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CONSTRUCTIONS IN SASAKIAN GEOMETRY

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1. Introduction

A Riemannian manifold (M,g) is called a *Sasakian manifold* if there exists a Killing vector field ξ of unit length on M so that the tensor field Φ of type $(1,1)$, defined by $\Phi(X) = -\nabla_X\xi$, satisfies the condition $(\nabla_X\Phi)(Y) = g(X,Y)\xi - g(\xi,Y)X$ fo any pair of vector fields X and Y on M. This is a curvature condition which can be easily expressed in terms the Riemann curvature tensor as $R(X,\xi)Y = g(\xi,Y)X - g(X,Y)\xi$. Equivalently, the Riemannian cone defined by $(\mathcal{C}(M),\bar{g},\Omega) = (\mathbb{R}_+\times M,\ dr^2+r^2g, d(r^2\eta))$ is Kähler with the Kähler form $\Omega = d(r^2\eta)$, where η is the dual 1-form of ξ. The 4-tuple $\mathcal{S} = (\xi,\eta,\Phi,g)$ is commonly called a *Sasakian structure* on M and ξ is its *characteristic* or *Reeb vector field*.

Sasakian geometry is a special kind of contact metric geometry such that the structure transverse to the Reeb vector field ξ is Kähler and invariant under the flow of ξ. In fact η is the contact 1-form, and Φ is a $(1,1)$ tensor field which defines a complex structure on the contact subbundle ker η which annihilates ξ. When both M and the leaves of the foliation generated by ξ are compact the Sasakian structure is called *quasi-regular*, and the space of leaves X^{orb} is a compact Kähler orbifold. In such a case M is the total space of a circle *orbi-bundle* (also called V-bundle) over X^{orb}. Moreover, the 2-form $d\eta$ pushes down to a Kähler form ω on X^{orb}. Now ω defines an integral class $[\omega]$ of the orbifold cohomology group $H^2(X^{orb},\mathbb{Z})$ which generally is only a rational class in the ordinary cohomology $H^2(X,\mathbb{Q})$.

This construction can be inverted in the sense that given a Kähler form ω on a compact complex orbifold X^{orb} which defines an element $[\omega]\in H^2(X^{orb},\mathbb{Z})$ one can construct a circle orbi-bundle on X^{orb} whose orbifold first Chern class is $[\omega]$. Then the total space M of this orbi-bundle has a natural Sasakian structure (ξ,η,Φ,g), where η is a connection 1-form whose curvature is ω. The tensor field Φ is obtained by lifting the almost complex structure I on X^{orb} to the horizontal distribution ker η and requiring that Φ annihilates ξ. Furthermore, the map $(M,g)\longrightarrow(X^{orb},h)$ is an orbifold Riemannian submersion. This is an orbifold version of a well-known construction of Kobayashi. For the essentials and more details on Sasakian geometry we refer the reader to the forthcoming book [BG06] of the first two authors.

The purpose of this paper is to describe in detail certain constructions of new Sasakian manifolds from old ones. In Section 2 we generalize the join construction introduced by the first two authors [BG00b] in the case of quasi-regular Sasakian-Einstein manifolds to arbitrary quasi-regular Sasakian spaces. This construction is far more flexible yielding a multitude of examples. Furthermore, owing to the recent Sasakian-Einstein metrics discovered on S^5 in [BGK05], we are able to prove the existence of families of Sasakian-Einstein metrics on manifolds homeomorphic to $S^2\times S^5$. However, determination of the smooth structure is rather subtle and would ultimately involve computation of the Kreck-Stolz invariants for these manifolds [KS88].

In Section 3 we show how the join construction emerges as a special case of Lerman's contact fibre bundle construction [Ler04b] which under some additional assumptions can be adapted to the

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Sasakian case. In particular, when both the base and the fiber of the contact fiber bundle are toric we show that the construction yields a new toric Sasakian manifold.

In the last section we study the toric Sasakian manifolds in dimension 5. All compact, smooth, simply-connected, oriented 5-manifolds were classified by fundamental theorems of Smale and Barden [Sma62, Bar65]. In particular, the manifold (and its unique smooth structure) is completely determined by $H_2(M^5,\mathbb{Z})$ together with the second Stiefel-Whitney class map $w_2 : H_2(M,\mathbb{Z})\longrightarrow\mathbb{Z}_2$. The second Betti number $b_2(M)$, the structure of the 2-torsion subgroup, and w_2 all provide obstruction to the existence of various geometric structures on such manifolds. For example, it is an elementary result that torsion in the second homology group is the obstruction to the existence of a free circle action on M^5. Moreover, vanishing of the torsion is also a necessary and sufficient condition for the existence of a regular contact structure as observed by Geiges [Gei91]. Remarkably, Oh proves the same condition is also a necessary and sufficient for the existence of an effective T^3 action on M^5 [Oh83] and Yamazaki shows that in such a case one can always choose a T^3 action with a compatible toric K-contact structure. We use these results to show that any simply-connected compact oriented 5-manifold with vanishing 2-torsion admits a toric Sasakian structure. Furthermore, we prove by explicitly constructing circle bundles over the blow-ups of Hirzebruch surfaces that one can always find toric Sasakian structures which are regular.

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2. The Join Construction

In this subsection we apply a construction due to Wang and Ziller [WZ90] to define a multiplication on the set of quasi-regular Sasakian orbifolds. This was done originally in [BG00b] in the case of Sasakian-Einstein orbifolds which is perhaps of more interest, but there is an easy generalization to the strict Sasakian case. The idea is quite simple and is based on the fact that product of Kähler orbifolds is a Kähler orbifold in a natural way.

Definition 2.1: *We denote by $\mathcal{SO}$ the set of compact quasi-regular Sasakian orbifolds, by $\mathcal{SM}$ the subset of $\mathcal{SO}$ that are smooth manifolds, and by $\mathcal{R}\subset\mathcal{SM}$ the subset of compact, simply connected, regular Sasakian manifolds. The set $\mathcal{SO}$ is topologized with the $C^{m,\alpha}$ topology, and the subsets are given the subspace topology.*

The set $\mathcal{SO}$ is graded by dimension, that is,

$$\mathcal{SO}=\bigoplus_{n=0}^{\infty}\mathcal{SO}_{2n+1},$$

and similarly for $\mathcal{SM}$ and $\mathcal{R}$. In the definition of Sasakian structure it is implicitly assumed that $n>0$. So we want to extend the definition of a Sasakian structure to the case when $n=0$. This can easily be done since a connected one dimensional orbifold is just an interval with possible boundary, or a circle. So we can just take $\xi=\frac{\partial}{\partial t},\eta=dt,\Phi=0$, with the flat metric $g=dt^2$. In this case the space of leaves $\mathcal{Z}$ of the characteristic foliation is just a point. The unit circle S^1 with this structure will play the role of the identity in our monoid. Notice that the identity is smooth.

For each pair of relatively prime positive integers (k_1,k_2) we define a graded multiplication

$$\star_{k_1,k_2}:\mathcal{SO}_{2n_1+1}\times\mathcal{SO}_{2n_2+1}\longrightarrow\mathcal{SO}_{2(n_1+n_2)+1} \tag{2.1}$$

