who proposed a precise correspondence between conformal field theory observables and those of supergravity [Wit]. It turns out, and this observation has recently been made by Figueroa [Fi], that M necessarily has real Killing spinors and the number of them determines the number of supersymmetries preserved. Depending on the dimension and the amount of supersymmetry, the following geometries are possible: spherical in any dimension, Sasakian-Einstein in dimension $2k+1$, 3-Sasakian in dimension $4k+3$, 7-manifolds with weak G_2-holonomy, and 6-dimensional nearly Kähler manifolds [AFHB]. The case when $\dim(M) = 5, 7$ seems to be of particular interest. Several other articles have begun an in-depth investigation of these ideas. In particular the paper of Klebanov and Witten [KW] examines this duality in the case of $M = S^2 \times S^3 = S^3 \star S^3$, Oh and Tatar do the same for $M = S^3 \star S^3 \star S^3$ [OhTa]. An article by Morrison and Plesser formulates an extension of Maldacena's Conjecture in the general case of Non-Spherical Horizons [MP]. These new developments are certainly a part of our motivation to embark on a more systematic study of Sasakian-Einstein geometry and this article takes the first few steps in this direction.

ACKNOWLEDGMENTS: We would like to thank José Figueroa-O'Farrill for telling us about the recent results concerning string branes at conical singularities, as well as Ben Mann and Jim Milgram for helpful discussions. The second named author would like to thank Max-Planck-Institute für Mathematik in Bonn for support and hospitality. This project started during his stay in Bonn in 1997.

§1. The Classifying Space of an Orbifold and V-bundles

An *orbifold* X is an important generalization of a manifold in which the locally Euclidean charts (U_i, ϕ_i) are replaced by the *local uniformizing systems* $\{\tilde{U}_i, \Gamma_i, \varphi_i\}$, where $\tilde{U}_i$ is an open subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$) containing the origin, Γ_i is a finite group of diffeomorphisms (or biholomorphisms) which can be taken to be elements of $O(n)$ (or $U(n)$), $\varphi_i : \tilde{U}_i \longrightarrow U_i$ is a continuous map onto an subset $U_i \subset X$ such that $\varphi_i \circ \gamma = \varphi_i$ for all $\gamma \in \Gamma_i$ and the induced natural map of $\tilde{U}_i/\Gamma_i$ onto U_i is a homeomorphism. The finite group Γ_i is called a *local uniformizing group.* When it is defined the least common multiple of the orders of the local uniformizing groups Γ_i is called the *order* of the orbifold X, and is denoted by $\mathrm{Ord}(X)$. We shall be particularly interested in the case when $X = \mathcal{Z}$ a compact *complex orbifold.* Notice that for a compact orbifold the order of the orbifold is always defined. We refer to our expository paper [BG2] as well as the original literature for more details.

As with manifolds an alternative definition of orbifold can be given in terms of groupoids. Following Haefliger [Hae]: Let $\mathcal{G}_X$ denote the groupoid of germs of diffeomorphisms generated by the germs of elements in Γ_i and the germs of the diffeomorphisms g_{ji} described above. Let $\tilde{X} = \sqcup_i \tilde{U}_i$ denote the disjoint union of the $\tilde{U}_i$. Then $x, y \in \tilde{X}$ are equivalent if there is a germ $\gamma \in \mathcal{G}_X$ such that $y = \gamma(x)$. The quotient space $X = \tilde{X}/\mathcal{G}_X$ defines an orbifold (actually an isomorphism class of orbifolds). In the case that an orbifold X is given as the space of leaves of a foliation $\mathcal{F}$ on a smooth manifold, the groupoid $\mathcal{G}_X$ is just the transverse holonomy groupoid of $\mathcal{F}$.

Next we review Haefliger's construction [Hae] of the classifying space of an orbifold X of dimension n. Let $\{\tilde{U}_i, \Gamma_i, \phi_i\}$ be the local uniformizing systems of X, and consider again the disjoint union $\tilde{X} = \sqcup \tilde{U}_i$. Let $\mathcal{G}_X$ denote the groupoid of X, that is the groupoid generated by germs of diffeomorphisms $g_{ij} : \tilde{U}_j \longrightarrow \tilde{U}_i$ and germs of diffeomorphisms of the action of the finite groups Γ_i. Consider the bundle of linear frames $\tilde{P}$ over $\tilde{X}$. The groupoid $\mathcal{G}_X$ acts freely on $\tilde{P}$ with quotient space P the frame bundle of X. For each positive integer N let $V^{(N)} = GL(n+N)/GL(N)$ denote the Stiefel manifold with the standard action of $GL(n)$. $V^{(N)}$ is N-universal as a principal $GL(n)$ bundle. The nested sequence $\cdots \subset V^{(N)} \subset V^{(N+1)} \cdots$ gives rise the direct limit $EGL(n)$. We form the associated bundles $EX^{(N)} = \tilde{P} \times_{GL(n)} V^{(N)}$ whose direct limit we denote by EX. The groupoid $\mathcal{G}_X$ acts freely on $EX^{(N)}$ for each N and thus on EX. So EX is universal for the groupoid $\mathcal{G}_X$. We denote the quotient $EX/\mathcal{G}_X$ by BX, and the finite pieces $EX^{(N)}/\mathcal{G}_X$ by $BX^{(N)}$. We also have that $EX = \tilde{P} \times_{GL(n)} EGL(n)$ and

1.1 $$BX = \tilde{P} \times_{GL(n)} EGL(n)/\mathcal{G}_X = P \times_{GL(n)} EGL(n).$$

Furthermore, there are natural projections

1.2 $$\begin{array}{ccc} & BX & \\ \swarrow & & \searrow \\ X & & BGL(n). \end{array}$$

The right arrow is a fibration, but the left arrow $p : BX \longrightarrow X$ is not. It has generically contractible fibers and encodes information from the local uniformizing groups on the singular strata. It is this map that is of most interest. Haefliger defines the orbifold cohomology, homology, and homotopy groups by

1.3 $$H^i_{orb}(X, \mathbb{Z}) = H^i(BX, \mathbb{Z}), \quad H^{orb}_i(X, \mathbb{Z}) = H_i(BX, \mathbb{Z}), \quad \pi^{orb}_i(X) = \pi_i(BX).$$

This definition of π^{orb}_1 is equivalent to Thurston's better known definition [Thu] in terms of orbifold deck transformations, and when X is a smooth manifold these orbifold groups coincide with the usual groups.

In the category of orbifolds the concept of a bundle is replaced by that of a V-bundle. This consists of bundles $B_{\tilde{U}_i}$ over the local uniformizing neighborhoods $\tilde{U}_i$ that patch together in a certain way. In particular, there are group homomorphisms $h_{\tilde{U}_i} : \Gamma_i \longrightarrow G$, where G is the group of the V-bundle that satisfy the condition that:

1.4 If $g_{ji} : \tilde{U}_i \longrightarrow \tilde{U}_j$ is a diffeomorphism onto an open set, then there is a bundle map $g^*_{ij} : B_{U_j}|g_{ji}(\tilde{U}_i) \longrightarrow B_{\tilde{U}_i}$ satisfying the condition that if $\gamma \in \Gamma_i$, and $\gamma' \in \Gamma_j$ is the unique element such that $g_{ji} \circ \gamma = \gamma' \circ g_{ji}$, then $h_{\tilde{U}_i}(\gamma) \circ g^*_{ji} = g^*_{ji} \circ h_{\tilde{U}_j}(\gamma')$.

If the fiber F is a vector space and G acts on F as linear transformations of F, then the V-bundle is called a *vector* V-bundle. Similarly, if F is the Lie group G with its right action, then the V-bundle is called a *principal* V-bundle. The *total space* of a V-bundle over X is an orbifold E with local uniformizing systems $\{B_{\tilde{U}_i}, \Gamma_i^*, \varphi_i^*\}$. By choosing the local uniformizing neighborhoods of X small enough, we can always take $B_{\tilde{U}_i}$ to be the product $\tilde{U}_i \times F$ which we shall heretofore assume. There is an action of the local uniformizing group Γ_i on $\tilde{U}_i \times F$ given by sending $(\tilde{x}_i, b) \in \tilde{U}_i \times F$ to $(\gamma^{-1}\tilde{x}_i, bh_{\tilde{U}_i}(\gamma))$, so the local uniformizing groups Γ_i^* can be taken to be subgroups of Γ_i. We are particularly interested in the case of a principal bundle. In the case the fiber is the Lie group G, so the image $h_{\tilde{U}_i}(\Gamma_i^*)$ acts freely on F. Thus the total space P of a principal V-bundle will be smooth if and only if $h_{\tilde{U}_i}$ is injective for all i. We shall often denote a V-bundle by the standard notation $\pi : P \longrightarrow X$ and think of this as an "orbifold fibration". It must be understood, however, that an orbifold fibration is not a fibration in the usual sense. Shortly, we shall show that it is a fibration rationally.

We want to have the analogue of an "atlas of charts" on BX. Let $\{\tilde{U}_i, \Gamma_i, \phi_i\}$ be a cover of uniformizing charts for the orbifold X, and let P_i denote the linear frame bundle over U_i. The group $GL(n)$ acts locally freely on P_i with isotropy group Γ_i fixing the frames over the center $a_i \in U_i$. So we have homeomorphisms $P_i/GL(n) \approx \tilde{U}_i/\Gamma_i \approx U_i$. Thus, we can cover BX by neighborhoods of the form $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ where Γ_i is viewed as a subgroup of $GL(n)$. In fact Γ_i can always be taken as a subgroup of $O(n)$. Now by refining the cover if necessary we have injection maps $g_{ji} : \tilde{U}_i \longrightarrow \tilde{U}_j$ and these induce the change of "charts" maps

1.5 $$G_{ji} : \tilde{U}_i \times_{\Gamma_i} EGL(n) \longrightarrow \tilde{U}_j \times_{\Gamma_j} EGL(n)$$

given by $G_{ji}([\tilde{x}_i, e]) = [g_{ji}(\tilde{x}_i), e]$. This is well-defined since the unique element γ_j is identified with γ_i under the identification of Γ_i as a subgroup of Γ_j. This will allow us to construct local data on $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ by considering smooth (or holomorphic) data on the $\tilde{U}_i$ and continuous data on $EGL(n)$ which is invariant under the Γ_i action. Since the g_{ji} are diffeomorphisms (or biholomorphisms) this local data will then patch to give global data on BX. For example we denote by $\mathcal{E}$ the sheaf of germs of complex-valued functions on BX that are smooth in $\tilde{U}_i$ and continuous in $EGL(n)$. We shall call global (local) sections of the sheaf $\mathcal{E}$ smooth functions on BX. Similarly we refer to "smooth" maps from BX. For example, if $f : X_1 \longrightarrow X_2$ is a smooth map of orbifolds, then this induces a smooth map $Bf : BX_1 \longrightarrow BX_2$. Similarly the map $p : BX \longrightarrow X$ is smooth, since its local covering maps are smooth.

Next we give a characterization of V-bundles.

THEOREM 1.6: *Let X be an orbifold. There is a one-to-one correspondence between isomorphism classes of V-bundles on X with group G and generic fiber F and isomorphism classes of bundles on BX with group G and fiber F.*

PROOF: A V-bundle on X is a bundle on $\tilde{U}_i$ for each local uniformizing neighborhood $\tilde{U}_i$ together with a group homomorphism $h_{\tilde{U}_i} \in Hom(\Gamma_i, G)$ that satisfy the compatibility condition 1.4. This gives an action of Γ_i on $B_{\tilde{U}_i}$. Now cover BX by neighborhoods of the form $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ where we make use of the fact that the local uniformizing groups Γ_i can be taken as subgroups of $O(n) \subset GL(n)$. There is an action of Γ_i on $\tilde{U}_i \times F \times EGL(n)$ given by $(\tilde{x}_i, u, e) \mapsto (\gamma^{-1}\tilde{x}_i, uh_{U_i}(\gamma), e\gamma)$, and this gives a G-bundle over $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ with fiber F for each i. Moreover, the compatibility condition 1.4 guarantees that these bundles patch together to give a G-bundle on BX with fiber F.

Conversely, given a G-bundle on BX with fiber F, restricting to $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ gives a G-bundle there. Since for each i $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ is the Eilenberg-MacLane space $K(\Gamma_i, 1)$, there is a one-to-one correspondence between isomorphism classes of G-bundles on $\tilde{U}_i \times_{\Gamma_i} EGL(n)$ and conjugacy classes of group homomorphisms $Hom(\Gamma_i, G)$. The fact that these G-bundles come from a global G-bundle on BX implies that the compatibility conditions are satisfied. The correspondence can be seen to be bijective. ∎

In what follows we shall not distinguish between V-bundles on X and bundles on BX. Thus, we have

PROPOSITION 1.7: *The isomorphism classes of V-bundles on X with group G are in one-to-one correspondence with elements of the non-Abelian cohomology set $H^1(BX, \mathcal{G}) = H^1_{orb}(X, \mathcal{G})$ where $\mathcal{G}$ is the sheaf of germs of maps to the group G.*

We now turn to the Abelian case with coefficients in a sheaf. Let $\mathcal{E}$ denote sheaf of germs of smooth complex-valued functions on BX and $\mathcal{E}^*$ the sub-sheaf of no-where vanishing complex valued smooth functions. The isomorphism classes of complex line bundles on BX, hence complex line V-bundles on X, are in one-to-one correspondence with the elements of the cohomology group $H^1_{orb}(X, \mathcal{E}^*)$. Now BX is an infinite paracompact CW complex, and $\mathcal{E}$ is a fine sheaf, so the exponential sequence gives an isomorphism

1.8
$$H^1_{orb}(X, \mathcal{E}^*) \xrightarrow{c_1} H^2_{orb}(X, \mathbb{Z}),$$

and we define the *Chern class* of a line V-bundle $\mathcal{L}$ to be the image $c_1(\mathcal{L}) \in H^2_{orb}(X, \mathbb{Z})$. More generally one can define the Chern classes of complex vector V-bundles by using the splitting principle.

We are particularly interested in holomorphic line V-bundles. In [BG1] the authors introduced the group $\mathrm{Pic}^{orb}(\mathcal{Z})$ of holomorphic line V-bundles on a complex orbifold $\mathcal{Z}$. On $B\mathcal{Z}$ define the sheaf $\mathcal{A}$ of germs of holomorphic functions on $B\mathcal{Z}$ to be the sub-sheaf of $\mathcal{E}$ consisting of functions that are holomorphic in the $\tilde{U}_i$. Similarly, $\mathcal{A}^* \subset$ denotes the sub-sheaf of germs of nowhere vanishing functions. As in Proposition 1.7 we have $\mathrm{Pic}^{orb}(\mathcal{Z}) \simeq H^1_{orb}(\mathcal{Z}, \mathcal{A}^*)$. Now let $\mathcal{Z}$ be a compact complex Fano orbifold, that is, the anti-canonical line V-bundle $K^{-1}_{\mathcal{Z}}$ is ample. In this case $c_1(K^{-1}_{\mathcal{Z}}) > 0$, and as in the smooth case we have

DEFINITION 1.9: *The index of a Fano orbifold $\mathcal{Z}$ is the largest positive integer m such that $\frac{c_1(K^{-1}_{\mathcal{Z}})}{m}$ is an element of $H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. The index of $\mathcal{Z}$ is denoted by $Ind(\mathcal{Z})$.*

In view of the fact that the requisite vanishing theorem is lacking in the singular case, we shall need

LEMMA 1.10: *Let $\mathcal{Z}$ be a Fano orbifold with $Ind(\mathcal{Z}) = m$. Then there is a holomorphic line V-bundle $\mathcal{L} \in \mathrm{Pic}^{orb}(\mathcal{Z})$ such that $\mathcal{L}^m = K^{-1}_{\mathcal{Z}}$.*

PROOF: The idea of the proof is simple, but since we are working on $B\mathcal{Z}$ we write out the details. First on $B\mathcal{Z}$ we define the following sheaf $\mathcal{E}^{p,q}$ of "differential forms": Let $(z_1, \cdots, z_n)$ be complex coordinates on $\tilde{U}_i$. Then using the standard multi-index notation, we construct the sheaf $\mathcal{E}^{p,q}$ whose stalks are spanned by elements of the form $f_{IJ}(z,e)dz_I \wedge d\bar{z}_J$ where $I = i_1 \cdots i_p$ and $J = j_1 \cdots j_q$ are the usual multi-indices, and f is a smooth function on $\tilde{U}_i \times EGL(2n)$ satisfying $f(\gamma^{-1}z, e\gamma) = f(z,e)$ for $\gamma \in \Gamma_i$. We have, as usual, differential operators $\bar{\partial}$ and the Dolbeault complex

1.14
$$\cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,q+1} \xrightarrow{\bar{\partial}} \cdots.$$

Notice that $\mathcal{E}^{0,0} = \mathcal{E}$ and $\ker(\bar{\partial} : \mathcal{E}^{0,0} \longrightarrow \mathcal{E}^{0,1}) = \mathcal{A}$. By the isomorphism 1.8 there is a complex line V-bundle $\mathcal{L}$ such that $\mathcal{L}^m = K^{-1}$. The transition function h_{ij} for $\mathcal{L}$ are nowhere vanishing local sections of $\mathcal{E}$ over $\tilde{U}_i \times_{\Gamma_i} EGL(2n) \cap \tilde{U}_j \times_{\Gamma_j} EGL(2n)$ and the transition functions for $\mathcal{L}^m = K^{-1}$ are h^m_{ij}. But since K^{-1} is holomorphic $\bar{\partial} h^m_{ij} = 0$; hence, $\bar{\partial} h_{ij} = 0$ implying that $\mathcal{L}$ is holomorphic. ∎

The groups $H^p_{orb}(\mathcal{Z}, \mathbb{Z})$ encode information from the local uniformizing groups as well as the ordinary cohomology groups. In fact by studying the Leray spectral sequence of the map $p : B\mathcal{Z} \longrightarrow \mathcal{Z}$ it is easy to see that these groups coincide with the ordinary groups in certain cases. For any sheaf $\mathcal{F}$ on $B\mathcal{Z}$ we let $R^q p(\mathcal{F})$ denote the derived functor sheaves, that is the sheaves associated to the presheaves $U \mapsto H^q(p^{-1}(U), \mathcal{F})$. Then Leray's theorem says that there is a spectral sequence $E_r^{p,q}$ with E_2 term given by $E_2^{p,q} = H^p(\mathcal{Z}, R^q p(\mathcal{F}))$ converging to $H^{p+q}_{orb}(\mathcal{Z}, \mathcal{F})$. If $\mathcal{F}$ is one of the constant sheaves $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ one easily sees that spectral sequence collapses and we recover a result of Haefliger [Hae], namely

COROLLARY 1.11: *$H^q_{orb}(\mathcal{Z}, A) \simeq H^q(\mathcal{Z}, A)$ for $A = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $\mathbb{Z}_p$, where p is relatively prime to $Ord(\mathcal{Z})$.*

§2. Sasakian-Einstein Geometry

Let us begin with a brief review of properties of Sasakian-Einstein spaces. There are several equivalent definitions of a Sasakian structure in the literature (See [Bl, YKon]), but the most geometric definition is also the most recent [BG2]:

DEFINITION 2.1: *Let $(\mathcal{S}, g)$ be a Riemannian manifold of real dimension m. We say that $(\mathcal{S}, g)$ is Sasakian if the holonomy group of the metric cone on $\mathcal{S}$ $(C(\mathcal{S}), \bar{g}) = (\mathbb{R}_+ \times \mathcal{S},\ dr^2 + r^2 g)$ reduces to a subgroup of $U(\frac{m+1}{2})$. In particular, $m = 2n+1, n \geq 1$ and $(C(\mathcal{S}), \bar{g})$ is Kähler.*

That this definition is equivalent to the other known definitions of a Sasakian structure given in terms of the triple $\{g, \xi, \Phi\}$, where g is Riemannian metric, ξ is a nowhere vanishing Killing vector field, and Φ is a tensor field on $\mathcal{S}$, was shown in our expository article [BG2]. The Killing vector field ξ is called the *characteristic* or *Reeb* vector field. The 1-form η defined to be the 1-form dual to ξ with respect to the metric g is called the *characteristic 1-form* of the Sasakian structure, and underscores the *contact* nature of a Sasakian structure. Indeed, a Sasakian structure is usually defined as a normal contact metric structure [Bl, YKon]. The terminology normal means that the almost CR-structure defined by Φ on the orthogonal complement to subbundle of $T\mathcal{S}$ defined by the characteristic vector field ξ is integrable. It is easy to generalize this definition to that of a Sasakian *orbifold*. One simply requires that $\{g, \xi, \Phi\}$ be invariant under the action of the local uniformizing groups of the orbifold.

We are interested in Sasakian-Einstein geometry. We have

DEFINITION-PROPOSITION 2.2: *A Sasakian manifold (orbifold) $(\mathcal{S}, g, \xi, \Phi)$ is Sasakian-Einstein if its Riemannian metric g is Einstein. The Ricci tensor Ric of any Sasakian manifold (orbifold) of dimension $2n+1$ satisfies $Ric(X, \xi) = 2n\eta(X)$. Thus, if the metric $(\mathcal{S}, g)$ is Sasakian-Einstein , then the scalar curvature of g is positive and equals $2n(2n+1)$.*

Furthermore, a Sasakian manifold $(\mathcal{S}, g)$ is Sasakian-Einstein if and only if the cone metric $\bar{g}$ is Ricci-flat, i.e., $(C(\mathcal{S}), \bar{g})$ is Calabi-Yau. In particular, it follows that the restricted holonomy group $\mathrm{Hol}^0(\bar{g}) \subset SU(n+1)$.

Then an immediate corollary of 2.2 and Myers Theorem is that any complete Sasakian-

Einstein manifold is compact with finite fundamental group. In fact there is a stronger result due to Hasegawa and Seino:

PROPOSITION 2.3 [HS]: *Let $\mathcal{S}$ be a complete Sasakian manifold such that $Ric(X,X) \geq \delta > -2$ for all unit vector fields X on $\mathcal{S}$. Then $\mathcal{S}$ is compact with finite fundamental group.*

The case when the characteristic vector field ξ of a compact Sasakian manifold $\mathcal{S}$ generates a free circle action has been well studied. In this case $\mathcal{S}$ is the total space of a principal S^1-bundle whose base space $\mathcal{Z}$ is a Hodge manifold; hence, a smooth projective algebraic variety. This is a special case of the well-known Boothby-Wang fibration and is due to Hatakeyama [Hat]. The contact form η is nothing but a connection 1-form on the bundle $\pi : \mathcal{S} \longrightarrow \mathcal{Z}$ and the curvature form $d\eta$ is just the pullback by π of the Kähler form on $\mathcal{Z}$. Moreover, the fibers of π are totally geodesic. Now one can bring to bare O'Neill's well-known formulae for Riemannian submersions, and these work equally as well in the quasi-regular case discussed below.

Now generally, since the Killing vector field ξ has unit norm, it defines a 1-dimensional foliation $\mathcal{F}$ on $\mathcal{S}$, and we are interested in the case that all the leaves of $\mathcal{F}$ are compact. The assumption that all leaves are compact is equivalent to the assumption that $\mathcal{F}$ is *quasi-regular*, *i.e.*, each point $p \in \mathcal{S}$ has a cubical neighborhood U such that any leaf $\mathcal{L}$ of $\mathcal{F}$ intersects a transversal through p at most a finite number of times $N(p)$. Furthermore, $\mathcal{S}$ is called *regular* if $N(p) = 1$ for all $p \in \mathcal{S}$. In this case, the foliation $\mathcal{F}$ is simple, and defines a global submersion. In the quasi-regular case it is well-known [Tho] that ξ generates a locally free circle action on $\mathcal{S}$, and that the space of leaves is a compact orbifold (or V-manifold)[Mol]. We shall denote the space of leaves of the foliation $\mathcal{F}$ on $\mathcal{S}$ by $\mathcal{Z}$. Then the natural projection $\pi : \mathcal{S} \longrightarrow \mathcal{Z}$ is an orbifold submersion and a Siefert fibration. Actually, much more is true.

THEOREM 2.4: *Let $(\mathcal{S}, g)$ be a compact quasi-regular Sasakian manifold of dimension $2n+1$, and let $\mathcal{Z}$ denote the space of leaves of the characteristic foliation. Then*

(i) The leaf space $\mathcal{Z}$ is a compact complex orbifold with a Kähler metric h and Kähler form ω which defines an integral class $[\omega]$ in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$ in such a way that $\pi : (\mathcal{S}, g) \longrightarrow (\mathcal{Z}, h)$ is an orbifold Riemannian submersion. The fibers of π are totally geodesic submanifolds of $\mathcal{S}$ diffeomorphic to S^1.

(ii) $\mathcal{Z}$ is also a normal projective algebraic variety which is $\mathbb{Q}$-factorial.

(iii) The orbifold $\mathcal{Z}$ is Fano if and only if $Ric_g > -2$. In this case $\mathcal{Z}$ as a topological space is simply connected, and as an algebraic variety is uniruled with Kodaira dimension $\kappa(\mathcal{Z}) = -\infty$.

(iv) $(\mathcal{S}, g)$ is Sasakian-Einstein if and only if $(\mathcal{Z}, h)$ is Kähler-Einstein with scalar curvature $4n(n+1)$.

PROOF: (i): It is well-known [Mol] that $\mathcal{Z}$ is a compact orbifold, and the remainder follows exactly as in the regular case except that now the Kähler class $[\omega]$ defines an integral class in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$ which is generally only a rational class in the ordinary cohomology $H^2(\mathcal{Z},\mathbb{Q})$. This proves (i). To prove (ii) we notice that as the usual case ω is a positive $(1,1)$-form and thus represents a holomorphic line V-bundle on $\mathcal{Z}$. (ii) now follows from the Kodaira-Baily embedding theorem [Ba2].

In the Sasakian case the O'Neill tensors T and N (See [Bes]) vanish, and the tensor A satisfies $A_X Y = -g(\Phi X, Y)\xi$ and $A_X \xi = \Phi X$ for X and Y horizontal. Then one easily sees from [Bes, 9.36] that

2.5 $$\mathrm{Ric}_g(X,Y) = \mathrm{Ric}_h(X,Y) - 2g(X,Y).$$

The first statement of (iii) now follows from this. Moreover, combining 2.5 with the last statement of 2.2 implies (iv). In (iii) the simple connectivity of $\mathcal{Z}$ is basically Kobayashi's argument [Kob1] with the usual Riemann-Roch replaced by the singular version of Baum, Fulton, and MacPherson (see [BG1] for details). The uniruledness and Kodaira dimension follow from Miyaoka and Mori [MiMo]. ∎

Next we give an inversion theorem to Theorem 2.4. In the regular case this goes back to a construction of Kobayashi [Kob2] together with Hatakeyama [Hat]. There is also a description in the context of Sasakian-Einstein geometry in [FrKat2,BFGK] in the regular case. The non-regular case follows by applying Hatakeyama's results to our more general setup. First we have

DEFINITION 2.6: *A compact Kähler orbifold is called a Hodge orbifold if the Kähler class* $[\omega]$ *lies in* $H^2_{orb}(\mathcal{Z},\mathbb{Z})$.

With this definition a restatement of the Kodaira-Baily embedding theorem [Ba2] is:

THEOREM [Baily]2.7: *A Hodge orbifold is a projective algebraic variety.*

Now our inversion theorem is:

THEOREM 2.8: *Let* $(\mathcal{Z},h)$ *be a Hodge orbifold. Let* $\pi:\mathcal{S}\longrightarrow\mathcal{Z}$ *be the* S^1 *V-bundle whose first Chern class is* $[\omega]$, *and let* η *be a connection 1-form in* $\mathcal{S}$ *whose curvature is* $2\pi^*\omega$, *then* $(\mathcal{S},\eta)$ *with the metric* $\pi^*h+\eta\otimes\eta$ *is a Sasakian orbifold. Furthermore, if all the local uniformizing groups inject into the group of the bundle* S^1, *the total space* $\mathcal{S}$ *is a smooth Sasakian manifold.*

REMARK 2.9: The orbifold structure of $\mathcal{Z}$ is crucial here. Consider the weighted projective space $\mathbb{CP}^2(p_1,p_2,p_3)$ defined by the usual weighted $\mathbb{C}^*$ action on $\mathbb{C}^3-\{0\}$ where p_1,p_2,p_3 are pairwise relatively prime integers. As an algebraic variety $\mathbb{CP}^2(p_1,p_2,p_3)$ is equivalent [Kol] to $\mathbb{CP}^2/\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\mathbb{Z}_{p_3}$. But as orbifolds these are distinct, since the former has $\pi_1^{orb}=0$, whereas the latter has $\pi_1^{orb}=\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\mathbb{Z}_{p_3}$. This is to be contrasted with the smooth case where equivalence as complex manifolds coincides with equivalence as algebraic varieties. Of course, the metrics are also different. In the latter case the metric is just the Fubini-Study metric pushed to the quotient which is Kähler-Einstein. The Sasakian structure on the corresponding V-bundle is just the standard Sasakian-Einstein structure on the lens space $S^5/\mathbb{Z}_{p_1}\times\mathbb{Z}_{p_2}\times\mathbb{Z}_{p_3}$. In the former case the Sasakian structure is one of the non-standard deformed structures on S^5 described in Example 7.1 of [YKon] which is ellipsoidal and not Einstein. Likewise, the metric h on $\mathbb{CP}^2(p_1,p_2,p_3)$ is not Kähler-Einstein.

We are particularly interested in constructing simply connected Sasakian-Einstein manifolds. Thus,

COROLLARY 2.10: *Let* $\mathcal{Z}$ *be a compact Fano orbifold with* $\pi_1^{orb}(\mathcal{Z})=0$. *Let* $\pi:\mathcal{S}\longrightarrow\mathcal{Z}$ *be the* S^1 *V-bundle whose first Chern class is* $\frac{c_1(\mathcal{Z})}{Ind(\mathcal{Z})}$. *Suppose further that the local uniformizing groups of* $\mathcal{Z}$ *inject into* S^1. *Then there is a metric* g *on the total space* $\mathcal{S}$ *such that* $\mathcal{S}$ *is a compact simply connected Sasakian manifold with* $Ric_g>-2$. *Furthermore, if* $\mathcal{Z}$ *is Kähler-Einstein then* $(\mathcal{S},g)$ *is Sasakian-Einstein.*

PROOF: The only part that does not follow immediately from our results is the simple connectivity. Suppose $\mathcal{S}$ were not simply connected, then by compactness and the bound on the Ricci tensor there would be at most a finite cover $\tilde{\mathcal{S}}$. But since $\pi_1^{orb}(\mathcal{Z})=0$ there is no nontrivial cover of $\mathcal{Z}$. So $\tilde{\mathcal{S}}$ must be the total space of a V-bundle on $\mathcal{Z}$ covering $\mathcal{S}$. But then we must have $\tilde{\mathcal{S}}=\mathcal{S}$ since the first Chern class of the V-bundle $\pi:\mathcal{S}\longrightarrow\mathcal{Z}$ is not divisible in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$ by the definition of $\mathrm{Ind}(\mathcal{Z})$. ∎

DEFINITION 2.11: *Let $(\mathcal{S},g)$ be a compact quasi-regular Sasakian orbifold with $\mathcal{Z}$ the space of leaves of the characteristic foliation. We define the order of $\mathcal{S}$, denoted by $Ord(\mathcal{S})$ to be the order of the orbifold $\mathcal{Z}$. When $Ric_g > -2$, we define the index of $\mathcal{S}$, denoted $Ind(\mathcal{S})$, to be the index of $\mathcal{Z}$.*

WARNING: The order of a quasi-regular Sasakian orbifold defined here is NOT the order of $\mathcal{S}$ as an orbifold. There are many smooth Sasakian manifolds with large order, whereas $\mathrm{Ord}(\mathcal{S})=1$ means that $\mathcal{S}$ is smooth and regular.