as follows: Let $\mathcal{S}_1,\mathcal{S}_2\in\mathcal{SO}$ of dimension $2n_1+1$ and $2n_2+1$ respectively. Since each orbifold $\mathcal{S}_i$ has a quasi-regular Sasakian structure, its Reeb vector field generates a locally free circle action, and the quotient space by this action has a natural orbifold structure $\mathcal{Z}_i$ [Mol88]. Thus, there is a locally free action of the 2-torus T^2 on the product orbifold $\mathcal{S}_1\times\mathcal{S}_2$, and the quotient orbifold is the product of the orbifolds $\mathcal{Z}_i$. (Locally free torus actions on orbifolds have been studied in [HS91]). Now the Sasakian structure on $\mathcal{S}_i$ determines a Kähler structure ω_i on the orbifold $\mathcal{Z}_i$, but in order to obtain an integral orbifold cohomology class $[\omega_i]\in H^2(\mathcal{Z}_i,\mathbb{Z})$ we need to assure that the period of a generic orbit is one. By a result of Wadsley [Wad75] the period function on a quasi-regular Sasakian orbifold is lower semi-continuous and constant on the dense open set of regular orbits. This is because on a Sasakian orbifold all Reeb orbits are geodesics. Thus, by a transverse homothety we can normalize the period function to be the constant 1 on the dense open set of regular orbits. In this case the Kähler forms ω_i define integer orbifold cohomology classes $[\omega_i]\in H^2_{orb}(\mathcal{Z}_i,\mathbb{Z})$. If Z_i denotes the underlying complex space associated with the orbifold $\mathcal{Z}_i$, one should not confuse $H^*_{orb}(\mathcal{Z}_i,\mathbb{Z})$ with $H^*(Z_i,\mathbb{Z})$ nor with the Chen-Ruan cohomology of an orbifold. $H^*_{orb}(\mathcal{Z}_i,\mathbb{Z})$ is the orbifold cohomology defined by Haefliger [Hae84] (see also [BG00b]). Notice, however, that we do have $H^*_{orb}(\mathcal{Z}_i,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}\approx H^*(Z_i,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$. Now each pair of positive integers k_1,k_2 give a Kähler form $k_1\omega_1+k_2\omega_2$ on the product. Furthermore, $[k_1\omega_1+k_2\omega_2]\in H^2_{orb}(\mathcal{Z}_1\times\mathcal{Z}_2,\mathbb{Z})$, and thus defines an S^1 V-bundle over the orbifold $\mathcal{Z}_1\times\mathcal{Z}_2$ whose total space is an orbifold that we denote by $\mathcal{S}_1\star_{k_1,k_2}\mathcal{S}_2$ and refer to as the (k_1,k_2)-*join* of $\mathcal{S}_1$ and $\mathcal{S}_2$. Furthermore, $\mathcal{S}_1\star_{k_1,k_2}\mathcal{S}_2$ admits a quasi-regular Sasakian structure [BG06] by choosing a connection 1-form on $\mathcal{S}_1\star_{k_1,k_2}\mathcal{S}_2$ whose curvature is $\pi^*(k_1\omega_1+k_2\omega_2)$. This Sasakian structure is unique up to a gauge transformation of the form $\eta\mapsto\eta+d\psi$ where ψ is a smooth basic function. This defines the maps in 2.1. If $\mathcal{S}_i$ are quasi-regular Sasakian structures on the compact manifolds M_i, respectively, we shall use the notation $\mathcal{S}_1\star_{k_1,k_2}\mathcal{S}_2$ and $M_1\star_{k_1,k_2}M_2$ interchangeably depending on whether we want to emphasize the Sasakian or manifold nature of the join. Notice also that if $\gcd(k_1,k_2)=m$ and we define $(k_1',k_2')=(\frac{k_1}{m},\frac{k_2}{m})$, then $\gcd(k_1',k_2')=1$ and $M_1\star_{k_1,k_2}M_2\approx(M_1\star_{k_1',k_2'}M_2)/\mathbb{Z}_m$. In this case the cohomology class $k_1'\omega_1+k_2'\omega_2$ is indivisible in $H^2_{orb}(\mathcal{Z}_1\times\mathcal{Z}_2,\mathbb{Z})$. Note also that $M_1\star_{k_1,k_2}M_2$ can be realized as the quotient space $(M_1\times M_2)/S^1(k_1,k_2)$ where the S^1 action is given by the map

$$(x,y)\mapsto(e^{ik_2\theta}x,e^{-ik_1\theta}y). \tag{2.2}$$

We are interested in restricting the map $\star_{k_1,k_2}$ of 2.1 to the subset of smooth Sasakian manifolds, that is in the map

$$\star_{k_1,k_2}:\mathcal{SM}_{2n_1+1}\times\mathcal{SM}_{2n_2+1}\longrightarrow\mathcal{SO}_{2(n_1+n_2)+1}. \tag{2.3}$$

If M_1 and M_2 are quasi-regular Sasakian manifolds, we are interested under what conditions the orbifold $M_1\star_{k_1,k_2}M_2$ is a smooth manifold. Let υ_i denote the order of the quasi-regular Sasakian manifolds M_i, that is, υ_i is the lcm of the orders of the leaf holonomy groups of M_i. The following proposition is essentially Proposition 4.1 of [BG00b]:

Proposition 2.2: *For each pair of relatively prime positive integers k_1,k_2, the orbifold $M_1\star_{k_1,k_2}M_2$ is a smooth quasi-regular Sasakian manifold if and only if* $\gcd(\upsilon_1k_2,\upsilon_2k_1)=1$. *In particular, if M_i are regular Sasakian manifolds, then so is $M_1\star_{k_1,k_2}M_2$.*

Generally, given two known Sasakian manifolds M_1 and M_2, it can be quite difficult to compute the diffeomorphism type of $M_1\star_{k_1,k_2}M_2$, but some information can be obtained. For example, we have

Proposition 2.3: *Let M_1 and M_2 be compact quasi-regular Sasakian manifolds and assume that* $\gcd(\upsilon_1k_2,\upsilon_2k_1)=1$. *Then $M_1\star_{k_1,k_2}M_2$ is the associated S^1 orbibundle over $\mathcal{Z}_1$ with fibre $M_2/\mathbb{Z}_{k_2}$. In particular, if M_1 is regular and $k_2=1$ then for each positive integer k, $M_1\star_{k,1}M_2$ is an M_2-bundle over the Kähler manifold Z_1.*

Proof. Following [WZ90] we break up the S^1 action on $M_1\times M_2$ into stages. First divide by the subgroup $\mathbb{Z}_{k_2}$ of the circle group $S^1(k_1,k_2)$ defined by Equation 2.2 giving $M_1\times(M_2/\mathbb{Z}_{k_2})$. Letting

$[y] \in M_2/\mathbb{Z}_{k_2}$ denote the equivalence class of $y \in M_2$, we see that the quotient group $S^1/\mathbb{Z}_{k_2}$ acts on $M_1 \times M_2/\mathbb{Z}_{k_2}$ by $(x,[y]) \mapsto (e^{i\theta}x,[e^{-i\frac{k_1}{k_2}\theta}y])$ which identifies $M_1 \star_{k_1,k_2} M_2$ as the orbibundle over $\mathcal{Z}_1$ with fibre $M_2/\mathbb{Z}_{k_2}$ associated to the principal S^1 orbibundle $\pi_1 : M_1 \longrightarrow \mathcal{Z}_1$. This proves the result. □

Recall [BG06] the *type* of a Sasakian structure. A Sasakian structure (ξ,η,Φ,g) is said to be of *positive (negative) type* if the first Chern class $c_1(\mathcal{F}_\xi)$ of the characteristic foliation is represented by a positive (negative) definite $(1,1)$-form. If either of these two conditions is satisfied (ξ,η,Φ,g) is said to be of *definite type*, and otherwise (ξ,η,Φ,g) is of *indefinite type.* (ξ,η,Φ,g) is said to be of *null type* if $c_1(\mathcal{F}_\xi)=0$. We often just say 'a positive Sasakian structure' instead of 'a Sasakian structure of positive type', etc. It will also be convenient to write $c_1(\mathcal{S})$ instead of $c_1(\mathcal{F}_\xi)$ even though c_1 is independent of the Sasakian structure in the deformation class $\mathfrak{F}(\xi)$ [BG06].

Proposition 2.4: *The (k_1,k_2)-join of two positive, negative, or null compact quasi-regular Sasakian manifolds is positive, negative, or null, respectively.*

Proof. This follows from the fact that for any quasi-regular Sasakian structure on a compact manifold M we have an orbifold submersion $\pi : M \longrightarrow \mathcal{Z}$ satisfying $c_1(\mathcal{F}_\xi) = \pi^* c_1^{orb}(\mathcal{Z})$ as real cohomology classes. So the sign or vanishing of $c_1(\mathcal{F}_\xi)$ and $c_1^{orb}(\mathcal{Z})$ coincide. Furthermore, for any pair of integers (k_1,k_2) we have

$$c_1(\mathcal{S}_1 \star_{k_1,k_2} \mathcal{S}_2) = \pi^* c_1^{orb}(\mathcal{Z}_1 \times \mathcal{Z}_2) = \pi^* c_1^{orb}(\mathcal{Z}_1) + \pi^* c_1^{orb}(\mathcal{Z}_2) = c_1(\mathcal{S}_1) + c_1(\mathcal{S}_2).$$

Now suppose that the Sasakian structures $\mathcal{S}_1$ and $\mathcal{S}_2$ are both definite of the same type. Then $c_1(\mathcal{S}_1)+c_1(\mathcal{S}_2)$ can be represented by either a positive definite or negative definite basic $(1,1)$ form. The null case is clear. □

Next we give examples of Wang and Ziller [WZ90] where the topology can be ascertained, even the homeomorphism and diffeomorphism type in certain cases.

EXAMPLE 2.5: The Wang-Ziller manifolds: Let $M^{p_1,p_2}_{k_1,k_2}$ denote $S^{2p_1+1} \star_{k_1,k_2} S^{2p_2+1}$. This is the S^1-bundle over $\mathbb{CP}^{p_1} \times \mathbb{CP}^{p_2}$ whose first Chern class is $k_1[\omega_1]+k_2[\omega_2]$ where ω_i is the standard Kähler class of $H^*(\mathbb{CP}^{p_i},\mathbb{Z})$ and $k_i \in \mathbb{Z}^+$. By Proposition 2.2 $M^{p_1,p_2}_{k_1,k_2}$ admits regular Sasakian structures, and by Proposition 2.4 they are positive. Furthermore, if $\gcd(k_1,k_2)=1$ (which we assume hereafter) the manifolds $M^{p_1,p_2}_{k_1,k_2}$ are simply connected. To analyze the manifolds $M^{p_1,p_2}_{k_1,k_2}$ we follow Wang and Ziller [WZ90] and consider the free T^2 action on $S^{2p_1+1} \times S^{2p_2+1}$ defined by $(\mathbf{x},\mathbf{y}) \mapsto (e^{i\theta_1}\mathbf{x}, e^{i\theta_2}\mathbf{y})$ where $(\mathbf{x},\mathbf{y}) \in \mathbb{C}^{p_1+1} \times \mathbb{C}^{p_2+1}$. The quotient space is $\mathbb{CP}^{p_1} \times \mathbb{CP}^{p_2}$, and $M^{p_1,p_2}_{k_1,k_2}$ can be identified with the quotient of $S^{2p_1+1} \times S^{2p_2+1}$ by the circle defined by $(\mathbf{x},\mathbf{y}) \mapsto (e^{ik_2\theta}\mathbf{x}, e^{-ik_1\theta}\mathbf{y})$. Now the free part of $H^2(M^{p_1,p_2}_{k_1,k_2},\mathbb{Z})$ has a single generator γ, and letting $\pi : M^{p_1,p_2}_{k_1,k_2} \longrightarrow \mathbb{CP}^{p_1} \times \mathbb{CP}^{p_2}$ denote the natural bundle projection, we see that the classes $[\omega_i]$ pull back as $\iota_*\pi^*[\omega_1]=k_2\gamma, \iota_*\pi^*[\omega_2]=-k_1\gamma$. Here by abuse of notation we let $\pi^*[\omega_i]$ also denote the basic classes in $H^2_B(\mathcal{F}_\xi)$. Furthermore, the basic first Chern class is $c_1(\mathcal{F}_\xi)=(p_1+1)\pi^*[\omega_1]+(p_2+1)\pi^*[\omega_2]$, so we get

$$c_1(\mathcal{D}) = \iota_* c_1(\mathcal{F}_\xi) = (p_1+1)\iota_*\pi^*[\omega_1] + (p_2+1)\iota_*\pi^*[\omega_2] = (k_2(p_1+1) - k_1(p_2+1))\gamma.$$

Thus, we have

$$w_2(M^{p_1,p_2}_{k_1,k_2}) = (k_2(p_1+1)+k_1(p_2+1))\gamma \mod 2. \tag{2.4}$$

In certain cases one can determine the manifold completely [WZ90]. For example, consider $p_1 = k_2 = 1, p_2 = q, k_1 = k$ in which case $M^{1,q}_{k,1}$ is an S^{2q+1}-bundle over S^2. The S^k-bundles over S^2 are classified by $\pi_1(SO(k+1)) \approx \mathbb{Z}_2$ [Ste51]. So there are precisely two S^{2q+1}-bundles over S^2, and they are distinguished by w_2. From Equation (2.4) we get $w_2(M^{1,q}_{k,1}) = k(q+1)\gamma \mod 2$. Thus, if q is odd or k is even, we get the trivial bundle $S^2 \times S^{2q+1}$; whereas, if q is even and k is odd, we get the unique non-trivial S^{2q+1}-bundle over S^2. This gives an infinite number of distinct deformation classes of regular positive Sasakian structures on these manifolds. In dimension five ($p_1 = p_2 = 1$)

we can do somewhat better. In fact for any pair of relatively prime positive integers (k_1,k_2) $M^{1,1}_{k_1,k_2}$ is diffeomorphic to $S^2\times S^3$; whereas later in Section we construct a positive Sasakian structure on the non-trivial S^3-bundle over S^2 as well as a family of indefinite Sasakian structures. Notice that in this case $c_1(\mathcal{D})=2(k_1-k_2)\gamma$.

Summarizing from this example gives

Corollary 2.6: *The manifolds $M^{p,q}_{k_1,k_2}$ all admit Sasakian metrics with positive Ricci curvature. In particular, the manifolds $S^2\times S^{2q+1}$ as well as the non-trivial S^{2q+1}-bundle over S^2 admit Sasakian metrics of positive Ricci curvature.*

Wang and Ziller [WZ90] were able to prove the existence of positive Einstein metrics on these manifolds. As we have shown these manifolds always admit positive Sasakian metrics, but the Einstein metrics on $S^2\times S^{2q+1}$ are not generally Sasakian-Einstein. For example, to get a Sasakian-Einstein metric a particular join is necessary [BG00b]. The Wang-Ziller construction gives Sasakian-Einstein metrics on $S^3\star_{2,q+1}S^{2q+1}$ for q even, and $S^3\star_{1,\frac{q+1}{2}}S^{2q+1}$ for q odd. These are all non-trivial fibre bundles over S^2 whose fibres are the appropriate lens spaces.

We are interested in when the join of two Sasakian η-Einstein manifolds of the same type is a Sasakian η-Einstein manifold of that type. In the case of Sasakian-Einstein structures this was described in [BG00b]. In that case we had to choose the pair (k_1,k_2) to be the relative Fano indices of the two Sasakian manifolds. The concept of index applies equally well to the negative definite case, but the name Fano is inappropriate. So assuming that $c_1^{orb}(\mathcal{Z})$ is either positive or negative definite, we write $|c_1^{orb}(\mathcal{Z})|$ to mean $c_1^{orb}(\mathcal{Z})$ in the positive case and $-c_1^{orb}(\mathcal{Z})$ in the negative case. Then we can define the *divisibility index* or just *index* of the polarized Kähler orbifold $(\mathcal{Z},|c_1^{orb}(\mathcal{Z})|)$ to be the largest integer $I=I(\mathcal{Z})$ such that $c_1^{orb}(\mathcal{Z})/I$ defines an integral class in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$. We also say that the polarized Kähler orbifold $(\mathcal{Z},|c_1^{orb}(\mathcal{Z})|)$ is *indivisible.* Thus, if $c_1^{orb}(\mathcal{Z})$ is represented by either a positive definite or negative definite $(1,1)$-form, there is a unique indivisible polarized Kähler orbifold $(\mathcal{Z},|c_1^{orb}(\mathcal{Z})|/I)$ that gives rise to a quasi-regular Sasakian orbifold structure $\mathcal{S}$. So we can consider the index I to be an invariant of the Sasakian structure $\mathcal{S}$ and write $I=I(\mathcal{S})$ as well. In this regard, we also say that a quasi-regular definite Sasakian structure $\mathcal{S}$ on M is *indivisible.* By Theorem 2.1 of [BG00b] indivisible positive quasi-regular Sasakian manifolds are simply connected. For $i=1,2$ we define the *relative indices* of a pair of quasi-regular definite Sasakian structures $(\mathcal{S}_1,\mathcal{S}_2)$ of the same type by

$$l_i=\frac{I(\mathcal{S}_i)}{\gcd(I(\mathcal{S}_1),I(\mathcal{S}_2))}.$$

Then, $\gcd(l_1,l_2)=1$, and if $\gcd(\upsilon_1l_2,\upsilon_2l_1)=1$, then $M_1\star_{l_1,l_2}M_2$ is a smooth indivisible Sasakian manifold. If in addition M_1 and M_2 are both positive Sasakian manifolds, $M_1\star_{l_1,l_2}M_2$ is a simply connected positive Sasakian manifold.