Notice that the index is always defined for any quasi-regular Sasakian-Einstein orbifold. Both the index $\mathrm{Ind}(\mathcal{S})$ and $\mathrm{Ord}(\mathcal{S})$ are invariants of the Sasakian structure on $\mathcal{S}$. If $\mathcal{S}$ is a 3-Sasakian manifold the space $\mathcal{Z}$ is independent, up to isomorphism, of the choice of characteristic vector field ξ in the Lie algebra $\mathfrak{su}(2)$. Thus, it makes sense to talk of both the index and order of a 3-Sasakian manifold. Moreover, from the existence of the contact line V-bundle on $\mathcal{Z}$ one has $\mathrm{Ind}(\mathcal{S})\geq n+1$ where $\dim\mathcal{S}=4n+3$. More generally, one easily sees that for the standard Sasakian structure on S^{2n+1} that $\mathrm{Ind}(S^{2n+1})=n+1$. Note that $\mathrm{Ind}(S^{4n+3})=2n+2$, and it was shown in [BG1] using Kawasaki's Riemann-Roch theorem for orbifolds [Kaw] that this is the only simply connected 3-Sasakian manifold with index $2n+2$. Furthermore, this is the largest possible index of a 3-Sasakian manifold. Summarizing we have

PROPOSITION 2.12: *The index $Ind(\mathcal{S})$ of a 3-Sasakian manifold $\mathcal{S}$ of dimension $4n+3$ is either $n+1$ or $2n+2$ and equals $2n+2$ if and only if the universal cover of $\mathcal{S}$ is S^{4n+3}.*

More detailed information about the index is available in the regular case which is discussed in the next section. Finally we end this section by mentioning a related result of Vaisman [Vai,DO]. If $\mathcal{S}$ is any Sasakian manifold then $\mathcal{S}\times S^1$ is locally conformal Kähler with parallel Lee form [Vai]. Such manifolds have been called generalized Hopf manifolds or Vaisman manifolds [DO]. Moreover, the universal cover of every generalized Hopf manifold is of the form $\mathcal{S}\times\mathbb{R}$ where $\mathcal{S}$ is Sasakian. The Sasakian-Einstein manifolds $\mathcal{S}$ discussed here give a subclass of generalized Hopf manifold with the property that the local Kähler metrics are also Ricci flat. Thus, one might refer to this subclass of manifolds as locally conformal Calibi-Yau manifolds. There are obvious translations of our results on Sasakian-Einstein manifolds to the class of locally conformal Calabi-Yau manifolds.

bigskip

§3. Regular Sasakian-Einstein manifolds

The following is a translation to regular Sasakian-Einstein geometry of known results [Ko, K-O, Wi1, Wi2] about the index of smooth Fano varieties:

THEOREM 3.1: *Let $\mathcal{S}$ be a regular Sasakian-Einstein manifold of dimension $2n+1$. Then*

(i) $Ind(\mathcal{S})\leq n+1$.

(ii) If $Ind(\mathcal{S})=n+1$ then the universal cover $\tilde{\mathcal{S}}$ of $\mathcal{S}$ is S^{2n+1} with its standard Sasakian structure.

(iii) If $Ind(\mathcal{S})=n$ then $\tilde{\mathcal{S}}$ is a circle bundle over the complex quadric $Q_n(\mathbb{C})$.

(iv) if $r=Ind(\mathcal{S})\geq\frac{n+2}{2}$, then $b_2(\mathcal{S})=0$ unless $\mathcal{S}$ is a circle bundle over $\mathbb{CP}^{r-1}\times\mathbb{CP}^{r-1}$.

(v) If $r=Ind(\mathcal{S})=\frac{n+1}{2}$, then $b_2(\mathcal{S})=0$ unless $\mathcal{S}$ is a circle bundle over $\mathcal{Z}$ where $\mathcal{Z}$ is either $\mathbb{CP}^{r-1}\times Q_r(\mathbb{C})$, $\mathbb{P}(T^*\mathbb{CP}^r)$, or possibly $\mathbb{CP}^{2r-1}$ blown up along a $\mathbb{CP}^{r-2}$.

Items (i)-(iii) follow from results of Kobayashi and Ochiai [KO], while items (iv) and (v), due to Wiśniewski [Wi1, Wi2], use Mori theory. The examples in [BGM1] and [BGMR] show that (iv) and (v) cannot hold in the non-regular case.

For dimensional reasons we have immediately the following

COROLLARY 3.2: *Let $\mathcal{S}$ be a regular Sasakian-Einstein manifold of dimension $4n+3$. Then if Ind($\mathcal{S}$) $> n+1$, we have $b_2(\mathcal{S})=0$.*

A related result follows from the boundedness of deformation types of smooth Fano varieties [Kol]. This implies that in any given dimension the Betti numbers of a regular Sasakian-Einstein manifold are bounded by constants only depending on the dimension. In particular, the bound on the second Betti number for regular Sasakian-Einstein 5-manifolds is 8, whereas for regular Sasakian-Einstein 7-manifolds it is 9. These bounds are actually sharp and are discussed in Theorem 3.5 and Proposition 4.12 below, respectively. The results of [BGMR] show that these bounds no longer hold in the non-regular case.

A special case of regular Sasakian-Einstein manifolds are the homogeneous ones. Recall the following well-known terminology. Let G be a complex semi-simple Lie group. A maximal solvable complex subgroup B is called a Borel subgroup, and B is unique up to conjugacy. Any complex subgroup P that contains B is called a parabolic subgroup. Then the homogeneous space G/P is called a generalized flag manifold. A well-known result of Borel and Remmert [BR] says that every connected homogeneous Kähler manifold is a product of a torus and a generalized flag manifold.

DEFINITION 3.3: *A Sasakian manifold $(\mathcal{S},g)$ is called a homogeneous Sasakian manifold if there is a transitive group K of isometries on $\mathcal{S}$ that preserve the Sasakian structure, that is, if $\phi^k\in$ Diff $\mathcal{S}$ corresponds to $k\in K$, then $\phi^k_*\xi=\xi$. (This implies that both Φ and η are also invariant under the action of K.)*

Note that if $\mathcal{S}$ is compact then K is a compact Lie group.

THEOREM 3.4: *Let $(\mathcal{S},g')$ be a complete homogeneous Sasakian manifold with $Ric_{g'}\geq\delta>-2$. Then $(\mathcal{S},g')$ is a compact regular homogeneous Sasakian manifold, and there is a homogeneous Sasakian-Einstein metric g on $\mathcal{S}$ that is compatible with the underlying normal contact structure. Moreover, $\mathcal{S}$ is the total space of an S^1-bundle over a generalized flag manifold G/P whose Chern class is $m\cdot\frac{c_1(G/P)}{Ind(G/P)}$ for some positive integer m. Conversely, given any generalized flag manifold G/P and any positive integer m, the total space $\mathcal{S}_m$ of the circle bundle $\pi:\mathcal{S}_m\longrightarrow G/P$ whose Chern class is given by the above formula has a homogeneous Sasakian-Einstein metric g.*

PROOF: By Boothby and Wang [BW] $\mathcal{S}$ is regular, and by 2.4 and 2.5, $\mathcal{S}$ is compact and fibers over a smooth simply connected Fano variety $\mathcal{Z}$. Moreover, since there is a transitive group of isometries on $(\mathcal{S},g')$ that commutes with the S^1 action generated by ξ, there is a positive Kähler metric h' on $\mathcal{Z}$ together with a transitive group of isometries K' preserving the Kähler structure. It follows from Borel and Remmert [BR] that $\mathcal{Z}$ is a generalized flag manifold G/P where the complex semi-simple Lie group G is the complexification of K' and $P\subset G$ is a parabolic subgroup. Then by a well-known result of Matsushima [Bes] there is a positive Kähler-Einstein metric h on G/P and an automorphism ϕ of the complex structure such that $h=\phi^*h'$. Furthermore, the group $K=\phi^{-1}K'\phi$ acts as a transitive group of isometries on $(G/P,h)$. Thus, by the Kobayashi construction (cf. Corollary 2.10) there is a Sasakian-Einstein metric g on $\mathcal{S}$ that is compatible with the normal contact structure, and the isometry group K lifts to a group $\check{K}$ on $\mathcal{S}$ that leaves π^*h invariant and commutes with the circle group S^1 generated by ξ. It follows after adjusting the scale of h that $\pi^*h+\eta\otimes\eta$ is a Sasakian-Einstein metric on $\mathcal{S}$.

The statement about the Chern class follows from Kobayashi's construction, and the converse follows easily as well. ∎

The classification of all simply connected regular Sasakian-Einstein manifolds depends on the classification of all smooth Fano varieties with a Kähler-Einstein metric. This is a deep and important problem (the well-known Calabi problem for c_1 positive) which has recently met with a great deal of success [TY, Ti1, Ti3, Sui] but whose complete solution is still at large. The resolution of this problem would also describe all regular Sasakian-Einstein manifolds. Indeed, until recently there was a folklore conjecture that stated that any (smooth) Fano variety with no holomorphic vector fields admits a Kähler-Einstein metric. This was first shown to be false in the orbifold category by Ding and Tian [DT] and more recently in the smooth manifold category by Tian [Ti3]. (The folklore conjecture is true in the case of smooth del Pezzo surfaces [TY, Ti2]).

In dimension 5, Friedrich and Kath [FrKat1, BFGK] reduce the classification of compact regular Sasakian-Einstein 5-manifolds to the classification of Fano surfaces with a Kähler-Einstein metric [TY, Ti2]. So Tian's later result [Ti2] actually proves a stronger theorem than appears in [FrKat1, BFGK].

THEOREM 3.5: *Let $(\mathcal{S}, g)$ be a regular Sasakian-Einstein 5-manifold. Then $\mathcal{S} = \tilde{\mathcal{S}}/\mathbb{Z}_m$ where the universal cover $(\tilde{\mathcal{S}}, g)$ is precisely one of the following:*

(1) S^5 with its standard (Sasakian) metric.

(2) The Stiefel manifold $V_2(\mathbb{R}^4)$ of 2-frames in $\mathbb{R}^4$ with the unique Sasakian metric that is a Riemannian submersion over $\mathbb{CP}^1 \times \mathbb{CP}^1$ with its standard complex structure and symmetric metric.

(3) The total space S_k of the S^1 bundles $S_k \to P_k$ for $3 \leq k \leq 8$, where P_k is $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$ with any of its complex structures for which c_1 is positive. Moreover, for each such complex structure there is a unique Sasakian-Einstein metric g on S_k.

We denote the circle bundle S_k with a fixed Sasakian-Einstein structure by $\mathcal{S}_k$. A well-known result of Smale says that any simply connected 5-manifold with H_2 torsion free is diffeomorphic to the connected sum $S^5 \# k(S^2 \times S^3)$ for some nonnegative integer k, which corresponds to the k above. Indeed, Theorem 3.5 implies that there is a regular Sasakian-Einstein structure on each of these manifolds for $k \leq 8$ with the exception of $k = 2$. Furthermore, for $5 \leq k \leq 8$ there are continuous families of Sasakian-Einstein structures, which are inequivalent (i.e. non-trivial deformations) since the Tian families on P_k are inequivalent. This should be contrasted with the 3-Sasakian case which is infinitesimally rigid [PP]. Let $\mathcal{R}_5(k)$ denote the set of regular Sasakian-Einstein structures on $S^5 \# k(S^2 \times S^3)$. Then summarizing we have

COROLLARY 3.6:

(1) $\mathcal{R}_5(k)$ is empty for $k = 2$ and $k \geq 9$.

(2) $\mathcal{R}_5(k) = \{point\}$ for $k = 0, 1, 3, 4$.

(3) Dim $\mathcal{R}_5(k) \geq k - 4$ for $5 \leq k \leq 8$.

An appropriate topology on $\mathcal{R}_5(k)$ will be discussed in section 5, so the concept of dimension is valid. Another corollary of [Ti2] is

COROLLARY 3.7: *Let $\mathcal{S}$ be a regular Sasakian 5-manifold and let $\mathcal{Z}$ denote the space of leaves of the characteristic foliation. Suppose also that $\mathcal{Z}$ is Fano. Then, $\mathcal{S}$ admits a compatible Sasakian-Einstein metric if and only if the Lie algebra $\mathfrak{aut}(\mathcal{Z})$ of holomorphic vector fields on $\mathcal{Z}$ is reductive.*

We finish this section with several examples in higher dimensions. These are S^1

bundles over Fano manifolds known to have Kähler-Einstein metrics.

EXAMPLE 3.8: *Fermat hypersurfaces.* Consider the Fermat hypersurfaces $F_{d,n+1}$ of degree d in $\mathbb{P}^{n+1}$ given in homogeneous coordinates by

$$z_0^d + \cdots + z_{n+1}^d = 0. \tag{3.9}$$

They are Fano for $d \leq n+1$ and Nadel has shown that $F_{d,n+1}$ has a Kähler-Einstein metric for $\frac{n+1}{2} \leq d \leq n+1$. (It was shown previously in [Sui] and [Ti1] that $F_{n,n+1}$ and $F_{n+1,n+1}$ are Kähler-Einstein.) Let $\mathcal{S}_{d,n+1}$ denote the S^1 bundle over $F_{d,n+1}$ determined by Corollary 2.10. The index of $F_{d,n+1}$, hence of $\mathcal{S}_{d,n+1}$, is $n+2-d$. Now the topology of smooth hypersurfaces is well-known [Dim]. The homology is torsion free and by the famous Lefschetz theorem $H_*(F_{d,n+1},\mathbb{Z}) = H_*(\mathbb{P}^n,\mathbb{Z})$ except in the middle dimension n where it is determined by its degree [Dim]. Moreover, $\mathcal{S}_{d,n+1}$ is just the link of the hypersurface 3.9 which is well-known to be $(n-1)$-connected [Dim]. Thus, from the Serre spectral sequence of the circle bundle one finds

$$H^i(\mathcal{S}_{d,n+1},\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i = 0, 2n+1; \\ \mathbb{Q}^{b_n} & \text{if } i = n, n+1; \\ 0 & \text{otherwise,} \end{cases}$$

where the nth Betti number is given by

$$b_n = b_n(\mathcal{S}_{d,n+1}) = (-1)^n\Big(1 + \frac{(1-d)^{n+2}-1}{d}\Big).$$

For example in dimension 7 ($n=3$) we have three Sasakian-Einstein manifolds, namely $\mathcal{S}_{4,4}, \mathcal{S}_{3,4}$, and $\mathcal{S}_{2,4}$. We find

$$b_3(\mathcal{S}_{4,4}) = 60, \quad b_3(\mathcal{S}_{3,4}) = 10, \quad b_3(\mathcal{S}_{2,4}) = 0.$$

To the best of the authors' knowledge the first two give the first Sasakian-Einstein manifolds with $b_3 \neq 0$. Thus, for topological reasons both $\mathcal{S}_{4,4}$ and $\mathcal{S}_{3,4}$ cannot admit a 3-Sasakian structure. More generally the Sasakian-Einstein manifold $\mathcal{S}_{n+2,2n+2}$ has the same dimension and index as a 3-Sasakian manifold, but it cannot admit a 3-Sasakian structure since $b_{2n+1} > 0$. Notice that $F_{2,n+1}$ is just the quadric (which is homogeneous and admits a Kähler-Einstein metric) discussed previously, and that if n is odd it has the cohomology groups of $\mathbb{CP}^n$, but differs in the ring structure. Notice also that generally $\mathcal{S}_{d,n+1}$ is just the Brieskorn manifold described by Equation 3.9 as a submanifold of S^{2n+1}.