We now consider the join of two Sasakian η-Einstein manifolds. The case of positive η-Einstein manifolds is essentially the same as for Sasakian-Einstein manifolds. Of course, in the positive case there are obstructions to satisfying the Monge-Ampère equation on a compact Kähler orbifold or the transverse Monge-Ampère equation on a Sasakian manifold; whereas, in both the negative case and null cases, there are no such obstructions. So for any null Sasakian structure or negative Sasakian structure such that $c_1(\mathcal{F}_\xi)$ is a multiple of $[d\eta]_B$ on a compact manifold there exists a compatible Sasakian η-Einstein metric.

Proposition 2.7: *Let l_i be the relative indices for a pair of quasi-regular definite Sasakian η-Einstein manifolds M_i of the same type, respectively. Then the join $M_1\star_{l_1,l_2}M_2$ admits a quasi-regular Sasakian η-Einstein structure of that type.*

We can now obtain new Sasakian-Einstein metrics by combining Proposition 2.3 with the results in [BGK05, BGKT05, GK05]. For example take $M_1=S^3$with its canonical round sphere Sasakian structure, and $M_2=S^5_{\mathbf{w}}$ with one of the 68 deformation classes of Sasakian-Einstein structures on

S^5 found in [BGK05] or one of the 12 Sasakian-Einstein structures in [GK05]. Let $L^5(l_2)$ denote the lens space S^5/C_{l_2} where $C_{l_2} \approx \mathbb{Z}_{l_2}$ is the cyclic subgroup of the circle group $S^1_{\mathbf{w}}$ generated by the Reeb vector field of the corresponding Sasakian structure, and l_1 is the reduced Fano index of $S^5_{\mathbf{w}}$ with respect to S^3. It is straightforward to compute $l_2 = l_2(\mathbf{w})$ as a function of $\mathbf{w}$. Then we have

Theorem 2.8: *If* $\gcd(l_2, \upsilon_2) = 1$ *then* $S^3 \star_{l_1,l_2} S^5_{\mathbf{w}}$ *is the total space of the fibre bundle over* S^2 *with fibre the lens space* $L^5(l_2)$*, and it admits Sasakian-Einstein metrics. In particular, for 16 different weight vectors* $\mathbf{w}$*, the manifold* $S^3 \star_{2,1} S^5_{\mathbf{w}}$ *is homeomorphic to* $S^2 \times S^5$ *and admits Sasakian-Einstein metrics including one 10-dimensional family. Moreover, for 3 different weight vectors* $\mathbf{w}$*, the manifold* $S^3 \star_{1,1} S^5_{\mathbf{w}}$ *is homeomorphic to* $S^2 \times S^5$ *and admits Sasakian-Einstein metrics.*

Proof. As mentioned above this is constructed using Proposition 2.3 with $M_1 = S^3$ with its standard round sphere Sasakian-Einstein structure, and M_2 one of the Sasakian-Einstein structures on S^5 mentioned above. So the first statement follows. To prove the second statement we need to compute the relative indices. Since $I(S^3) = 2$ for the standard Sasakian structure on S^3, we need to consider two cases, namely when $I(S^5_{\mathbf{w}}) = 1$, or 2. In both cases the relative index $l_2 = 1$, so we have $S^3 \star_{l_1,1} S^5_{\mathbf{w}}$ which is an S^5 bundle over S^2. These are classified by their second Stiefel-Whitney class w_2. But by construction the orbifold first Chern class of $\mathbb{CP}^1 \times \mathcal{Z}$ is proportional to the first Chern class of the S^1 orbibundle defining $S^3 \star_{l_1,1} S^5_{\mathbf{w}}$. This implies that $w_2(S^3 \star_{l_1,1} S^5_{\mathbf{w}})$ vanishes [BGN03c]. Now all of the Sasakian structures on S^5 can be represented as links of Brieskorn-Pham polynomials of the form $f = z_0^{a_0} + z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ or deformations thereof. So we need to compute the Fano index I for the 68 Brieskorn polynomials representing S^5 that admit Sasakian-Einstein metrics found in [BGK05], and the 12 found in [GK05]. Now it is easy to see that in terms of the Brieskorn exponents the Fano index takes the form

$$I = \mathrm{lcm}(a_0, a_1, a_2, a_3)\big(\sum_{i=0}^{3} \frac{1}{a_i} - 1\big).$$

It is easy to write a Maple program to determine the Fano index of the 80 cases. There are 16 with $I = 1$ and 3 with $I = 2$ giving 19 in all. For example the 10 parameter family given by Example 41 in [BGK05] has $\mathbf{a} = (2, 3, 7, 35)$, and one easily sees that $I = 1$. □

REMARKS 2.1: We remark that only the $S^5_{\mathbf{w}}$ found in [BGK05] give rise to Sasakian-Einstein metrics on $S^2 \times S^5$; the ones found in [GK05] all have Fano index greater than 2 and give non-trivial lens spaces. In fact the largest l_2 obtained is 89. Generally, assuming $\gcd(l_2, \upsilon_2) = 1$, an easy spectral sequence argument shows that the manifolds $S^3 \star_{l_1,l_2} S^5_{\mathbf{w}}$ are simply connected with the rational homology type of $S^2 \times S^5$, but with $H^4(S^3 \star_{l_1,l_2} S^5_{\mathbf{w}}, \mathbb{Z}) \approx \mathbb{Z}_{l_2}$.

It is now quite straightforward to apply Proposition 2.3 to many other cases. For example, we can consider the join $S^3 \star_{2,1} k(S^2 \times S^3)_{\mathbf{w}}$ or $S^3 \star_{1,1} k(S^2 \times S^3)_{\mathbf{w}}$ where $k(S^2 \times S^3)_{\mathbf{w}}$ is any of the Sasakian-Einstein manifolds consider in [BGN03a, BGN02, BG03] with Fano index $I = 1$ in the first case, and $I = 2$ in the second. This gives Sasakian-Einstein metrics on manifolds whose rational cohomology can be determined as in [BG00b]. The higher index cases in [BGN03a] can also be treated as long as the relative index l_2 is relatively prime to the order of the Sasakian structure of $k(S^2 \times S^3)_{\mathbf{w}}$.

Recall (cf. [BG06, BGM06]) that the real Heisenberg group $\mathfrak{H}_{2n+1}(\mathbb{R})$ admits a homogeneous Sasakian structure with its standard 1-form $\eta = dz - \sum_i y^i dx^i$. As a manifold $\mathfrak{H}_{2n+1}(\mathbb{R})$ is just $\mathbb{R}^{2n+1}$ which can be realized in terms of $n+2$ by $n+2$ nilpotent matrices of the form

$$\begin{pmatrix} 1 & x_1 & \cdots & x_n & z \\ 0 & 1 & 0 & \cdots & y_1 \\ \vdots & & \ddots & \cdots & \vdots \\ 0 & \cdots & 0 & 1 & y_n \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \tag{2.5}$$

If we consider the discrete subgroup $\mathfrak{H}_{2n+1}(\mathbb{Z})$ of $\mathfrak{H}_{2n+1}(\mathbb{R})$ defined by the matrices 2.5 with integer entrees, the quotient manifold $\mathcal{N}_{2n+1} = \mathfrak{H}_{2n+1}(\mathbb{R})/\mathfrak{H}_{2n+1}(\mathbb{Z})$ is a nilmanifold with an induced Sasakian structure. As a coset space it is also a homogeneous manifold, but the homogeneous structure and Sasakian structure are incompatible. The Reeb vector field generates the one dimensional center $\mathfrak{Z}(\mathfrak{H}_{2n+1}(\mathbb{R}))$ of the group $\mathfrak{H}_{2n+1}(\mathbb{R})$, and thus $\mathfrak{Z}(\mathfrak{H}_{2n+1}(\mathbb{R}))$ induces an S^1 action on the quotient space $\mathcal{N}_{2n+1}$. This S^1 is the connected component of the group $\mathfrak{Aut}$ of Sasakian automorphisms of $\mathcal{N}_{2n+1}$ and makes $\mathcal{N}_{2n+1}$ the total space of an S^1 bundle over the principally polarized Abelian variety $\mathcal{A} = T^{2n}$ with its standard complex structure. One can obtain many regular Sasakian structures on $\mathcal{N}_{2n+1}$ by considering it as a circle bundle over a polarized Abelian variety and deforming the complex structure. Now we can form the join $\mathcal{N}_{2n+1} \star_{k_1,k_2} M$ where M is a regular Sasakian manifold which by Proposition 2.3 can be thought of as an $M/\mathbb{Z}_{k_2}$-bundle over T^{2n}.

Recall that a locally conformal Kähler manifold is a complex manifold which admits a covering endowed with a Kähler metric with respect to which the group of deck transformations acts by holomorphic homotheties (cf. [DO98]). The subclass of Vaisman manifolds can be characterized in terms of Sasakian geometry as follows (cf. [OV03]): Any compact Vaisman manifold P is a suspension over a circle, with fibre a Sasakian manifold M. Moreover, there exist a Sasakian automorphism φ of M and a positive q such that P is isomorphic with the quotient of the Riemannian cone $(M \times \mathbb{R}_+, t^2 g_M + dt^2)$ by the cyclic group generated by $(x,t) \mapsto (\varphi(x), qt)$.It is clear from the definition that the product of two l.c.K. structures is not, in general, l.c.K. Moreover, the product of two Vaisman manifolds might not be Vaisman: *e.g.* the product of two Hopf surfaces has $b_1 = 2$ which prevents it to admit a Vaisman structure, for which b_1 should be odd. Instead, we can combine the join construction with the structure theorem to define a join of quasi-regular compact Vaisman manifolds. Note that, in fact, this is not restrictive, since any compact Vaisman structure can be deformed to a quasi-regular one [OV05]. Summing up, we have:

Definition 2.9: *Let P_1, P_2 be two compact, quasi-regular Vaisman manifolds and let M_1, M_2 be the respective Sasakian manifolds provided by the structure theorem. Then, for each $k_1, k_2 \in \mathbb{Z}$, the suspension over the circle with fibre $M_1 \star_{k_1,k_2} M_2$ is the* **join** *of P_1 and P_2.*

3. Contact Fibre Bundles and Toric Sasakian Structures

The aim of this section is to briefly discuss a construction due to Lerman [Ler04a] that allows one to construct K-contact structures on the total space of a fibre bundle whose fibres are K-contact. This construction generalizes the join construction described in Section 2 as well as the fibre join construction of Yamazaki [Yam99]. Actually one can work within the pure contact setting, and it is the contact analog of symplectic fibre bundles described in [GLS96]. Recall that a *contact structure* on an oriented manifold M is an equivalence class of 1-forms η which satisfy $\eta \wedge (d\eta)^n \neq 0$, where two such forms η, η' are equivalent if there is a nowhere vanishing smooth function f such that $\eta' = f\eta$. Alternatively, a contact structure is a maximally non-integrable codimension one subbundle $\mathcal{D}$ of the tangent bundle TM. The relation between the two descriptions is $\mathcal{D} = \ker\ \eta$. We denote a contact manifold by the pair $(M, \mathcal{D})$, and let $\mathfrak{Con}(M, \mathcal{D})$ denote the group of contactomorphisms of $(M, \mathcal{D})$, that is, the subgroup of the group of diffeomorphisms of M leaving the contact bundle $\mathcal{D}$ invariant. The contact manifold $(M, \mathcal{D})$ is said to be *co-oriented* if the subbundle $\mathcal{D}$ is oriented. The subgroup of $\mathfrak{Con}(M, \mathcal{D})$ which fixes the orientation is denoted by $\mathfrak{Con}(M, \mathcal{D})^+$. Here is Lerman's definition of a contact fibre bundle.

Definition 3.1: *A fibre bundle $F \longrightarrow M \xrightarrow{\pi} B$ is called a* **contact fibre bundle** *if*

(i) *F is a co-oriented contact manifold with contact bundle $\mathcal{D}$.*
(ii) *There are an open cover $\{U_i\}$ of B and local trivializations $\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times F$ such that for every point $p \in U_i \cap U_j$ the transition functions $\phi_j \circ \phi_i^{-1}|_{\{p\}\times F}$ are elements of $\mathfrak{Con}(F, \mathcal{D})^+$.*

We need the notion of fatness of a bundle due to Weinstein [Wei80]. Let α be a connection 1-form in a principal bundle $P(M,G)$ with Lie group G, and let $\Omega = D\alpha$ denote its curvature 2-form. Let $S \subset \mathfrak{g}^*$ be any subset in the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G. We say that the connection α is *fat on* S if the bilinear map

$$\mu \circ \Omega : \mathcal{H}P \times \mathcal{H}P : \longrightarrow \mathbb{R} \tag{3.1}$$

is non-degenerate for all $\mu \in S$. In particular, if G is a torus, the bundle $\pi : P \to B$ is identified, up to a gauge transformation, by a connection form A such that $dA = \pi^*\omega$ with $[\omega] \in H^2(B,\mathbb{Z})$. Then, if ω is non-degenerate, that is a symplectic form, A is certainly fat on the image of the moment map.

We first recall the main lines of the construction, not in full generality, but adapted to our needs. Let $\pi : P \to B$ be a principal G-bundle endowed with a connection A (we don't distinguish between the connection and its 1-form). Let F be a K-contact manifold, with fixed contact form η_F and Reeb field ξ_F. Suppose $G \subset \mathrm{Aut}(F,\eta)$, *i.e.* it acts (from the left) on F by strong contactomorphisms and denote by $\Psi : F \to \mathfrak{g}^*$ the associated momentum map. Then Lerman proves:

Theorem 3.2: [Ler04a] *In the above setting, if the connection A is fat at all the points of the image of the momentum map Ψ, then the total space M of the associated bundle $P \times_G F$ admits a K-contact structure.*

We are interested in the case that the underlying almost CR structure of the K-contact structure is integrable. In this case the manifold $P \times_G F$ will be Sasakian. One way of guarantying this occurs in the toric setting, so we now give a brief review of toric contact geometry. This was begun in [BM93], continued in [BG00a], and completed in [Ler02]. Let $(M,\mathcal{D})$ be a co-oriented contact manifold,

Definition 3.3: *A* **toric contact manifold** *is a triple $(M,\mathcal{D},T^{n+1})$ where $(M,\mathcal{D})$ is an a co-oriented contact manifold of dimension $2n+1$ with an effective action*

$$\mathcal{A} : T^{n+1} \longrightarrow \mathfrak{Con}(M,\mathcal{D})^+$$

of a $(n+1)$-torus T^{n+1}.

In [BG00a] the first two authors introduced the notion of a contact toric structure of Reeb type.

Definition 3.4: *We say that a torus action $\mathcal{A} : T^{n+1} \longrightarrow \mathfrak{Con}(M,\mathcal{D})^+$ is of* **Reeb type** *if there are a contact 1-form η of the contact structure $\mathcal{D}$ and an element $\varsigma \in \mathfrak{g}$ such that X^ς is the Reeb vector field of η.*

Fixing a contact form η it is easy to see that the action $\mathcal{A}$ of T^{n+1} is of Reeb type if and only if there is an element τ in the Lie algebra $\mathfrak{t}_{n+1}$ of T^{n+1} such that $\eta(X^\tau) > 0$. Note that when T^{n+1} acts properly we can always fix a contact 1-form η without loss of generality by using a slice theorem. In this case the relevant group is the subgroup $\mathfrak{Con}(M,\eta) \subset \mathfrak{Con}(M,\mathcal{D})^+$ of contactomorphisms leaving η invariant. We are now ready for

Theorem 3.5: *Let F^{2n+1} be a compact toric contact manifold of Reeb type and with torus $T^{n+1} \subset \mathfrak{Con}(F,\eta)$. Let $\pi : P \to B$ be a principal T^{n+1} bundle over a toric compact symplectic manifold B. Then $P \times_{T^{n+1}} F$ is a toric Sasakian manifold.*

Proof. Choosing a connection A as above whose curvature is non-degenerate, this will be fat and Lerman's construction applies. From [BG00a], F has a compatible Sasakian structure, in particular it is K-contact. Note that this Sasakian structure is toric. Then by [Ler04a], in our hypothesis, $P \times_{T^{n+1}} F$ has a K-contact structure. We claim that this is toric, of Reeb type, and hence the result follows by applying again [BG00a]. To prove our claim, we show that:

(i) The Hamiltonian action of T^m on B ($\dim B = 2m$) lifts to a T^m action on P.
(ii) This lifted action extends to $P \times_{T^{n+1}} F$ preserving the contact form.
(iii) The action of T^m on $P \times_{T^{n+1}} F$ commutes with the action of T^{n+1}, hence, as it leaves the contact form η invariant, it induces an action of T^{n+m+1} on $P \times_{T^{n+1}} F$. Then we only need to see that the Reeb field of $P \times_{T^{n+1}} F$ is generated by the T^{n+1} action.