A similar analysis can be used to discuss other examples of circle bundles over known Kähler-Einstein Fano manifolds. For example Nadel [Na] shows the existence of Kähler-Einstein metrics on certain Fano complete intersections. Also Koiso and Sakane [Sa, KS1, KS2] proved the existence of Kähler-Einstein metrics on certain almost homogeneous Fano manifolds, and in particular toric Kähler-Einstein Fano manifolds [Mab]. We intend to study the Sasakian-Einstein circle bundles over these manifolds elsewhere.

§4. The Sasakian-Einstein Monoid

In this section we apply a construction due to Wang and Ziller to define a multiplication on the set of quasi-regular Sasakian-Einstein orbifolds.

Definition 4.1: *We denote by $\mathcal{SE}$ the set of compact quasi-regular Sasakian-Einstein orbifolds with $\pi_1^{orb}=0$, by $\mathcal{SE}^s$ the subset of $\mathcal{SE}$ that are smooth manifolds, and by $\mathcal{R}\subset\mathcal{SE}^s$ the subset of compact, simply connected, regular Sasakian-Einstein manifolds. The set $\mathcal{SE}$ is topologized with the $C^{m,\alpha}$ topology, and the subsets are given the subspace topology.*

The condition $\pi_1^{orb}=0$ is made to avoid complications, and since every Sasakian-Einstein orbifold is covered by one with $\pi_1^{orb}=0$, there is no loss of generality. The set $\mathcal{SE}$ is graded by dimension, that is,

$$\mathcal{SE}=\bigoplus_{n=0}^{\infty}\mathcal{SE}_{2n+1},$$

and similarly for $\mathcal{SE}^s$ and $\mathcal{R}$. In the definition of Sasakian structure it is implicitly assumed that $n>0$. So we want to extend the definition of a Sasakian-Einstein structure to the case when $n=0$. This can easily be done since a connected one dimensional orbifold is just an interval with possible boundary, or a circle. So we can just take $\xi=\frac{\partial}{\partial t},\eta=dt,\Phi=0$, with the flat metric $g=dt^2$. In this case the space of leaves $\mathcal{Z}$ of the characteristic foliation is just a point. The unit circle S^1 with this structure will play the role of the identity in our monoid. We define the *index* of any element of $\mathcal{SE}_1$ to be 0 and understand that $\gcd(0,m)=m$ for any integer m. Then we see that $\mathcal{SE}^s_1=\mathcal{R}_1$ is a single point, namely the unit circle with its flat metric. Also it follows from a theorem of Hamilton [Bes] that $\mathcal{SE}^s_3=\mathcal{R}_3=\{pt\}$, namely S^3 with its standard Sasakian-Einstein structure. Moreover, we know of no examples of elements of $\mathcal{SE}^s_5$ that are not in $\mathcal{R}_5$.

We now define a graded multiplication

$$\mathcal{SE}_{2n_1+1}\times\mathcal{SE}_{2n_2+1}\longrightarrow\mathcal{SE}_{2(n_1+n_2)+1}$$

as follows: Let $\mathcal{S}_1,\mathcal{S}_2\in\mathcal{SE}$ of dimension $2n_1+1$ and $2n_2+1$ respectively. Their respective space of leaves $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are Kähler-Einstein Fano orbifolds of complex dimension n_1,n_2, and metrics h_1,h_2, respectively. Moreover, the scalar curvature of $(\mathcal{Z}_i,h_i)$ is $4n_i(n_i+1)$. Now the product orbifold $(\mathcal{Z}_1\times\mathcal{Z}_2,h_1+h_2)$ is Kähler, but in general not Kähler-Einstein. However, by rescaling we see that the metric

$$h'=\frac{(n_1+1)h_1+(n_2+1)h_2}{n_1+n_2+1} \tag{4.2}$$

is Kähler-Einstein and Fano with scalar curvature $4(n_1+n_2)(n_1+n_2+1)$. Furthermore, if $\mathrm{Ind}(\mathcal{Z}_i)$ denotes the indices of $\mathcal{Z}_i$, then the index of $\mathcal{Z}_1\times\mathcal{Z}_2$ is just $\gcd(\mathrm{Ind}(\mathcal{Z}_1),\mathrm{Ind}(\mathcal{Z}_2))$. Thus, by Corollary 2.10 the S^1 V-bundle on $\mathcal{Z}_1\times\mathcal{Z}_2$ whose first Chern class is

$$\frac{c_1(\mathcal{Z}_1\times\mathcal{Z}_2)}{\mathrm{Ind}(\mathcal{Z}_1\times\mathcal{Z}_2)}=\frac{c_1(\mathcal{Z}_1)+c_1(\mathcal{Z}_2)}{\gcd(\mathrm{Ind}(\mathcal{Z}_1),\mathrm{Ind}(\mathcal{Z}_2))} \tag{4.3}$$

is simply connected in the orbifold sense with a Sasakian-Einstein structure determined by Theorem 2.8. We denote this multiplication by $\star$, and refer to it as the *join*. It is clear from the construction that if $\mathcal{S}_1,\mathcal{S}_2\in\mathcal{SE}$ then also $\mathcal{S}_1\star\mathcal{S}_2\in\mathcal{SE}$ with the Sasakian-Einstein structure as defined above, and that this procedure can be iterated in such a way that $\star$ is associative. It is also commutative up to isomorphism. Furthermore, $\star$ is continuous in both factors. We have arrived at:

THEOREM 4.4: *The operation $\star$ defined above gives $\mathcal{SE}$ the structure of a commutative associative topological monoid.*

Actually we are interested in the subset $\mathcal{SE}^s$ of smooth Sasakian-Einstein manifolds. However, it is certainly not true that $\mathcal{SE}^s$ is a submonoid. We want conditions that guarantee that if $\mathcal{S}_i \in \mathcal{SE}^s$ for $i = 1,2$ then $\mathcal{S}_1 \star \mathcal{S}_2 \in \mathcal{SE}^s$. For $i = 1,2$ we define the *relative indices* of the pair $(\mathcal{S}_1, \mathcal{S}_2)$ by

4.5
$$l_i = \frac{\mathrm{Ind}(\mathcal{S}_i)}{\gcd(\mathrm{Ind}(\mathcal{S}_1), \mathrm{Ind}(\mathcal{S}_2))}.$$

Then, $\gcd(l_1, l_2) = 1$, and we have,

PROPOSITION 4.6: *Let $\mathcal{S}_i \in \mathcal{SE}^s$ with orders m_i, and relative indices l_i, respectively. Then*

(i) $Ind(\mathcal{S}_1 \star \mathcal{S}_2) = \gcd(Ind(\mathcal{S}_1), Ind(\mathcal{S}_2))$.

(ii) $\mathcal{S}_1 \star \mathcal{S}_2 \in \mathcal{SE}^s$ if and only if $\gcd(m_1 l_2, m_2 l_1) = 1$.

(iii) $\mathcal{R}$ is a submonoid.

(iv) If $\mathcal{S}_1$ is regular (i.e., $m_1 = 1$) and $Ind(\mathcal{S}_2)$ divides $Ind(\mathcal{S}_1)$, then $\mathcal{S}_1 \star \mathcal{S}_2 \in \mathcal{SE}^s$ independently of m_2.

PROOF: (i) and (iii) are clear from the discussion above, and (iv) is a special case of (ii), which we now prove. Since $\mathcal{S}_1 \times \mathcal{S}_2$ is a 2-torus V-bundle over $\mathcal{Z}_1 \times \mathcal{Z}_2$, the S^1 V-bundle $\mathcal{S}_1 \star \mathcal{S}_2$ can be realized as a quotient $(\mathcal{S}_1 \times \mathcal{S}_2)/S^1$ by choosing a certain homomorphism $S^1 \longrightarrow S^1 \times S^1$. For the V-bundle with Chern class given by 4.3 we have the action on $\mathcal{S}_1 \times \mathcal{S}_2$ given by $(x,y) \mapsto (\tau^{l_2} x, \tau^{-l_1} y)$. The condition that $\mathcal{S}_1 \star \mathcal{S}_2$ is in $\mathcal{SE}^s$ is that there are no fixed points (x,y) under the above action. If m_x, m_y denote the orders of the local uniformizing groups (leaf holonomy groups) of x, y, respectively, then the condition that (x,y) be a fixed point is that $\gcd(m_x l_2, m_y l_1) = g > 1$. That this condition never hold for all pairs (x,y) is precisely the condition $\gcd(m_1 l_2, m_2 l_1) = 1$. ∎

This proposition will allow us to construct smooth examples of quasi-regular Sasakian-Einstein manifolds in all odd dimensions greater than or equal to 9. Before discussing some important examples, we introduce some more terminology.

DEFINITION 4.7: *We say that $\mathcal{S} \in \mathcal{SE}$ is $\mathcal{SE}$-irreducible if writing $\mathcal{S} = \mathcal{S}_1 \star \mathcal{S}_2$ implies that either $\mathcal{S}_1$ or $\mathcal{S}_2$ is S^1 with its flat structure. $\mathcal{S}$ is $\mathcal{SE}$-reducible if it is not $\mathcal{SE}$-irreducible.*

It is clear that $\mathcal{SE}$-irreducibility corresponds to Riemannian irreducibility on the space of leaves. Reducibility first occurs in dimension 5, and up to diffeomorphism there is precisely one regular $\mathcal{SE}$-reducible 5-manifold, namely the homogeneous Stiefel manifold $V_2(\mathbb{R}^4) \approx S^2 \times S^3 \approx S^3 \star S^3$. In terms of the homogeneous Sasakian-Einstein manifolds, reducibility has a group theoretic interpretation. A well-known theorem of Borel [Bo] states a simply connected homogeneous Kähler manifold corresponding to a complex semi-simple Lie group G is Riemannian irreducible if and only if G is simple. It is easy to see that in terms of Sasakian-Einstein geometry this implies

PROPOSITION 4.8: *A simply connected Sasakian-Einstein homogeneous manifold corresponding to a semi-simple Lie group G is $\mathcal{SE}$-irreducible if and only if G is simple.*

Moreover, our construction also implies

PROPOSITION 4.9: *Let $\mathcal{S}_1, \mathcal{S}_2$ be simply connected homogeneous Sasakian-Einstein manifolds, then $\mathcal{S}_1 \star \mathcal{S}_2$ is a simply connected homogeneous Sasakian-Einstein manifold.*

This proposition states that the subset $\mathcal{H}$ of simply connected homogeneous Sasakian-Einstein manifolds forms a submonoid of $\mathcal{R}$.

As noted by Theorem 3.5, the regular Sasakian-Einstein 5-manifolds have been classified. This immediately gives a classification of the regular $\mathcal{SE}$-reducible 7-manifolds.

PROPOSITION 4.10: *Any simply connected regular $\mathcal{SE}$-reducible 7-manifold is one of the following:*

$$S^3 \star S^3 \star S^3, \quad S^3 \star S^5, \quad S^3 \star \mathcal{S}_k$$

for $3 \leq k \leq 8$ where $\mathcal{S}_k$ is one of the Sasakian-Einstein circle bundles over the del Pezzo surface discussed in Theorem 3.5.

These examples have already been noted in [BFGK]. Non simply connected examples are obtained by quotienting by a cyclic subgroup of the circle generated by the characteristic vector field. Notice also that for $5 \leq k \leq 8$ the 7-manifolds $S^3 \star S_k$ have continuous families of Sasakian-Einstein structures on them. We shall discuss these shortly, but first we mention that in the list given in Proposition 4.10 only the first two are homogeneous. In fact as already noted in [BG2] we have

PROPOSITION 4.11: *The simply connected homogeneous Sasakian-Einstein 7-manifolds are precisely one of the following:*

(i) An $\mathcal{SE}$-reducible manifold $S^3 \star S^3 \star S^3$ or $S^3 \star S^5$.

(ii) The real Stiefel manifold $V_{5,2}$.

(iii) A homogeneous 3-Sasakian 7-manifold S^7 or $\mathcal{S}(1,1,1)$.

Of course, there are other regular $\mathcal{SE}$-irreducible 7-manifolds as noted in Examples 3.8 and 3.10, but so far a complete classification of regular Sasakian-Einstein 7-manifolds is lacking. Actually the results of Mori and Mukai [MoMu] on the classification of smooth Fano 3-folds with $b_2 \geq 2$ go quite far toward a classification of regular Sasakian-Einstein 7-manifolds, but this is beyond the scope of the present paper, and is currently under investigation. Nevertheless, there are some immediate consequences of their work worth mentioning here, namely

PROPOSITION 4.12: *Let $\mathcal{S}$ be a compact regular Sasakian-Einstein 7-manifold. Then*

(i) $b_2(\mathcal{S}) \leq 9$.

(ii) If $b_2(\mathcal{S}) \geq 5$ then $\mathcal{S}$ is reducible. Explicitly $\mathcal{S} \approx S^3 \star \mathcal{S}_k$, where $4 \leq k \leq 8$.

Notice that the upper bound on the second Betti number for regular Sasakian-Einstein 7-manifolds is realized by $S^3 \star S_8$, and there is a continuous 4-dimensional family of Sasakian-Einstein structures on these manifolds. It is interesting to contemplate whether it is generally true that the bound on b_2 is realized by a $\mathcal{SE}$-reducible element in $\mathcal{R}$. If this were true then the regular Sasakian-Einstein 9-manifold with the largest second Betti number would be $S_8 \star S_8$ with $b_2 = 17$. The answer probably lies in Mori theory. For general odd dimension greater than 3 our join construction gives

PROPOSITION 4.13: *For each positive integer $n \geq 2$ there exists smooth manifolds of dimension $2n+1$ that admit continuous families $\mathcal{F}$ of regular Sasakian-Einstein structures. Examples of such the manifolds are*

$$S_k, \quad \overbrace{S^3 \star \cdots \star S^3}^{n-2 \text{ times}} \star S_k$$

for $5 \leq k \leq 8$, as well joins with other regular Sasakian-Einstein manifolds. Moreover, the dimension of the family $\mathcal{F}$ of Sasakian-Einstein structures on S_k is $\geq k-4$.

This result should be contrasted with the 3-Sasakian case which is infinitesimally rigid [PP]. Manifolds that admit continuous families of non-regular Sasakian-Einstein structures also exist as will be discussed below. However, they begin in dimension 11.