To prove (i), it is enough to show that T^m lifts to an action that preserves the fat connection A. Denote $\{Y_i\}$ the generators of the T^m action on B. We need to construct lifts $\tilde{Y}_i$ on P such that $\mathcal{L}_{\tilde{Y}_i} A = 0$. Define them as

$$\tilde{Y}_i = \hat{Y}_i + a_i^\alpha X_\alpha,$$

where the $\hat{}$ refers to horizontal lifts, X_α are vertical fields and the functions a_i^α need to be determined. If we let $\{e_\alpha\}$ be a basis of the Lie algebra $\mathfrak{t}^{n+1}$ of T^{n+1}, then, as A is a T^{n+1}-connection, we have

$$A(\tilde{Y}_i) = a_i^\alpha A(X_\alpha) = a_i^\alpha e_\alpha.$$

On the other hand, we want the functions a_i^α to be solutions of the equation

$$\tilde{Y}_i \rfloor dA + d(\tilde{Y}_i \rfloor A) = 0,$$

and hence

$$d(a_i^\alpha) = -\tilde{Y}_i \rfloor A.$$

As Y_i are Hamiltonian, $Y_i \rfloor \omega^\alpha = dF_i^\alpha$, thus

$$dF_i^\alpha = Y_i \rfloor \omega^\alpha = \tilde{Y}_i \rfloor A.$$

All in all $da_i^\alpha = -dF_i^\alpha$ and we may take $a_i^\alpha = -F_i^\alpha$.

For (ii), as T^m preserves A and η_F (the action of T^m on F is trivial), T^m preserves the contact form η on $P \times_{T^{n+1}} F$. □

Remark 3.1: Suppose now that F is regular and let B_F the basis of its Boothby-Wang fibration. Also, suppose that for a torus T (not necessarily of maximal dimension) $P \times_T F$ has a Sasakian structure. As we have seen above, this is the case when P and B are toric. Lerman only constructs the contact structure on $P \times_T F$, but it can be seen that a contact form adapted to this is written on $P \times F$ as $\eta = f_A \cdot A + \eta_F$ for some function f_A and hence

$$\begin{aligned} d\eta &= df_A \wedge A + f_A dA + d\eta_F \\ &= df_A \wedge A + f_A \omega_P + \omega_{B_F} \end{aligned}$$

Here ω_P and ω_{B_F} are $(1,1)$ forms. Splitting df_A and A into their $(0,1)$ and $(1,0)$ components, we see that $d\eta$ has a $(2,0)$ component, namely $df_A^{(1,0)} \wedge A^{(1,0)}$, if and only if $df_A^{(1,0)}$ and $A^{(1,0)}$ are linearly independent, which clearly happens when on $B \times B_F$ we take the product complex structure. But in this case $d\eta = \pi^*\omega$ for the Kähler form of $B \times B_F$ and it has to be of type $(1,1)$. So, if the complex structure on $B \times B_F$ is the product one, then necessarily $f_A = const$. For $T = S^1$, this corresponds to the above described join. Hence, the join is a particular case of Lerman's construction.

4. Toric Sasakian 5-Manifolds

We begin this section by recalling fundamental results of Smale and Barden concerning classification of compact smooth simply-connected 5-manifolds [Sma62, Bar65]. Remarkably, any such manifold is completely determined by $H_2(M,\mathbb{Z})$ and the second Stiefel-Whitney class map w_2. In particular, the smooth structure on a closed simply-connected 5-manifold is unique.

Theorem/Definition 4.1: *Let M be a compact, smooth, oriented, 1-connected 5-manifold. Write $H_2(M,\mathbb{Z})$ as a direct sum of cyclic groups of prime power order*

$$H_2(M,\mathbb{Z}) = \mathbb{Z}^k \bigoplus_{p,i} (\mathbb{Z}_{p^i})^{c(p^i)} \tag{4.1}$$

where $k = b_2(M)$, and $c(p^i) = c(p^i, M)$. The non-negative integers $k, c(p^i)$ are determined by $H_2(M,\mathbb{Z})$ but the subgroups $\mathbb{Z}_{p^i} \subset H_2(M,\mathbb{Z})$ are not unique. One can choose the decomposition (4.1) such that the second Stiefel–Whitney class map

$$w_2 : H_2(M,\mathbb{Z}) \to \mathbb{Z}_2$$

is zero on all but one summand $\mathbb{Z}_{2^j}$. The value j is unique, denoted by $i(M)$, and called the **Barden invariant** *of M. It can take on any value j for which $c(2^j) \neq 0$, besides 0 and ∞. Alternatively, $i(M)$ is the smallest j such that there is an $\alpha \in H_2(M,\mathbb{Z})$ such that $w_2(\alpha) \neq 0$ and α has order 2^j.*

The following theorem was proved by Smale [Sma62] in the spin case in when $w_2 = 0$ implying $i = 0$. Subsequent generalization with no assumption on w_2 is due to Barden [Bar65]. We shall formulate it here using Barden's notation.

Theorem 4.2: *The class $\mathcal{B}$ of simply connected, closed, oriented, smooth, 5-manifolds is classifiable under diffeomorphism. Furthermore, any such M is diffeomorphic to one of the spaces*

$$M_{j;k_1,\ldots,k_s} = X_j \# M_{k_1} \# \cdots \# M_{k_s}, \tag{4.2}$$

where $-1 \leqslant j \leqslant \infty$, $s \geqslant 0$, $1 < k_1$ and k_i divides k_{i+1} or $k_{i+1} = \infty$. A complete set of invariants is provided by $H_2(M,\mathbb{Z})$ and $i(M)$ and the manifolds $X_{-1}, X_0, X_j, X_\infty, M_j, M_\infty$ are characterized as follows

M	$H_2(M,\mathbb{Z})$	$i(M)$
$X_{-1} = SU(3)/SO(3)$	$\mathbb{Z}_2$	1
$M_0 = X_0 = S^5$	0	0
$X_j, \quad 0 < j < \infty$	$\mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j}$	j
X_∞	$\mathbb{Z}$	∞
$M_k, \quad 0 < k < \infty$	$\mathbb{Z}_k \oplus \mathbb{Z}_k$	0
$M_\infty = S^2 \times S^3$	$\mathbb{Z}$	0

In this section we would like to investigate the question which of the manifolds in $\mathcal{B}$ admit toric Sasakian structures. We begin with an important example.

Example 4.3: **[Circle Bundles over Hirzebruch surfaces]** We shall construct infinite families of deformation classes of toric Sasakian structures on circle bundles over Hirzebruch surfaces. The total space M of any such circle bundle must have $b_1(M) = 0$ and $b_2(M) = 1$. Theorem 4.2 implies that there are precisely two 5-manifolds in $\mathcal{B}$ with $b_2 = 1$. They are $M_\infty = S^2 \times S^3$ and X_∞, the non-trivial S^3-bundle over S^2. They both have $H_2(M,\mathbb{Z}) = \mathbb{Z}$ and are distinguished by their Barden invariant.

Recall the Hirzebruch surfaces S_n (cf. [GH78], pgs 517-520) are realized as the projectivizations of the sum of two line bundles over $\mathbb{CP}^1$, which we can take as

$$S_n = \mathbb{P}\big(\mathcal{O} \oplus \mathcal{O}(n)\big).$$

They are diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ if n is even, and to the blow-up of $\mathbb{CP}^2$ at one point, which we denote as $\widetilde{\mathbb{CP}}^2$, if n is odd. For $n = 0$ and 1 we get $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\widetilde{\mathbb{CP}}^2$, respectively. Now $\mathrm{Pic}(S_n) \approx \mathbb{Z} \oplus \mathbb{Z}$, and we can take the Poincaré duals of a section of $\mathcal{O}(n)$ and the homology class of the fibre as its generators. The corresponding divisors can be represented by rational curves which we denote by C and F, respectively satisfying

$$C \cdot C = n, \qquad F \cdot F = 0, \qquad C \cdot F = 1.$$

Let α_1 and α_2 denote the Poincaré duals of C and F, respectively. The classes α_1 and α_2 can be represented by $(1,1)$ forms ω_1 and ω_2, respectively, so that the $(1,1)$ form $\omega_{l_1,l_2} = l_1\omega_1 + l_2\omega_2$ determines a circle bundle over S_n whose first Chern class is $[\omega_{l_1,l_2}]$. We thus have circle bundles depending on a triple of integers (l_1, l_2, n), with n non-negative,

$$S^1 \longrightarrow M_{l_1,l_2,n} \xrightarrow{\pi} S_n.$$

Now in order that $M_{l_1,l_2,n}$ admit a Sasakian structure it is necessary that ω_{l_1,l_2} be a positive $(1,1)$ form, that is, ω_{l_1,l_2} must lie within the Kähler cone $\mathcal{K}(S_n)$. The conditions for positivity are by Nakai's criterion,

(i) $\omega^2_{l_1,l_2} > 0$,

(ii) $\int_D \omega_{l_1,l_2} > 0$ for all holomorphic curves D,

which in our case give $l_1, l_2 > 0$.

Next we determine the diffeomorphism type of $M_{l_1,l_2,n}$. Since the Kähler class $[\omega_{l_1,l_2}]$ transgresses to the derivative of the contact form, $d\eta_{l_1,l_2}$, we see that $\pi^*\alpha_1 = -l_2\gamma$ and $\pi^*\alpha_2 = l_1\gamma$ where γ is a generator of $H^2(M_{l_1,l_2,n},\mathbb{Z}) \approx \mathbb{Z}$. Now the first Chern class of S_n is [GH78]

$$c_1(S_n) = 2\alpha_1 - (n-2)\alpha_2, \tag{4.3}$$

which pulls back to the basic first Chern class on $M_{l_1,l_2,n}$. So the first Chern class of the contact line bundle $\mathcal{D}$ is given by

$$c_1(\mathcal{D}) = -[2l_2 - l_1(2-n)]\gamma. \tag{4.4}$$

If we take the integers l_1, l_2 to be relatively prime then the manifold $M_{l_1,l_2,n}$ will be simply connected. Furthermore, since $M_{l_1,l_2,n}$ has a regular contact structure, there is no torsion in $H_2(M_{l_1,l_2,n},\mathbb{Z})$ [Gei91]. Thus, by Theorem 4.2 $M_{l_1,l_2,n}$ is either $S^2\times S^3$ or X_∞, depending on whether $M_{l_1,l_2,n}$ is spin or not. But we have $w_2(M_{l_1,l_2,n}) \equiv nl_1 \mod 2$, so $M_{l_1,l_2,n}$ is diffeomorphic to $S^2\times S^3$ if nl_1 is even, and to X_∞ if nl_1 is odd. Now that we have identified the manifolds $M_{l_1,l_2,n}$, we wish to distinguish the deformation classes of Sasakian structures that live on them. Thus, we consider the equivalence classes of regular homologous Sasakian structures $\mathfrak{F}_{l_1,l_2,n}$ [BGN03b, BG06]. We have arrived at

Theorem 4.4: *For each triple of positive integers (l_1, l_2, m) satisfying $\gcd(l_1, l_2) = 1$, the manifold $S^2\times S^3$ admits the following deformation classes of regular Sasakian structures $\mathfrak{F}_{l_1,l_2,2m}$ and $\mathfrak{F}_{2l_1,l_2,2m+1}$.*

Theorem 4.5: *For each triple of positive integers (l_1, l_2, m) satisfying $\gcd(l_1, l_2) = 1$, the manifold X_∞ admits the deformation classes of regular Sasakian structures $\mathfrak{F}_{2l_1-1,l_2,2m-1}$.*

Now we have

Theorem 4.6: *The deformation classes of Sasakian structures described in Theorems 4.4 and 4.5 are all toric.*

Proof. As in [Aud94] we describe Hirzebruch surfaces S_n as smooth algebraic subvarieties of $\mathbb{CP}^1\times\mathbb{CP}^2$, viz.

$$S_n = \{[u_0,u_1],[v_0,v_1,v_2] \mid u_0^n v_1 = u_1^n v_0\} \subset \mathbb{CP}^1\times\mathbb{CP}^2.$$

Now S_n admits the action of a complex 2-torus by

$$([u_0,u_1],[v_0,v_1,v_2]) \mapsto ([\tau u_0,\zeta u_1],[\zeta^{-n}v_0,\tau^{-n}v_1,\tau^{-n}\zeta^{-n}v_2]).$$

The homology class F is represented by the rational curve $([a,b],[0,0,1])$, while C is represented by $([a,b],[a^n,b^n,0])$, and it is easy to check that these rational curves are invariant under $\mathbb{C}^*\times\mathbb{C}^*$-action given above. It follows that for each admissible value of (l_1,l_2) the Kähler form $\omega = l_1\omega_1 + l_2\omega_2$ is invariant under the toral subgroup $\mathfrak{T}_2$ of $\mathbb{C}^*\times\mathbb{C}^*$. Furthermore, the action of $\mathfrak{T}_2$ is Hamiltonian and hence, it lifts to a $\mathfrak{T}_2$ in the automorphism group of the Sasakian structure. This together with the S^1 generated by the Reeb vector field ξ_{l_1,l_2} gives $M_{l_1,l_2,n}$ a toric Sasakian structure. □

Remark 4.1: The existence of toric Sasakian structures on $S^2\times S^3$ and X_∞ also follows from Theorem 3.5 as well as Theorem 4.9 below.

Next we briefly discuss which toric Sasakian structures $\mathfrak{F}_{l_1,l_2,n}$ belong to equivalent contact structures. It is convenient to make a change of basis of $H^2(S_n,\mathbb{Z})$. For simplicity we consider the case $n = 2m$ so S_{2m} is diffeomorphic to $S^2\times S^2$. For $i = 1, 2$ we let σ_i denote the classes in $H^2(S^2\times S^2,\mathbb{Z})$ given by pulling back the volume form on the i^{th} factor. Writing the Kähler class $[\omega] = a_1\sigma_1 + a_2\sigma_2$ in terms of the this basis, we see that

$$a_1 = l_1 m + l_2, \qquad a_2 = l_1,$$

and the positivity condition becomes $a_1 > ma_2 > 0$. We denote the corresponding deformation classes of toric Sasakian structures on $S^2 \times S^3$ by $\mathfrak{F}(a_1, a_2, m)$. The integers a_1, a_2 are written as a, b in [Kar03] and [Ler03]. In terms of the a_i the first Chern class 4.4 simplifies to $c_1(\mathcal{D}) = 2(a_1 - a_2)\gamma$. Thus, $\mathfrak{F}(a_1, a_2, m)$ and $\mathfrak{F}(a_1', a_2', m')$ belong to non-isomorphic contact structures if $a_1' - a_2' \neq a_1 - a_2$. The following theorem is due to Lerman [Ler03].

Theorem 4.7: *For every pair of relatively prime integers (a_1, a_2) there are $\lceil \frac{a_2}{a_1} \rceil$ inequivalent regular toric Sasakian structures on $S^2 \times S^3$ having the same contact form η_{a_1,a_2}. However, for each integer $m = 0, \cdots, \lceil \frac{a_2}{a_1} \rceil$ the structures $\mathfrak{F}(a_1, a_2, m)$ are inequivalent as toric contact structures.*

Here $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. The essence of Theorem 4.7 is that $\lceil \frac{a_2}{a_1} \rceil$ is precisely the number of non-conjugate maximal tori in the contactomorphism group $\mathfrak{Con}(S^2 \times S^3, \eta_{a_1,a_2})$ [Kar03, Ler03].

We now begin the discussion of the general toric case. It turns out that even asking for just an effective T^3 action on M severely restricts its topology. Recall the following classification theorem of Oh [Oh83]:

Theorem 4.8: *Let M be a closed simply connected 5-manifold with an effective T^3-action. Then M has no 2-torsion. In particular, M is diffeomorphic to $S^5, k(S^2 \times S^3)$ or $X_\infty \# (k-1)(S^2 \times S^3)$, where $k = b_2(M) \geqslant 1$. Conversely, all these manifolds admit effective T^3 actions.*

In particular, there are infinitely many Sasakian 5-manifolds (even Sasakian-Einstein) which do not admit any toric contact structure. In [Yam01] Yamazaki proved that all Oh's toric 5-manifolds also admit compatible K-contact structures. But then the main theorem in [BG00a] can be used to strengthen this to

Theorem 4.9: *Let M be a closed simply connected 5-manifold with an effective T^3-action. Then M admits toric Sasakian structures and is diffeomorphic to $S^5, k(S^2 \times S^3)$ or $X_\infty \# (k-1)(S^2 \times S^3)$, where $k = b_2(M) \geqslant 1$.*

Further recall that Geiges [Gei91] showed that the torsion in $H_2(M^5, \mathbb{Z})$ is the only obstruction to the existence of a *regular* contact structure on M. So the question arises whether, for a given M in Theorem 4.9 there exist a regular Sasakian structures compatible with some toric contact structure. We answer this in the affirmative by giving an explicit construction as circle bundles over the blow-ups of the Hirzebruch surfaces.