Next we turn to the more lucrative non-regular case. Propositions 2.12 and 4.6 imply the following:

COROLLARY 4.14: *Let $\mathcal{S}$ be any compact simply connected 3-Sasakian manifold of dimension $4n+3$ that is not a sphere. Then for any positive integer r the join $S^{2(n+1)r-1}\star\mathcal{S}$ is a smooth Sasakian-Einstein manifold of dimension $4n+2nr+2r+1$. In particular, if $n=1$, so that $\dim\mathcal{S}=7$, then $S^3\star\mathcal{S}$ is a smooth Sasakian-Einstein 9-manifold.*

COROLLARY 4.15: *Let $\mathcal{S}$ be any compact simply connected Sasakian-Einstein 7-manifold of index 2, for example, a 3-Sasakian 7-manifold that is not S^7. Then $S^{2m+1}\star\mathcal{S}$ is smooth if m is odd or if m is even and the order of $\mathcal{S}$ is odd.*

The last case in Corollary 4.14 will prove to be of much interest to us. Notice that $\mathrm{Ind}(S^3\star\mathcal{S})=\gcd(\mathrm{Ind}(S^3),\mathrm{Ind}(\mathcal{S}))=2$, so that the procedure iterates arriving at:

COROLLARY 4.16: *Let $\mathcal{S}$ be any compact simply connected 3-Sasakian manifold of dimension 7 that is not a 7-sphere. Then for any positive integer r the r-fold join $S^3\star\cdots\star S^3\star\mathcal{S}$ is a smooth Sasakian-Einstein $2r+7$-manifold of index 2.*

The examples of a 3-Sasakian 7-manifold that we have in mind are the toric 3-Sasakian manifolds $\mathcal{S}(\Omega_k)$ of [BGMR] where Ω_k is a k by $k+2$ matrix of integers which satisfy certain gcd conditions (see [BGMR] for details), and $b_2(\mathcal{S}(\Omega_k))=k$. In the case of the join of two non-regular Sasakian-Einstein manifolds and/or when the relative indices are different, the gcd conditions in Proposition 4.6 are generally fairly restrictive. For example, let $\mathcal{S}_1,\mathcal{S}_2\in\mathcal{SE}^s_{4n+3}$ be two 3-Sasakian manifolds neither of which are spheres. Then $l_1=l_2=1$, and the smoothness conditions become $\gcd(m_1,m_2)=1$. Even this is restrictive since orders tend to be large and have many divisors. However, consider the 3-Sasakian manifolds $\mathcal{S}(1,\cdots,1,2p_i+1)$ discussed in [BGM2]. In these cases $m_i=p_i+1$, so if we choose $\gcd(p_1+1,p_2+1)=1$, which is easy to satisfy, we get a smooth join. Notice, however, that $\mathcal{S}\star\mathcal{S}$ is never smooth for $\mathcal{S}$ non-regular.

Finally we consider some non-regular examples of manifolds that admit continuous families of Sasakian-Einstein structures. Then Proposition 4.6 implies

PROPOSITION 4.17: *Let $\mathcal{S}$ be a simply connected Sasakian-Einstein manifold, and S_k the circle bundle over the del Pezzo surface defined in Theorem 3.5. Then*

(i) $S_k\star\mathcal{S}$ admits a continuous family of dimension $\geq k-4$ of inequivalent non-regular Sasakian-Einstein structures when $5\leq k\leq 8$.

(ii) $S_k\star\mathcal{S}$ is a smooth manifold if and only if $\gcd(Ind(\mathcal{S}),Ord(\mathcal{S}))=1$. In particular, $S_k\star\mathcal{S}(\Omega_k)$ is smooth if and only if $Ord(\mathcal{S}(\Omega_k))$ is odd.

We have only computed the orders of the 3-Sasakian manifolds $\mathcal{S}(\Omega_k)$ in the case $k=1$. These are the 3-Sasakian 7-manifolds $\mathcal{S}(p_1,p_2,p_3)$ of [BGM2] where the p_i's are pairwise relatively prime. Furthermore, the order is the least common multiple of $(\frac{p_1+p_2}{2})(\frac{p+p_3}{2})(\frac{p_2+p_3}{2})$ if all p_i's are odd, and the least common multiple of the product $(p_1+p_2)(p_1+p_3)(p_2+p_3)$ if one of the p_i's is even. In the last case one of the sums is even so this can be eliminated. In the first case we find a solution if $p_i=4r_i+1$ for some natural numbers r_i. Summarizing we have

COROLLARY 4.18: *The 11 dimensional orbifolds $S_k\star\mathcal{S}(p_1,p_2,p_3)$ are smooth manifolds if and only if $p_i=4r_i+1$ for some natural numbers r_i which satisfy $\gcd(4r_i+1,4r_j+1)=1$. Furthermore, if $5\leq k\leq 8$ the manifolds $S_k\star\mathcal{S}(p_1,p_2,p_3)$ admit a continuous family of dimension $\geq k-4$ of inequivalent non-regular Sasakian-Einstein structures. In particular, for each natural number r and for $5\leq k\leq 8$ the 11-manifolds $S_k\star\mathcal{S}(1,1,4r+1)$ admit such continuous families of non-regular Sasakian-Einstein structures.*

The gcd condition in Proposition 4.17 seemingly becomes less restrictive when the dimension of $\mathcal{S}$ is such that $n+1$ is a large prime. However, it is in dimension 7 that we have the most interesting examples of 3-Sasakian manifolds, those having arbitrary second Betti number constructed in [BGMR]. However, as previously mentioned it is only in the case $b_2=k=1$ that we have computed the order. It would be interesting to see whether there exist continuous families of Sasakian-Einstein structures on manifolds with any second Betti number.

Notice that once we find solutions $\mathcal{S}$ to the conditions in Proposition 4.16, we can easily construct solutions in higher dimension, for example by joining with S^3. Since $\mathrm{Ind}(\mathcal{S}_k\star\mathcal{S})=1$ the smoothness conditions for $S^3\star\mathcal{S}_k\star\mathcal{S}$ are automatically satisfied. Thus,

COROLLARY 4.19: *Let $\mathcal{S}$ be a simply connected Sasakian-Einstein manifold of dimension $2n+1$ satisfying $\gcd(Ind(\mathcal{S}),Ord(\mathcal{S}))=1$, then $\overbrace{S^3\star\cdots\star S^3}^{m\text{ times}}\star S_k\star\mathcal{S}$ is a smooth $2(m+n)+5$-manifold that admits a continuous family of dimension $\geq k-4$ of Sasakian-Einstein structures. In particular, the $(2m+11)$-manifolds $\overbrace{S^3\star\cdots\star S^3}^{m\text{ times}}\star S_l\star\mathcal{S}(\Omega_k)$ with $Ord(\mathcal{S}(\Omega_k))$ odd has second Betti number $b_2=k$ and continuous families of inequivalent Sasakian-Einstein structures.*

We should mention that we do not know whether the condition $\mathrm{Ord}(\mathcal{S}(\Omega_k))$ be odd can be satisfied for arbitrary k.

§5. The Cohomology of Some Joins

We first obtain some general information about the low Betti numbers of the join $\mathcal{S}_1\star\mathcal{S}_2$ of two Sasakian-Einstein manifolds (orbifolds). The following lemma follows from harmonic theory, and was given in [BG1]:

LEMMA 5.1: *Let $\mathcal{S}$ be a Sasakian-Einstein orbifold of dimension $2n+1$, then for $0\leq r\leq n$ we have $b_r(\mathcal{S})=b_r(\mathcal{Z})-b_{r-2}(\mathcal{Z})$.*

This lemma can be used to show:

LEMMA 5.2: *Let $\mathcal{S}_i\in\mathcal{SE}_{2n_i+1}$, then*

(1) $b_2(\mathcal{S}_1\star\mathcal{S}_2)=b_2(\mathcal{S}_1)+b_2(\mathcal{S}_2)+1$ *if* $n_i\geq 1$,

(2) $b_3(\mathcal{S}_1\star\mathcal{S}_2)=b_3(\mathcal{S}_1)+b_3(\mathcal{S}_2)$ *if* $n_i\geq 3$,

(3) $b_4(\mathcal{S}_1\star\mathcal{S}_2)=b_4(\mathcal{S}_1)+b_4(\mathcal{S}_2)+b_2(\mathcal{S}_1)b_2(\mathcal{S}_2)+b_2(\mathcal{S}_1)+b_2(\mathcal{S}_2)+1$ *if* $n_i\geq 4$.

When n_i is outside the indicated range, the formula is slightly different, but is easily worked out. For general b_r the formulas are increasingly more complicated and are different depending on whether r is even or odd, or whether the range conditions are satisfied or not. In order to determine the cohomology of $\mathcal{S}_1\star\mathcal{S}_2$ in specific cases, we shall employ a more elegant technique using spectral sequences used by Wang and Ziller [WZ].

Let $\mathcal{S}_i\in\mathcal{SE}^s$, and consider the commutative diagram of fibrations

5.3
$$\begin{array}{ccccc}
\mathcal{S}_1\times\mathcal{S}_2 & \longrightarrow & B(\mathcal{S}_1\star\mathcal{S}_2) & \longrightarrow & BS^1\\
\Big\downarrow{\scriptstyle =} & & \Big\downarrow & & \Big\downarrow{\scriptstyle\psi}\\
\mathcal{S}_1\times\mathcal{S}_2 & \longrightarrow & B\mathcal{Z}_1\times B\mathcal{Z}_2 & \longrightarrow & BS^1\times BS^1.
\end{array}$$

The maps are all the obvious ones. In particular, ψ is determined by the S^1 action of the previous section, namely, $\psi(\tau)=(\tau^{l_2},\tau^{-l_1})$. The point is that the differentials in the Serre spectral sequence of the top fibration are determined through naturality by the differentials in the Serre spectral sequence of the bottom fibration. Wang and Ziller apply this method to computing the integral cohomology ring of more general circle bundles (torus bundles as well) over products of projective spaces. For us, this corresponds to the case $S^{2m+1}\star S^{2n+1}$, which is homogeneous. We refer to [WZ] for the cohomology ring in this case. At first we shall apply this method to a more general situation where only rational information can be obtained; however, there are several cases of interest to us where we have enough information about the differentials to compute the integral cohomology groups. Recall that from its definition the join $\mathcal{S}_1\star\mathcal{S}_2$ of two simply connected Sasakian-Einstein manifolds is necessarily simply connected (in the orbifold sense if $\mathcal{S}_1\star\mathcal{S}_2$ is not smooth). Nevertheless, we can easily obtain non-simply connected Sasakian-Einstein manifolds with cyclic fundamental group by dividing by a cyclic subgroup of the circle generated by the characteristic vector field ξ. Hereafter, in this section all joins are simply connected.

THEOREM 5.4: *Let $\mathcal{S}$ be any simply connected 3-Sasakian 7-manifold which is not the 7-sphere. Then*

$$H^q(S^{2m+1}\star\mathcal{S},\mathbb{Q})\approx\begin{cases}H^q(S^{2m+1}\times\mathcal{Z},\mathbb{Q}) & \text{if } m>2;\\ H^q(S^2\times\mathcal{S},\mathbb{Q}) & \text{if } m=1;\\ \begin{cases}\mathbb{Q} & \text{if } q=0,11;\\ \mathbb{Q}^{k+1} & \text{if } q=2,4,7,9;\\ 0 & \text{if } q=1,10,3,8;\end{cases} & \text{if } m=2,\end{cases}$$

where $k=b_2(\mathcal{S})$. Moreover, for the case when $m=2$ if $k>1$ then $H^5(S^5\star\mathcal{S},\mathbb{Q})\approx H^6(S^5\star\mathcal{S},\mathbb{Q})\approx 0$; whereas if $k=1$ there are two possibilities; either $H^5(S^5\star\mathcal{S},\mathbb{Q})\approx H^6(S^5\star\mathcal{S},\mathbb{Q})\approx 0$, or $H^5(S^5\star\mathcal{S},\mathbb{Q})\approx H^6(S^5\star\mathcal{S},\mathbb{Q})\approx\mathbb{Q}$.

Notice in this case that the rational cohomology of $S^{2m+1}\star\mathcal{S}$ depends only on $b_2(\mathcal{S})$. Moreover, when $m=2$ and $k>1$ it does not have the rational cohomology of a product. Even in other cases $S^{2m+1}\star\mathcal{S}$ cannot be a product. For example, if m is odd and greater than 1, then by Corollary 4.8 $S^{2m+1}\star\mathcal{S}$ is smooth, but if $\mathcal{S}$ is non-regular, $S^{2m+1}\times\mathcal{Z}$ is not. However, in the second possibility for $m=2$ and $k=1$ we see that $H^q(S^5\star\mathcal{S},\mathbb{Q})\approx H^q(\mathbb{CP}^2\times\mathcal{S},\mathbb{Q})$.

PROOF OF THEOREM 5.4: We consider the Serre spectral sequence of the fibrations in diagram 5.3. We also make use of the fact (Corollary 1.17) that rationally H^q and H^q_{orb} coincide. If $m>3$ then the orientation class u of S^{2m+1} in the fiber $S^{2m+1}\times\mathcal{S}$ occurs after the orientation class of $\mathcal{S}$, so by commutativity of diagram 5.3, $E_2^{p,q}$ of the Serre spectral sequence for the top fibration in 5.3 coincides rationally with the E_2 term for the spectral sequence for the fibration $\mathcal{S}\longrightarrow B\mathcal{Z}\longrightarrow BS^1$ for $q<2m+1$. It follows that the E_2 term of the former converges to the cohomology of the product $S^{2m+1}\times\mathcal{Z}$. For $m=3$ there are two 7-classes in the cohomology of the fiber, say u and v. By naturality of the spectral sequences, we have that $d_8(u)=l_2^4s^4$ and $d_8(v)=l_1^4s^4$ where l_i are given by 4.5. The same argument goes through as before, but now it is the 7-class $l_1^4u-l_2^4v$ that survives to the limit. So again rationally the cohomology is the cohomology of the product $S^7\times\mathcal{Z}$.

Next consider the case $S^3\star\mathcal{S}$ for which $(l_1,l_2)=(1,1)$. Here we discuss this case integrally with the added assumption that $H^3(\mathcal{S},\mathbb{Z})=0$, since it will be treated in Theorem 5.6 below. By the Leray-Serre Theorem the E_2 term of the spectral sequence for the fibration $S^3\times\mathcal{S}\longrightarrow S^3\star\mathcal{S}\longrightarrow BS^1$ is given by

$$
\begin{array}{c|ccccccc}
q & & & & & & & \\
 & \mathbb{Z}^{2k} & 0 & \mathbb{Z}^{2k} & 0 & & & \\
 & T & 0 & T & 0 & T & & \\
 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \\
 & \mathbb{Z}^{k} & 0 & \mathbb{Z}^{k} & 0 & \mathbb{Z}^{k} & 0 & \mathbb{Z}^{k} \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} \\
\hline
 & & & & & & & & p
\end{array}
$$

Diagram 5.5: $E_2^{p,q}$ for $S^3 \times \mathcal{S} \longrightarrow S^3 \star \mathcal{S} \longrightarrow BS^1$

Again let u denote the orientation class of S^3 and s the 2-class of BS^1, then by naturality of the diagram 5.3, we have $d_4(u) = s^2$. Now $S^3 \star \mathcal{S}$ is a 9-manifold, so by Poincaré duality it suffices to consider the diagram up to and including dimension 4. But it is easily seen from Diagram 5.5 that no other differentials can occur in this range. Furthermore, the torsion groups T in row 4 do not occur rationally, and the result follows.

Finally we consider the case $m = 2$. Over $\mathbb{Q}$ the E_2 term of the spectral sequence in this case is obtained from Diagram 5.5 by tensoring with $\mathbb{Q}$, putting 0's in the fourth row (*i.e.*, $E_2^{p,3} = 0$), and changing the $\mathbb{Q}^{2k}$'s in row six to $\mathbb{Q}^{k+1}$. Now there are no differentials below level 5, so by Poincaré duality we need only concern ourselves with the $E_2^{0,5}$ term. Here there are $k+1$ classes, namely the orientation class u of S^5 and the 5-classes $\beta_i \in H^5(\mathcal{S}, \mathbb{Q})$ that are the Poincaré duals of the 2-classes in $H^2(\mathcal{S}, \mathbb{Q})$. By looking at the rational spectral sequence of the fibration $\mathcal{S} \longrightarrow B\mathcal{Z} \longrightarrow BS^1$, one sees that there are two possibilities (details will be given elsewhere). The first possibility is that $d_4(\beta_i) = \sum_j a_{ij} s^2 \otimes \alpha_j$ where the rank of the matrix (a_{ij}) is k. In this case we also have $d_6(u) = s^3$. This implies that $H^5(S^5 \star \mathcal{S}, \mathbb{Q}) \approx H^6(S^5 \star \mathcal{S}, \mathbb{Q}) \approx 0$. The second possibility only occurs if $k = 1$, and in this case we get $H^5(S^5 \star \mathcal{S}, \mathbb{Q}) \approx H^6(S^5 \star \mathcal{S}, \mathbb{Q}) \approx \mathbb{Q}$. ∎

Generally, this only determines the rational cohomology; however, for $S^3 \star \mathcal{S}$, since $(l_1, l_2) = (1, 1)$ the only torsion class in dimension 4 occurs in the $\mathcal{S}$ factor on the fiber. This class will survive to E_∞. Thus, it remains to solve an extension problem to determine the integral cohomology in this case. Below dimension 4 integral information follows easily from Diagram 5.5; however, since determination of the full integral cohomology requires some more detailed knowledge we shall specialize to the case of the 3-Sasakian manifolds $\mathcal{S} = \mathcal{S}(p_1, p_2, p_3)$ described in [BGM2] for the complete picture. Actually we do not directly solve the extension problem, but rather take a different tact.

THEOREM 5.6: *Let $\mathcal{S}$ be a simply connected Sasakian-Einstein 7-manifold of index 2 with $H^3(\mathcal{S}, \mathbb{Z}) \approx 0$ and second Betti number $b_2(\mathcal{S}) = k$. Then we have $H^2(S^3 \star \mathcal{S}, \mathbb{Z}) \approx \mathbb{Z}^{k+1}$ and $H^3(S^3 \star \mathcal{S}, \mathbb{Z}) \approx 0$. More specifically, if $\mathcal{S}(p_1, p_2, p_3)$ is one of the simply connected 3-Sasakian 7-manifolds described in [BGM2] with the p_i's pairwise relatively prime. Then* *there is an isomorphism*

$$H^q(S^3 \star \mathcal{S}(p_1,p_2,p_3),\mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \textit{if } q = 0,9,5; \\ \mathbb{Z}^2 & \textit{if } q = 2,7; \\ \mathbb{Z} \oplus \mathbb{Z}_{\sigma_2} & \textit{if } q = 4; \\ \mathbb{Z}_{\sigma_2} & \textit{if } q = 6; \\ 0 & \textit{otherwise,} \end{cases}$$

where $\sigma_2 = p_1p_2+p_1p_3+p_2p_3$. *In fact there is a ring isomorphism* $H^*(S^3 \star \mathcal{S}(p_1,p_2,p_3),\mathbb{Z}) \approx H^*(S^2 \times \mathcal{S}(p_1,p_2,p_3),\mathbb{Z})$.

Proof: As mentioned above the first statement follows easily from Diagram 5.5. However, the proof of the remainder is much more involved. By Theorem 5.4 and Poincaré duality it suffices to show $H^4(S^3 \star \mathcal{S}(p_1,p_2,p_3),\mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z}_{\sigma_2}$. Our approach is to consider $S^3 \times SU(3)$ as a 2-torus bundle over $S^3 \star \mathcal{S}(p_1,p_2,p_3)$. Consider the action of $S^1 \times S^1$ on $S^3 \times SU(3)$ given by

5.7 $$(q,\mathbb{A}) \longmapsto (q\rho, \operatorname{diag}(\tau^{p_1},\tau^{p_2},\tau^{p_3})\mathbb{A}\ \operatorname{diag}(\rho,\rho^{-1},\tau^{-p_1-p_2-p_3})).$$

It is well-known that the cohomology ring of $SU(3)$ is isomorphic to the exterior algebra over 2 generators e_3, e_5 in cohomological dimension 3 and 5, respectively. So the E_2 term of the Leray-Serre spectral sequence for the fibration

5.8 $$S^3 \times SU(3) \longrightarrow S^3 \star \mathcal{S}(p_1,p_2,p_3) \longrightarrow BS^1 \times BS^1$$

is given by

$$\begin{array}{c|ccccccc} q & & & & & & & \\ & \mathbb{Z} & 0 & \mathbb{Z} & 0 & & & \\ & 0 & 0 & 0 & 0 & 0 & & \\ & \mathbb{Z}^2 & 0 & \mathbb{Z}^4 & 0 & \mathbb{Z}^6 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \mathbb{Z} & 0 & \mathbb{Z}^2 & 0 & \mathbb{Z}^3 & 0 & \mathbb{Z}^4 \\ \hline & & & & & & & p \end{array}$$

Diagram 5.9: $E_2^{p,q}$ for $S^3 \times SU(3) \longrightarrow S^3 \star \mathcal{S}(p_1,p_2,p_3) \longrightarrow BS^1 \times BS^1$

Let u, e_3 denote the 3-classes in $E_2^{0,3}$. We need to determine the differentials $d_4(u)$ and $d_4(e_3)$. The theorem follows easily from

Lemma 5.10: *The differentials satisfy*

(i) $d_4(u)=t^2$,

(ii) $d_4(e_3)=\sigma_2 s^2-t^2$, *where t and s are the corresponding 2-classes in $BS^1\times BS^1$.*

PROOF: First we notice from the form of the action 5.8 that the fibration 5.8 factors through the fibration

5.11 $$S^3\times SU(3)\longrightarrow S^3\times \mathcal{S}(p_1,p_2,p_3)\longrightarrow BS^1,$$

where the S^1 action is defined by the subgroup obtained by putting $\rho=\mathrm{id}$ in 5.7. This S^1 acts only on the $SU(3)$ factor, and the $d_4(e_3)$ was determined in [BGM2] to be $d_4(e_3)=\pm\sigma_2 s^2$. Thus, by naturality for the fibration 5.8 we must have the form $d_4(e_3)=\sigma_2 s^2+at^2+bst$ where a and b are to be determined. Actually, since the fibration 5.8 factors through both the fibration 5.11 and the fibration

5.12 $$S^3\times SU(3)\longrightarrow (S^3\times SU(3))/S^1(\rho)\longrightarrow BS^1,$$

where the action $S^1(\rho)$ is obtained from 5.8 by putting $\tau=\mathrm{id}$, it follows that $b=0$. Now the action $S^1(\rho)$ is a subgroup of a 2-torus action which acts on the S^3 and $SU(3)$ factors separately. So we have a commutative diagram of fibrations

5.13 $$\begin{array}{ccccc} S^3\times SU(3) & \longrightarrow & (S^3\times SU(3))/S^1(\rho) & \longrightarrow & BS^1 \\ \Big\downarrow = & & \Big\downarrow & & \Big\downarrow \\ S^3\times SU(3) & \longrightarrow & S^2\times M_{1,-1} & \longrightarrow & BS^1\times BS^1, \end{array}$$

where $M_{1,-1}$ is a certain Aloff-Wallach manifold described in Eschenburg [Esch1-2]. Now for the bottom row we have the Hopf fibration on the first factor, so $d_4(u)=t_1^2$, and Eschenburg shows that $d_4(e_3)=t_2^2$ for the second factor. By naturality these pull back to $d_4(u)=t^2$ and $d_4(e_3)=t^2$ for the fibration on the top row of 5.13. Now combining 5.13 with 5.9 we have the commutative diagram of fibrations

5.14 $$\begin{array}{ccccc} S^3\times SU(3) & \longrightarrow & S^3\star\mathcal{S}(p_1,p_2,p_3) & \longrightarrow & BS^1\times BS^1 \\ \Big\downarrow = & & \Big\downarrow & & \Big\downarrow \psi \\ S^3\times SU(3) & \longrightarrow & S^2\times \mathcal{Z}(p_1,p_2,p_3) & \longrightarrow & BS^1\times BS^1\times BS^1. \end{array}$$

Here the map ψ is induced by $(\rho,\tau)\mapsto(\rho,\rho^{-1},\tau)$. Pulling back by this map proves the lemma, and hence, Theorem 5.6. ∎

Now let $\mathcal{S}$ be a simply connected Sasakian-Einstein 7-manifold of index 2 with vanishing $H^3(\mathcal{S},\mathbb{Z})$, and consider the r-fold iterates $S^3\star\cdots\star S^3\star\mathcal{S}$ which by Proposition 4.6 are all have smooth of index 2. For general $\mathcal{S}$ the argument is with rational coefficients, but the argument will work integrally for any such $\mathcal{S}$ that satisfies $H^*(S^3\star\mathcal{S},\mathbb{Z})\approx H^*(S^2\times\mathcal{S},\mathbb{Z})$. So assume by induction that in the fibration

$$S^3\times(S^3\star\cdots\star S^3\star\mathcal{S})\longrightarrow S^3\star\cdots\star S^3\star\mathcal{S}\longrightarrow BS^1$$

the fiber is homologically $S^3\times S^2\times\cdots S^2\times\mathcal{S}$. Then in the spectral sequence there are no differentials off of any of the S^2 factors, so all the differentials come from the $S^3\times\mathcal{S}$ term as in Theorems 5.4 and 5.7. It follows that the total space is homologically $S^2\times\cdots S^2\times\mathcal{S}$, and we have arrived at

Theorem 5.15: *Let $\mathcal{S}$ be a simply connected Sasakian-Einstein 7-manifold of index 2 with $H^3(\mathcal{S},\mathbb{Z})=0$. Then there are group isomorphisms*

$$H^2(S^3\star\cdots\star S^3\star\mathcal{S},\mathbb{Z})\approx H^2(S^2\times\cdots\times S^2\times\mathcal{S},\mathbb{Z}),\quad H^3(S^3\star\cdots\star S^3\star\mathcal{S},\mathbb{Q})\approx 0,$$

and ring isomorphisms

$$H^*(S^3\star\cdots\star S^3\star\mathcal{S},\mathbb{Q})\approx H^*(S^2\times\cdots\times S^2\times\mathcal{S},\mathbb{Q}),$$

and

$$H^*(S^3\star\cdots\star S^3\star\mathcal{S}(p_1,p_2,p_3),\mathbb{Z})\approx H^*(S^2\times\cdots\times S^2\times\mathcal{S}(p_1,p_2,p_3),\mathbb{Z}).$$

It is interesting to speculate as to whether $S^3\star\cdots\star S^3\star\mathcal{S}$ is actually homeomorphic (or even diffeomorphic) to $S^2\times\cdots\times S^2\times\mathcal{S}$ in this theorem. It is not true for the homogeneous space $S^3\star S^7$, as it was shown by Wang and Ziller [WZ], where in their notation $S^3\star S^7$ is $M^{1,3}_{1,2}$, that it is a nontrivial $\mathbb{RP}^7$ bundle over S^2. But this has more to do with the fact that the relative indices for $S^3\star S^7$ are $(1,2)$ instead of $(1,1)$. Indeed, Wang and Ziller show that for the circle action corresponding to $(1,1)$ their $M^{1,3}_{1,1}$ is diffeomorphic to $S^2\times S^7$. However, their proof does not seem to generalize to our more general (replacing S^7 by $\mathcal{S}$) situation. Even if $S^3\star\cdots\star S^3\star\mathcal{S}$ were diffeomorphic to $S^2\times\cdots\times S^2\times\mathcal{S}$ the Sasakian-Einstein metric could not be the product metric.

In [BGMR] the authors constructed “toric” 3-Sasakian 7-manifolds $\mathcal{S}(\Omega_k)$ with arbitrary second Betti number. As mentioned previously these manifolds depend on a certain k by $k+2$ matrix Ω_k. Moreover, it was shown in [BGM6] that these manifolds are simply connected and that $H^3(\mathcal{S}(\Omega_k),\mathbb{Z})=0$. Thus the results in [BGMR] and [BGM6] together with Theorem 5.7 imply

Corollary 5.16: *There exist compact simply connected Sasakian-Einstein manifolds with arbitrary second Betti number b_2 and $b_3=0$ in every odd dimension greater than 5, namely the manifolds $S^3\star\cdots\star S^3\star\mathcal{S}(\Omega_k)$.*

It is still an open question whether this corollary holds in dimension 5 as well. For the 3-Sasakian 7-manifolds $\mathcal{S}(p_1,p_2,p_3)$ of [BGM2], Theorem 5.7 implies

Corollary 5.17: *Let p_1,p_2,p_3 be pairwise relatively prime positive integers. Then among the Sasakian-Einstein manifolds $S^3\star\cdots\star S^3\star\mathcal{S}(p_1,p_2,p_3)$ there are infinitely many that are homotopically distinct in every odd dimension greater than 5. Furthermore, all these manifolds have $b_2=1$ and $b_3=0$.*

Next we consider Sasakian-Einstein manifolds $S^3\star\mathcal{S}_{3,4}$ and $\mathcal{S}_{3,4}\star\mathcal{S}$ where again $\mathcal{S}$ is a simply connected Sasakian-Einstein 7-manifold of index 2 with $H^3(\mathcal{S},\mathbb{Z})=0$, and $\mathcal{S}_{3,4}$ is the Sasakian-Einstein circle bundle over the cubic Fermat surface $F_{3,4}$. As mentioned in Example 3.8, $\mathcal{S}_{3,4}$ has index 2, so again we get $(l_1,l_2)=(1,1)$. From the spectral sequence for the fibration $\mathcal{S}_{3,4}\longrightarrow F_{3,4}\longrightarrow BS^1$ we see that the differentials must satisfy $d_2(u_a)=d_4(u_a)=0$ and $d_2(v_a)=t\otimes u_a$ where u_a are the 3-classes in $\mathcal{S}_{3,4}$, and v_a are their Poincare duals. We can then analyze the spectral sequences as before using naturality of the diagram 5.3. We find

PROPOSITION 5.19: *Let $\mathcal{S}$ be a simply connected Sasakian-Einstein 7-manifold of index 2 with $H^3(\mathcal{S},\mathbb{Z})=0$ and $b_2(\mathcal{S})=k$. Let $\mathcal{S}_{3,4}$ denote the Sasakian-Einstein circle bundle over the cubic Fermat surface $F_{3,4}$. Then the Sasakian-Einstein 9-manifolds $S^3\star S_{3,4}$ satisfy*

$$H^q(S^3\star\mathcal{S}_{3,4},\mathbb{Z})\approx\begin{cases}\mathbb{Z} & \text{if } q=0,9,2,7;\\ \mathbb{Z}^{10} & \text{if } q=3,6;\\ 0 & \text{otherwise,}\end{cases}$$

whereas for the Sasakian-Einstein 13-manifolds $S_{3,4}\star\mathcal{S}$ satisfy

$$H^2(S_{3,4}\star\mathcal{S},\mathbb{Z})\approx\mathbb{Z}^{k+1},\qquad H^3(S_{3,4}\star\mathcal{S},\mathbb{Z})\approx\mathbb{Z}^{10}.$$

We can now consider the iterated join of S^3 with these manifolds, beginning with $S^3\star\cdots\star S^3\star\mathcal{S}_{3,4}$. Notice that $S^3\star\mathcal{S}_{3,4}$ is not cohomologically $S^2\times\mathcal{S}_{3,4}$. In fact, they are the same through dimension 3, but differ at 4. However, looking at the spectral sequence for the fibration

$$S^3\times S^3\star\mathcal{S}_{3,4}\longrightarrow S^3\star S^3\star\mathcal{S}_{3,4}\longrightarrow BS^1,$$

we see that indeed $S^3\star S^3\star\mathcal{S}_{3,4}$ is cohomologically $S^2\times S^3\star\mathcal{S}_{3,4}$. It is now easy to see that the induction procedure now works in this case. In the case of $S^3\star\cdots\star S^3\star\mathcal{S}_{3,4}\star\mathcal{S}$ where as usual $\mathcal{S}$ is any simply connected Sasakian-Einstein 7-manifold of index 2 with vanishing $H^3(\mathcal{S},\mathbb{Z})$, we notice that the iteration argument works through cohomology dimension 3, and we arrive at:

THEOREM 5.20: *For the Sasakian-Einstein $2n+7$-manifolds $\overbrace{S^3\star\cdots\star S^3}^{n\text{ times}}\star\mathcal{S}_{3,4}$ there is a ring isomorphism*

$$H^*(\overbrace{S^3\star\cdots\star S^3}^{n\text{ times}}\star\mathcal{S}_{3,4},\mathbb{Z})\approx H^*(\overbrace{S^2\times\cdots\times S^2}^{n-1\text{ times}}\times S^3\star\mathcal{S}_{3,4},\mathbb{Z}).$$

Let $\mathcal{S}$ be a simply connected Sasakian-Einstein 7-manifold of index 2 with $H^3(\mathcal{S},\mathbb{Z})=0$. Then for $q=1,2,3$ there are group isomorphisms

$$H^q(\overbrace{S^3\star\cdots\star S^3}^{n\text{ times}}\star\mathcal{S}_{3,4}\star\mathcal{S},\mathbb{Z})\approx H^q(\overbrace{S^2\times\cdots\times S^2}^{n-1\text{ times}}\times S^3\star\mathcal{S}_{3,4}\star\mathcal{S},\mathbb{Z}).$$

Proposition 5.19 and Theorem 5.20 have the following corollary:

COROLLARY 5.21: *There exist compact simply connected Sasakian-Einstein manifolds with arbitrary second Betti number and non-vanishing third Betti number in every odd dimension greater than 11, namely, the manifolds $S^3\star\cdots S^3\star\mathcal{S}_{3,4}\star\mathcal{S}(\Omega_k)$. In particular, for every integer $n>2$ there exist compact simply connected Sasakian-Einstein manifolds with arbitrary second Betti number of dimension $4n+3$ which cannot admit a 3-Sasakian structure.*

It is interesting to ask whether in the case of $k=b_2(\mathcal{S})=1$ we can find infinitely many homotopically distinct manifolds with $b_3\neq0$. One can certainly imitate the proof of Theorem 5.7 with S^3 replaced by the 7-manifold $S_{3,4}$. However, in the present case there is only one 3-class in $\mathcal{S}_{3,4}\times SU(3)$ that does not survive to E_∞, namely e_3 in $SU(3)$, and this transgresses to a generator in $E_4^{0,4}$. Thus, there is no torsion produced at this stage as in Theorem 5.7.