It is well-known [Ful93] that the smooth toric surfaces are all obtained by blowing-up Hirzebruch surfaces at the fixed points of the T^2 action. Begin with a Hirzebruch surface S_n and blow-up S_n at one of the 4 fixed points of the T^2 action. This gives a smooth toric surface $S_{n,1}$ which can be represented by a Delzant polytope with 5 vertices. Repeat this procedure inductively to obtain smooth algebraic toric surfaces $S_{n,k}$ whose Delzant polytope has $k+4$ vertices. Choose a Kähler class $[\omega]$ lying on the Neron-Severi lattice, and construct the circle bundle $\pi_{n,k} : M_{n,k} \longrightarrow S_{n,k}$ whose Euler class is $[\omega]$. The Kähler form can be chosen to be invariant under the T^2 action, and we can choose a T^2 invariant connection η in $\pi_{n,k} : M_{n,k} \longrightarrow S_{n,k}$ whose curvature form satisfies $d\eta = \pi_{n,k}^* \omega$. This gives a regular Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ on $M_{n,k}$, and by Theorem 4.2 $M_{n,k}$ is diffeomorphic to either $(k+1)(S^2 \times S^3)$ or $X_\infty \# k(S^2 \times S^3)$. Since $H^1(S_{n,k}, \mathbb{Z}) = 0$ the torus action T^2 lifts to a T^2 action in the automorphism group of the Sasakian structure $\mathcal{S}$ (cf. [BG06]) which together with the circle group generated by the Reeb field ξ makes $\mathcal{S}$ a regular toric Sasakian structure.

It remains to show that all of the manifolds in $\mathcal{B}$ with no 2-torsion occur. For this we need to compute the second Stiefel-Whitney class $w_2(M_{n,k})$ which is the mod 2 reduction of the first Chern class of contact bundle $\mathcal{D}_{n,k}$ of $M_{n,k}$. First we give a Kähler form of the complex manifolds $S_{n,k}$ constructed in the paragraph above. Let $\tilde{\omega}_{l_1,l_2}, \tilde{\omega}_1, \tilde{\omega}_2$ denote the proper transform of $\omega_{l_1,l_2}, \omega_1, \omega_2$ respectively. Then the Kähler form on $S_{n,k}$ can be written as

$$\tilde{\omega}_{l_1,\cdots,l_{k+2}} = \sum_i l_i \tilde{\omega}_i \tag{4.5}$$

where $\tilde{\omega}_{i+2}$ is the $(1,1)$ form representing the Poincaré duals $\tilde{\alpha_{i+2}}$ of the exceptional divisors E_i. Then writing the Kähler class as $\sum_{i=1}^k l_i\tilde{\alpha}_i$, the positivity condition becomes

$$0<\sum_{i=1}^k l_i\tilde{\alpha}_i\cup\sum_{i=1}^k l_i\tilde{\alpha}_i=l_1(2l_2+nl_1)-\sum_{i=1}^k l_{i+1}^2. \tag{4.6}$$

So $\tilde{\omega}_{l_1,\cdots,l_{k+2}}$ defines a Kähler metric if this inequality is satisfied. Here we have used the fact the exceptional divisor is in the kernel of the corresponding blow-up map. Define the integer valued $k+2$-vector $\mathbf{l}=(l_1,\cdots,l_{k+2})$. It is convenient to choose $\mathbf{l}=(1,l_2,1,\cdots,1)$ in which case the positivity condition becomes $2l_2+n>k$. For simplicity we denote the corresponding Kähler form by $\tilde{\omega}_{l_2}$.

Next we need to compute the first Chern class of $\mathcal{D}_{n,k}$. This is $\pi_{n,k}^*c_1(S_{n,k})$ modulo the transgression of the Kähler class on $S_{n,k}$, that is, modulo the relation $\pi_{n,k}^*\sum_{i=1}^{k+2}l_i\tilde{\alpha}_i=0$. Let $\beta_1,\cdots,\beta_{k+1}$ be a basis for $H^2(M_{n,k},\mathbb{Z})$, and write $\pi_{n,k}^*\tilde{\alpha}_i=\sum_{j=1}^{k+1}m_{ij}\beta_j$. We need to choose the $k+2$ by $k+1$ matrix (m_{ij}) such that

$$\sum_{i=1}^{k+2}l_im_{ij}=0. \tag{4.7}$$

Now using Equation 4.3 we have

$$c_1(S_{n,k})=2\tilde{\alpha}_1-(n-2)\tilde{\alpha}_2-\sum_{i=1}^k\tilde{\alpha}_{k+2} \tag{4.8}$$

which gives

$$\pi_{n,k}^*c_1(S_{n,k})=2\sum_j m_{1j}\beta_j-(n-2)\sum_j m_{2j}\beta_j-\sum_{i=3}^{k+2}\sum_j m_{ij}\beta_j. \tag{4.9}$$

We now make a judicious choice of the matrix (m_{ij}).

$$(m_{ij})=\begin{pmatrix}-l_2 & 2 & 2 & \cdots & 2\\ 1 & 0 & 0 & \cdots & 0\\ 0 & -2 & \ddots & \vdots & \vdots\\ \vdots & 0 & \ddots & 0 & 0\\ \vdots & \vdots & \cdots & -2 & 0\\ 0 & 0 & \cdots & 0 & -2\end{pmatrix} \tag{4.10}$$

The orthogonality condition 4.7 is satisfied and Equation 4.8 becomes

$$\pi_{n,k}^*c_1(S_{n,k})=2l_2\beta_1-6(\beta_2+\cdots\beta_{k+1})-(n-2)\beta_1.$$

It follows that $w_2(M_{n,k})\equiv n \mod 2$.

We have arrived at

Theorem 4.10: *Let $S_{n,k}$ be the equivariant k-fold blow-up of the Hirzebruch surface S_n. Let $\pi_{n,k}:M_{n,k}\longrightarrow S_{n,k}$ be the circle bundle defined by the integral Kähler form $\tilde{\omega}_{l_2}$. Then for each positive integer l_2 satisfying $2l_2+n>k$, the manifold $M_{n,k}$ admits a toric regular Sasakian structure, $M_{n,k}$ is diffeomorphic to $k(S^2\times S^3)$ if n is even, and if n is odd it is diffeomorphic to $X_\infty\#(k-1)(S^2\times S^3)$. Thus, every regular contact 5-manifold admits a toric regular Sasakian structure.*

Remark 4.2: With more analysis one can make a count of inequivalent deformation classes of toric Sasakian structures on the manifolds $k(S^2\times S^3)$ and $X_\infty\#(k-1)(S^2\times S^3)$ as done in Theorems 4.4 and 4.5.

We close this section with a brief discussion of toric Sasakian-Einstein structures. It is well-known that S^5 and $S^2\times S^3$ have homogeneous, hence regular, Sasakian-Einstein structures. Both of these are clearly toric. First inhomogeneous examples of explicit toric Sasakian-Einstein structures on $S^2 \times S^3$ were obtained by Gauntlett et al. [GMSW04, MS06]. These metrics are of cohomogeneity 1 with $U(2) \times U(1)$ acting by isometries. In fact, Gauntlett et al. construct infinite families of toric Sasakian-Einstein structures parameterized by two relatively prime integers $p > q$. When $4p^2 - 3q^2 = n^2$ their examples are quasi-regular, i.e., the Reeb vector field has closed orbits. Otherwise the Sasakian-Einstein structure is not quasi-regular. These new examples were further generalized by Cvetič et al. [CLPP05] (see also [MS05]) who found toric Sasakian-Einstein metrics on $S^2 \times S^3$ of cohomogeneity 2.

This raises a natural question: Do all spin manifolds of Theorem 4.9 admit a toric Sasakian-Einstein structure? Since simply connected Sasakian-Einstein spaces are necessarily spin, $X_\infty \#(k-1)(S^2 \times S^3)$ must be excluded. It is known that S^5 and $k(S^2 \times S^3)$ admit families of quasi-regular Sasakian-Einstein structures for any k [BGN03a, BGK05, Kol04]. But most of these metrics have only a one-dimensional isometry group. However, it turns out that $S^2 \times S^3$ is by no means special in this respect: there exist families of toric Sasakian-Einstein structures on $k(S^2 \times S^3)$ for arbitrary k. This has just recently been proven in [CFO07, FOW06], and a bit earlier van Coevering [vC] proved this result for k odd.

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