Finally we consider the case of certain manifolds which admit continuous families of Sasakian-Einstein structures. In general it is difficult to determine the necessary differentials in the relevant spectral sequences, but for the case of the regular 7-manifolds $S^3\star\mathcal{S}_k$ it is quite tractable. Of course as discussed previously it is only for the range $3\leq k\leq 8$ that we have a Sasakian-Einstein structure, and only in the range $5\leq k\leq 8$, where there are continuous families.

THEOREM 5.22: *The integral cohomology ring of the 7-manifolds $S^3\star\mathcal{S}_k$ is given by*

$$H^q(S^3\star\mathcal{S}_k,\mathbb{Z})\approx\begin{cases}\mathbb{Z} & \text{if } q=0,7;\\ \mathbb{Z}^{k+1} & \text{if } q=2,5;\\ \mathbb{Z}_2^k & \text{if } q=4;\\ 0 & \text{otherwise,}\end{cases}$$

with the ring relations determined by $\alpha_i\cup\alpha_j=0, s^2=0, 2\alpha_i\cup s=0$, where α_i, s are the $k+1$ two classes with $i=1,\cdots k$.

PROOF: A result of Smale [Sm, BFGK] says that $\mathcal{S}_k$ is diffeomorphic to $\#_k(S^2\times S^3)$. So the E_2 term of the spectral sequence for the fibration $S^3\times\mathcal{S}_k\longrightarrow S^3\star\mathcal{S}_k\longrightarrow BS^1$ is that of Diagram 5.5 with the $\mathbb{Z}$ in row four replaced by $\mathbb{Z}^{k+1}$ and the $\mathbb{Z}^{2k}$ in row six replaced by $\mathbb{Z}^{k+1}$. By the Kobayashi-Ochiai Theorem [KO] we have $\mathrm{Ind}(\mathcal{S}_k)=1$. Thus, we have $(l_1,l_2)=(2,1)$. As usual we can compute the relevant differentials by naturality using Diagram 5.3. Letting u denote orientation class of S^3, and β_i the 3-classes in $\mathcal{S}_k$, we find $d_4(u)=s^2$, and $d_2(\beta_i)=-2s\otimes\alpha_i$. This gives rise to torsion classes $E_3^{2,2}\approx\mathbb{Z}_2^k$ which survive to E_∞. The remainder now follows by naturality and Poincaré duality. ∎

It would be interesting to construct infinite sequences of homotopically distinct manifolds of a given dimension that admit continuous families of Sasakian-Einstein structure. For example, the 11-manifolds $\mathcal{S}_k\star\mathcal{S}(1,1,4r+1)$ are good candidates. However, at this time we do not appear to have enough knowledge of the differentials in the spectral sequence to determine whether the torsion classes $H^4(\mathcal{S}(1,1,4r+1),\mathbb{Z})\approx\mathbb{Z}_{8r+3}$ give arise to enough torsion in the total space to homotopically distinguish these manifolds.

Of course, we can construct higher dimensional examples with continuous families of Sasakian-Einstein structures, for example by joining with more copies of S^3. However, since the relative indices in this case are $(l_1,l_2)=(2,1)$ more 2-torsion is produced with each iteration making the spectral sequence more difficult to analyze. Particular cases where this does not happen are with the 9-manifolds $S_k\star S_{k'}$ and the 11-manifolds $S_k\star\mathcal{S}_{4,4}$ and further iterates, where we recall the Sasakian-Einstein circle bundle $\mathcal{S}_{4,4}$ over the quartic Fermat surface. These all have relative indices $(1,1)$ and will have no torsion. For example our methods give

THEOREM 5.23: *The integral cohomology ring of the 11-manifolds $\mathcal{S}_k\star\mathcal{S}_{4,4}$ is given by*

$$H^q(S_k\star\mathcal{S}_{4,4},\mathbb{Z})\approx\begin{cases}\mathbb{Z} & \text{if } q=0,11,4,7;\\ \mathbb{Z}^{k+1} & \text{if } q=2,9;\\ \mathbb{Z}^{60} & \text{if } q=3,8;\\ \mathbb{Z}^{60k} & \text{if } q=5,6;\\ 0 & \text{otherwise,}\end{cases}$$

with the ring relations determined by $\alpha_i\cup\alpha_j=0, s^3=0, \alpha_i\cup s=0, u_a\cup u_b=0, s\cup u_a=0$, where α_i, s are the $k+1$ 2 classes with $i=1,\cdots k$, and u_a are the 3 classes with $a=1,\cdots 60$.

and

THEOREM 5.24: *The integral cohomology ring of the 9-manifolds $\mathcal{S}_k \star \mathcal{S}_{k'}$ is given by*

$$H^q(S_k \star S_{k'},\mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{if } q=0,9; \\ \mathbb{Z}^{k+k'+1} & \text{if } q=2,7; \\ \mathbb{Z}^{kk'+1} & \text{if } q=4,5; \\ 0 & \text{otherwise,} \end{cases}$$

with the ring relations determined by $\alpha_i \cup \alpha_j = 0, s^3 = 0, \alpha_i \cup s = 0, \alpha'_i \cup s = 0, \alpha'_i \cup \alpha'_j = 0$ *where* α_i, α'_j, s *are the* $k+k'+1$ *2 classes with* $i = 1,\cdots k; j = 1\cdots k'$.

It is straightforward to work out further iterations all of which have continuous families of Sasakian-Einstein structures. In particular, the $(2m+11)$-orbifolds $\overbrace{S^3 \star \cdots \star S^3}^{m \text{ times}} \star S_l \star \mathcal{S}(\Omega_k)$ discussed in Corollary 4.18 satisfy $b_2 = l+k+m+1$ and $b_3 = 0$.

§6. A Lattice of Einstein Orbifolds

In their paper [WZ] Wang and Ziller constructed Einstein metrics on the total space of many torus bundles over Kähler-Einstein manifolds. The purpose of the this section is to discuss the various Einstein manifolds that arise as a result of our extension of their method to the orbifold category.

Given any two Sasakian-Einstein manifolds $\mathcal{S}_1, \mathcal{S}_2$ one can define an orbifold with an Einstein metric by generalizing the Wang-Ziller procedure to treat circle V-bundles over Kähler-Einstein orbifolds. We simply relax the condition in Proposition 4.6 that the l_i be the respective relative indices of $\mathcal{S}_i$. For any pair (l,k) of positive integers we let $S^1(l,k)$ denote the circle action on $\mathcal{S}_1 \times \mathcal{S}_2$ given by $(x,y) \mapsto (\tau^k x, \tau^{-l} y)$. Then we define orbifolds as the quotient

6.1
$$M(\mathcal{S}_1,\mathcal{S}_2;l,k) = \frac{\mathcal{S}_1 \times \mathcal{S}_2}{S^1(l,k)}.$$

It is clear that when (l,k) are the relative indices (l_1,l_2) we recover our join operation, that is $M(\mathcal{S}_1,\mathcal{S}_2;l_1,l_2) = \mathcal{S}_1 \star \mathcal{S}_2$. The following result follows directly from our results together with those of Wang and Ziller [WZ].

THEOREM 6.2: *Let* $\mathcal{S}_i \in \mathcal{SE}^s$ *with orders* m_i*, respectively. Then*

(i) $M(\mathcal{S}_1,\mathcal{S}_2;l,k)$ *is a compact orbifold.*

(ii) $\pi_1^{orb}(M(\mathcal{S}_1,\mathcal{S}_2;l,k)) = 0$ *if and only if* $\gcd(l,k) = 1$.

(iii) If $\gcd(m_1k, m_2l) = 1$ *then* $M(\mathcal{S}_1,\mathcal{S}_2;l,k)$ *is a compact simply connected manifold.*

(iv) $H^q(M(\mathcal{S}_1,\mathcal{S}_2;l,k),\mathbb{Q}) \approx H^q(\mathcal{S}_1 \star \mathcal{S}_2,\mathbb{Q})$ *for all* q *and for all* l,k.

(v) $M(\mathcal{S}_1,\mathcal{S}_2;l,k)$ *admits a homothety class of Einstein metrics of positive scalar curvature.*

(vi) $M(\mathcal{S}_1,\mathcal{S}_2;l,k)$ *has a Sasakian structure that is Sasakian-Einstein if and only if* $(l,k) = m(l_1,l_2)$ *for* $m \in \mathbb{Z}^+$.

PROOF: (i) through (iv) and (vi) are clear from the discussion in previous sections. The homothety class of Einstein metrics on $M(\mathcal{S}_1,\mathcal{S}_2;l,k)$ in (v) is obtained as follows: One considers the product $\mathcal{S}_1 \times \mathcal{S}_2$ with a Riemannian metric given by $g(\lambda_1, \lambda_2) = \lambda_1 g_1 + \lambda_2 g_2$, where g_1, g_2 are the Sasakian-Einstein metrics on each factor. Then a calculation identical to that of [WZ] shows that there exists a ratio λ_1/λ_2 for which the quotient metric $\hat{g}(\lambda_1, \lambda_2)$ on $M(\mathcal{S}_1, \mathcal{S}_2; l, k)$ obtained by Riemannian submersion via the quotient map

$$\pi(\lambda_1, \lambda_2) : \mathcal{S}_1 \times \mathcal{S}_2 \longrightarrow M(\mathcal{S}_1, \mathcal{S}_2; l, k)$$

is Einstein. This ratio depends on (n_1, n_2), (l_1, l_2) and (l, k). In the Sasakian-Einstein case of (vi) the scaling factors (λ_1, λ_2) are in addition completely determined by the condition that submerssed Einstein metric on the associated leaf space of the characteristic foliation be of correct Einstein constant (see 4.4). ∎

The set $\{M(\mathcal{S}_1, \mathcal{S}_2; l, k) | (l, k) \in \mathbb{Z}^+ \times \mathbb{Z}^+\}$ can be given the structure of a lattice as follows:

(1) $M(\mathcal{S}_1, \mathcal{S}_2; l, k) \leq M(\mathcal{S}_1, \mathcal{S}_2; l', k')$ if and only if $l \leq l'$ and $k \leq k'$;

(2) $M(\mathcal{S}_1, \mathcal{S}_2; l, k) \vee M(\mathcal{S}_1, \mathcal{S}_2; l', k') = M(\mathcal{S}_1, \mathcal{S}_2; \mathrm{lcm}(l, l'), \mathrm{lcm}(k, k'))$;

(3) $M(\mathcal{S}_1, \mathcal{S}_2; l, k) \wedge M(\mathcal{S}_1, \mathcal{S}_2; l', k') = M(\mathcal{S}_1, \mathcal{S}_2; \gcd(l, l'), \gcd(k, k'))$.

This lattice is isomorphic to the lattice $\mathbb{Z}^+ \times \mathbb{Z}^+$ with the same operations.

Hence, associated to every pair $(\mathcal{S}_1, \mathcal{S}_2)$ of Sasakian-Einstein manifolds there is a lattice $L(\mathcal{S}_1, \mathcal{S}_2)$ of compact Einstein orbifolds, all of positive scalar curvature of course, and all having the same rational cohomology. Notice that if $\gcd(m_1, m_2) > 1$ then no member of the lattice is a smooth manifold. If however, $\gcd(m_1, m_2) = 1$ then there will be infinitely many smooth manifolds in the lattice. There is a half-line of Sasakian-Einstein orbifolds in the lattice $L(\mathcal{S}_1, \mathcal{S}_2)$ given by $\{M(\mathcal{S}_1, \mathcal{S}_2; ml_1, ml_2) | m = 1, 2, \cdots\}$, precisely one of which is simply connected in the orbifold sense, namely $M(\mathcal{S}_1, \mathcal{S}_2; l_1, l_2)$. We want to extend the lattice $L(\mathcal{S}_1, \mathcal{S}_2)$ in two ways. First we allow either l or k to take the values 0, but not both simultaneously. Then we see that $M(\mathcal{S}_1, \mathcal{S}_2; l, 0) = \mathcal{S}_1 \times \mathcal{Z}_2$ and $M(\mathcal{S}_1, \mathcal{S}_2; 0, k) = \mathcal{Z}_1 \times \mathcal{S}_2$. Notice that these added elements do not necessarily have the same rational cohomology as the elements of $L(\mathcal{S}_1, \mathcal{S}_2)$. Second and more importantly we want to include limit points of sequences. To do this we consider the Cheeger ρ^*-topology as discussed for example in Wang and Ziller [WZ] by noticing that this easily extends to the orbifold category. This topology defines a distance between two Riemannian orbifolds that basically depends on the curvature and its covariant derivative. Thus, two Riemannian orbifolds can be close even though their topology may be quite different. Using Proposition 4.3 of [WZ] we find

PROPOSITION 6.3: *Any element $M(\mathcal{S}_1, \mathcal{S}_2; l, k) \in L(\mathcal{S}_1, \mathcal{S}_2)$ can be written as the limit*

$$M(\mathcal{S}_1, \mathcal{S}_2; l, k) = \lim_{t \to \infty} M(\mathcal{S}_1, \mathcal{S}_2; lt + a, kt + b)$$

for arbitrary $(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+$.

This proposition says that the completion $\check{L}(\mathcal{S}_1, \mathcal{S}_2)$ of $L(\mathcal{S}_1, \mathcal{S}_2)$ in the Cheeger ρ^*-topology behaves like the projective scheme over the lattice $L(\mathcal{S}_1, \mathcal{S}_2)$.

We are mainly interested in those points of the lattice that are represented by smooth manifolds. Necessary condition for the smoothness of the join $\mathcal{S}_1 \star \mathcal{S}_2$ are that $\gcd(m_i, l_i) = 1$ and $\gcd(m_1, m_2) = 1$. Thus, if $\gcd(m_1, m_2) = 1$ there will be infinitely many pairs (l, k) such that $M(\mathcal{S}_1, \mathcal{S}_2; l, k)$ is a smooth Einstein manifold even though $\mathcal{S}_1 \star \mathcal{S}_2$ may not be smooth. An example is given by taking $\mathcal{S}_1 = S^{4n+1}$ and $\mathcal{S}_2$ any 3-Sasakian 7-manifold whose order is even. Then $S^{4n+1}\star\mathcal{S}$ is not smooth, but $M(S^{4n+1},\mathcal{S};l,k)$ will be smooth for infinitely many (l,k), namely those that satisfy $\gcd(k,m_2l)=1$.

EXAMPLE 6.4: As a special case let us take $\mathcal{S}_1$ to be regular and $k=1$. Then the gcd condition in (iii) above is automatically satisfied, so for each l, $M(\mathcal{S}_1,\mathcal{S}_2;l,1)$ is a smooth Einstein manifold of dimension dim $\mathcal{S}_1$ + dim $\mathcal{S}_2-1$. Now let us further specialize by putting $\mathcal{S}_1=S^3$ and taking $\mathcal{S}_2$ to be any simply connected 3-Sasakian 7-manifold $\mathcal{S}$ that is not S^7. In this case $M(S^3,\mathcal{S};1,1)=S^3\star\mathcal{S}$ and we get a sequence $\{M(S^3,\mathcal{S};l,1)\}_{l=1}^{\infty}$ of compact simply connected Einstein manifolds of dimension 9. Furthermore, with respect to the Cheeger ρ^*-topology we see this sequence converges to $S^2\times\mathcal{S}$ with the product Einstein metric. One can also see that this sequence of Einstein manifolds contains a subsequence of homotopically distinct Einstein manifolds.

EXAMPLE 6.5: In the case of $\mathcal{S}_1=\mathcal{S}_l$ a circle bundle over a del Pezzo surface, and $\mathcal{S}_2=\mathcal{S}(\Omega_k)$, we know that $\mathcal{S}_l\star\mathcal{S}(\Omega_k)$ will be a smooth manifold if and only if $\mathrm{Ord}(\mathcal{S}(\Omega_k))$ is odd. However, even if $\mathcal{S}_l\star\mathcal{S}(\Omega_k)$ is not a smooth manifold, $M(S_l,\mathcal{S}(\Omega_k);1,1)$ is always a smooth Einstein manifold. Of course, it is not Sasakian-Einstein . It is interesting to ponder the question whether the continuous family of Sasakian-Einstein structures on S_l when $5\leq l\leq 8$ induces an effective continuous family of Einstein metrics on $M(S_l,\mathcal{S}(\Omega_k);1,1)$.

An interesting question is whether any elements in a given lattice $L(\mathcal{S}_1,\mathcal{S}_2)$ are homeomorphic or diffeomorphic. This is a difficult question in general; however, Wang and Ziller give some examples in the case of spheres. For example, it follows directly from Smale's classification of simply connected 5-manifolds with spin [Sm] that $M(S^3,S^3;l,k)$ are all diffeomorphic to $S^2\times S^3$ for all relatively prime (l,k).

REMARK 6.6: Clearly Theorem 6.2 can be further extended to the case of n compact Sasakian-Einstein orbifolds $(\mathcal{S}_i,g_i), i=1,\ldots,n$. Let $\mathbf{x}=(x_1\ldots,x_n)$ and consider the product $\mathcal{S}_1\times\cdots\times\mathcal{S}_n$ with a metric $g(\mathbf{x})=x_1g_1+\cdots+x_ng_n$. Now, there is a canonical n-torus action on $\mathcal{S}_1\times\cdots\times\mathcal{S}_n$. A choice of an arbitrary circle subgroup in this torus is obtained via a homomorphism $h(\mathbf{p}):S^1\to T^n$, where $\mathbf{p}=(p_1,\ldots,p_n)$, and $p_i\in\mathbb{Z}$ are the "winding numbers" on each factor. Now, the quotient of T^n by this circle gives a $T^{n-1}(\mathbf{p})$-torus action on the product $\mathcal{S}_1\times\cdots\times\mathcal{S}_n$ with the quotient space $M(\mathcal{S}_1,\ldots,\mathcal{S}_n;\mathbf{p};\mathbf{x})$ being a compact orbifold. The combinatorial conditions for smoothness are considerably more complicated here, but in principle one certainly expects to find an analogue of (iii) of Theorem 6.2. In any case one can produce many examples of smooth quotients. As in the $n=2$ case we do get an n-dimensional lattice $L(\mathcal{S}_1,\ldots,\mathcal{S}_n)$ of compact orbifolds which all admit Einstein metrics of positive scalar curvature by the argument identical to that of [WZ]. Also, there is a unique simply-connected Sasakian-Einstein space $\mathcal{S}_1\star\cdots\star\mathcal{S}_n$ at some lattice point of L determined by the relative indices of the factors. Hence, we can think of $L(\mathcal{S}_1,\ldots,\mathcal{S}_n)$ as completely determined by $\mathcal{S}_1\star\cdots\star\mathcal{S}_n$.

REMARK 6.7: Finally, even Wang and Ziller's torus bundle construction can be reformulated and generalized in the language of the monoid $(\mathcal{SE},\star)$. In the discussion of the previous remark one can consider a homomorphism $h(\Omega):T^s\to T^n$, where $\Omega\in\mathcal{M}(s\times n;\mathbb{Z})$ is a "weight matrix". This yields a $T^{n-s}(\Omega)$-torus action on the product $\mathcal{S}_1\times\cdots\times\mathcal{S}_n$ with the quotient space $M(\mathcal{S}_1,\ldots,\mathcal{S}_n;\Omega;\mathbf{x})$ being a compact orbifold and a T^s-bundle over the product $\mathcal{Z}_1\times\cdots\times\mathcal{Z}_n$. Taking all weight matrices Ω defines an ns-dimensional lattice $L(\mathcal{S}_1,\ldots,\mathcal{S}_n;s)$ whose points are all compact Einstein orbifolds.

Bibliography

[AFHS] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull, and B. Spence, *Branes at Conical Singularities and Holography*, preprint, August 1998; hep-th/9808014.

[An] M. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. 102 (1990), 429-445.

[Ba1] W. L. Baily, *The decomposition theorem for V-manifolds*, Amer. J. Math. 78 (1956), 862-888.

[Ba2] W. L. Baily, *On the imbedding of V-manifolds in projective space*, Amer. J. Math. 79 (1957), 403-430.

[Bes] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, New York (1987).

[BFGK] H. Baum, T. Friedrich, R. Grunewald, and I. Kath, *Twistors and Killing Spinors on Riemannian Manifolds*, Teubner-Texte für Mathematik, vol. 124, Teubner, Stuttgart, Leipzig, 1991.

[BG1] C. P. Boyer and K. Galicki, *The twistor space of a 3-Sasakian manifolds*, Int. J. Math. 8 (1997), 31-60.

[BG2] C. P. Boyer and K. Galicki, *3-Sasakian Manifolds*, preprint, January 1998, MPI-98/19; (http://www.mpim-bonn.mpg.de/html/preprints/preprints.html); to appear in *Essays on Einstein Manifolds*, International Press; C. LeBrun and M. Wang, Eds.

[BGM1] C. P. Boyer, K. Galicki, and B. M. Mann, *Quaternionic reduction and Einstein manifolds*, Comm. Anal. Geom. 1 (1993), 1-51.

[BGM2] C. P. Boyer, K. Galicki, and B. M. Mann, *The geometry and topology of 3-Sasakian manifolds*, J. reine angew. Math. 455 (1994), 183-220.

[BGM3] C. P. Boyer, K. Galicki, and B. M. Mann, *On strongly inhomogeneous Einstein manifolds*, Bull. London Math. Soc. 28 (1996), 401-408.

[BGM4] C. P. Boyer, K. Galicki, and B. M. Mann, *Hypercomplex structures on Stiefel manifolds*, Ann. Global Anal. Geom. 14 (1996), 81-105.

[BGM5] C. P. Boyer, K. Galicki, and B. M. Mann, *A note on smooth toral reductions of spheres*, Manuscripta Math. 95 (1998), 149-158.

[BGM6] C. P. Boyer, K. Galicki, and B. M. Mann, *Hypercomplex structures from 3-Sasakian structures*, J. reine angew. Math. 501 (1998), 115-141.

[BGMR] C. P. Boyer, K. Galicki, B. M. Mann, and E. Rees, *Compact 3-Sasakian 7-manifolds with arbitrary second Betti number*, Invent. Math. 131 (1998), 321-344.

[Bl] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics 509, Springer-Verlag, New Yrok 1976.

[Bo] A. Borel, *Kählerian coset spaces of semi-simple Lie groups*, Proc. Nat. Acad. Sci. USA 40 (1954), 1147-1151. Reprinted in A. Borel, Oeuvres, Collected Papers, Vol. 1, Springer-Verlag, Berlin Heidelberg, 1983.

[BR] A. Borel and R. Remmert, *Über kompakte homogene Kählersche Mannigfaltigkeiten*, Math. Ann. 145 (1962), 429-439.

[BW] W.M. Boothby and H.C. Wang, *On Contact Manifolds*, Ann. of Math. 68 (1958), 721-734.

[Dim] A. Dimca, *Singularities and Topology of Hypersurfaces*, Springer-Verlag, New York, 1992.

[DO] S. Dragomir and L. Ornea, *Locally Conformal Kähler Geometry*, Progress in Mathematics 155, Birkhäuser, Boston, 1998.

[DT] W. Ding and G. Tian, *Kähler-Einstein metrics and the generalized Futaki invariants*, Invent. Math. 110 (1992), 315-335.

[Esch1] J. H. Eschenburg, *New examples of manifolds with strictly positive curvature*, Invent. Math. 66 (1982), 469-480.

[Esch2] J. H. Eschenburg, *Cohomology of biquotients*, Manuscripta Math. 75 (1992), 151-166.

[Fi] J. M. Figueroa-O'Farrill, *Near-Horizon Geometries of Supersymmetric Branes*, preprint, July 1998; hep-th/9807149.

[FrKat1] T. Friedrich and I. Kath, *Einstein manifolds of dimension five with small first eigenvalue of the Dirac operator*, J. Diff. Geom. 29 (1989), 263-279.

[FrKat2] T. Friedrich and I. Kath, *Compact seven-dimensional manifolds with Killing spinors*, Comm. Math. Phys. 133 (1990), 543-561.

[Gro] M. Gromov, *Curvature, diameter and Betti numbers*, Comment. Math. Helvetici 56 (1981) 179-195.

[GS] K. Galicki and S. Salamon, *On Betti numbers of 3-Sasakian manifolds*, Geom. Ded. 63

(1996), 45-68.

[Hae] A. Haefliger, *Groupoides d'holonomie et classifiants*, Astérisque 116 (1984), 70-97.

[Hat] Y. Hatakeyama, *Some notes on differentiable manifolds with almost contact structures*, Tôhuku Math. J. 15 (1963), 176-181.

[HS] I. Hasegawa and M. Seino, *Some remarks on Sasakian geometry–applications of Myers' theorem and the canonical affine connection*, J. Hokkaido Univ. Education 32 (1981), 1-7.

[Kaw] T. Kawasaki, *The Riemann-Roch theorem for complex V-manifolds*, Osaka J. Math. 16 (1979), 151-159.

[KO] S. Kobayashi and T. Ochiai, *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto. Univ. 13 (1973), 31-47.

[Kob1] S. Kobayashi, *On compact Kähler manifolds with positive Ricci tensor*, Ann. Math. 74 (1961), 381-385.

[Kob2] S. Kobayashi, *Topology of positively pinched Kaehler manifolds*, Tôhoku Math. J. 15 (1963), 121-139.

[Kol] J. Kollár, *Rational Curves on Algebraic Varieties*, Springer-Verlag, New York, 1996.

[KS1] N. Koiso and Y. Sakane, *Non-homogeneous Kähler-Einstein metrics on compact complex manifolds* in Curvature and Topology of Riemannian manifolds, Lecture Notes in Mathematics 1201, Springer-Verlag, 1986, 165-179.

[KS2] N. Koiso and Y. Sakane, *Non-homogeneous Kähler-Einstein metrics on compact complex manifolds II*, Osaka J. Math. 25 (1988), 933-959.

[KW] I. R. Klebanov and E. Witten, *Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity*, preprint, July 1998; hep-th/9807080.

[Ma] J. Maldacena, *The large N limit of superconformal field theories and supergravity*, preprint, 1997; hep-th/9711200.

[Mab] T. Mabuchi, *Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties*, Osaka J. Math. 24 (1987), 705-737.

[Mol] P. Molino, *Riemannian Foliations*, Progress in Mathematics 73, Birkhäuser, Boston, 1988.

[MiMo] Y. Miyaoka and S. Mori, *A numerical criterion for uniruledness*, Ann. Math. 124 (1986), 65-69.

[MoMu] S. Mori and S. Mukai, *Classification of Fano 3-folds with $B_2 \geq 2$.*, Manuscripta Math. 36 (1981), 147-162.

[MP] D. R. Morrison and M. R. Plesser, *Non-Spherical Horizons, I*, preprint, October 1998; hep-th/9810201.

[Na] A.M. Nadel *Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature*, Ann. Math. 138 (1990), 549-596.

[OhTa] K. Oh and R. Tatar, *Three Dimensional SCFT from M2 Branes at Conical Singularities*, preprint, October 1998; hep-th/9810244.

[PP] H. Pedersen and Y. S. Poon, *A note on rigidity of 3-Sasakian manifolds*, Proc. Am. Math. Soc., to appear (1998).

[Sa] Y. Sakane, *Examples of compact Einstein Kähler manifolds with positive Ricci tensor*, Osaka J. Math. 23 (1986), 585-617.

[Sas] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure*, Tôhoku Math. J. 2 (1960), 459-476.

[Siu] Y.-T. Siu, *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. Math. 127 (1988), 585-627.

[Sm] S. Smale, *On the structure of 5-manifolds*, Ann. Math. 75 (1962), 38-46.

[Tan] S. Tanno, *Geodesic flows on C_L-manifolds and Einstein metrics on $S^3 \times S^2$*, in *Minimal submanifolds and geodesics (Proc. Japan-United States Sem., Tokyo, 1977)*, pp. 283-292, North Holland, Amsterdam-New York, 1979.

[Thu] W. Thurston, *The Geometry and Topology of 3-Manifolds*, Mimeographed Notes, Princeton Univ. Chapt. 13 (1979).

[Tho] C.B. Thomas, *Almost regular contact manifolds*, J. Differential Geom. 11 (1976), 521-533.

[Ti1] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$*, Invent. Math. 89 (1987), 225-246.

[Ti2] G. Tian, *On Calabi's Conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101 (1990), 101-172.

[Ti3] G. Tian, *Kähler-Einstein metrics with positve scalar curvature*, Invent. Math. 137 (1997), 1-37.

[TY] G. Tian and S.-T. Yau, *Kähler-Einstein metrics on complex surfaces with $c_1 > 0$*, Comm. Math. Phys. 112 (1987), 175-203.

[Vai] I. Vaisman, *Locally conformal Kähler manifolds with parallel Lee form*, Rend. Matem. Roma 12 (1979), 263-284.

[WZ] M. Wang and W. Ziller, *Einstein metrics on principal torus bundles*, J. Diff. Geom. 31 (1990), 215-248.

[Wi1] J.A. Wiśniewski, *On a conjecture of Mukai*, Manus. Math. 68 (1990), 135-141.

[Wi2] J.A. Wiśniewski, *On Fano manifolds of large index*, Manus. Math. 70 (1991), 145-152.

[Wit] E. Witten, *Anti-de Sitter space and holography*, preprint, 1998; hep-th/9802150.

[YKon] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics 3, World Scientific Pub. Co., Singapore, 1984.

Department of Mathematics and Statistics
University of New Mexico
Albuquerque, NM 87131
email: cboyer@math.unm.edu, galicki@math.unm.edu

October 1999