EINSTEIN METRICS ON RATIONAL HOMOLOGY SPHERES

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1. Introduction

In this paper we prove the existence of Einstein metrics, actually Sasakian-Einstein metrics, on nontrivial rational homology spheres in all odd dimensions greater than 3. It appears as though little is known about the existence of Einstein metrics on rational homology spheres, and the known ones are typically homogeneous. The are two exception known to the authors. Both involve Sasakian geometry and both occur in dimension 7. In [BGN02] the two authors and M. Nakamaye gave a list of 184 rational homology 7-spheres with Sasakian-Einstein metrics. The result was based on a theorem of Johnson and Kollár proving the existance of Kähler-Einstein metric on cartain Fano 3-folds with orbifold singularities [JK01]. More recently, Grove, Wilking and Ziller constructed infinitely many rational homology 7-spheres with 3-Sasakian metrics of cohomogeneity one under an action of $S^3\times S^3$ [Zil]. Being 3-Sasakian these metrics are necessarily Sasakian-Einstein. Their construction involves orbifold self-dual Einstein metrics discovered by N. Hitchin [Hit96] a decade ago. Hitchin constructed a family of positive self-dual Einstein metrics, indexed by integers $k\geq 4$ which live on $S^4\setminus \mathbb{RP}^2$. They are complete in the orbifold sense, i.e., they can be viewed as metrics defined on compact Riemannian orbifolds M^4_k with a $\mathbb{Z}_k$ quotient singularities along $\mathbb{RP}^2$. Any positive self-dual Einstein orbifold $\mathcal{O}$ admits a compact 3-Sasakian orbifold V-bundle $\mathcal{S}{\longrightarrow}\mathcal{O}$ over it [BGM94]. Grove, Wilking and Ziller showed that in the case of Hitchin's metrics the bundle $N^7_k{\longrightarrow}M^4_k$ is actually a smooth 7-manifold and, moreover, computed $H^*(N^7_k,\mathbb{Z})$. When $k=2m-1$ the manifold N^7_k turns out to be a rational homology 7-sphere with $H_3(N^7_k,\mathbb{Z})=\mathbb{Z}_m$.

In dimension 5, aside from the standard 5-sphere S^5 there is only one known case of an Einstein metric on a simply connected rational homology 5-sphere. It is the homogeneous space $SU(3)/SO(3)$ which is a simply connected non-spin manifold with $H_2(SU(3)/SO(3),\mathbb{Z})=\mathbb{Z}_2$. On the other hand the 5-manifolds M^5 considered in this paper are all spin, i.e. $w_2(M)=0$. A well known theorem of Smale [Sma62] says that for any simply connected 5-manifold with spin, the torsion group in H_2 is of the form $G\oplus G$ for some Abelian group G. Until recently [BG02] this was the only simply connected nontrivial rational homology sphere known to admit Riemannian metrics of positive Ricci curvature. In this note we go much further by proving the existence of Sasakian-Einstein metrics on infinitely many simply connected rational homology 5-spheres. Furthermore, the metrics typically depend on parameters, that is, there is non-trivial moduli.

We prove the following two theorems.

Theorem 1. *There exists continuous parameter families of Sasakian-Einstein metrics on infinitely many simply connected rational homology spheres in every odd* *dimension greater than 3. For some of these families the number of effective parameters grows exponentially with dimension.*

Dimension 5 is treated separately since there is a classification of simply connected 5-manifolds due to Smale [Sma62] for spin manifolds and Barden [Bar65] more generally, and our results are somewhat sharper. It is the spin manifold case that is relevant here, as any simply connected Sasakian-Einstein manifold is necesarily spin. Our results cover all orders of the torsion group $H_2(M_k,\mathbb{Z})$, that is not 4 nor divisible by 6. We have

Theorem 2. *For every integer $k>2$ that is either relatively prime to 3 or 2, there exist Sasakian-Einstein metrics, depending on two real parameters, on a simply connected rational homology 5-sphere M^5_k with $w_2(M^5_k)=0$, and $H_2(M^5_k,\mathbb{Z})$ having order k^2.*

This result is probably not optimal. Furthermore, there are many more examples than those given in Section 4.

2. Branched Covers

Let $f=f(z_1,\cdots,z_m)$ be a quasi-smooth weighted homogeneous polynomial of degree $d_f=w(f)$ in m complex variables, and let L_f denote its link. Let $\mathbf{w}_f=(w_1,\cdots,w_m)$ be the corresponding weight vector, and for a given weight vector $\mathbf{w}$ we let $|\mathbf{w}|=\sum_{i=1}^m w_i$ be its norm. We consider branched covers constructed as the link L_F of the polynomial

$$F=z_0^k+f(z_1,\cdots,z_m).$$

Then L_F is a k-fold branched cover of S^{2m+1} branched over the link L_f. The degree of L_F is $d_F=\mathrm{lcm}(k,d_f)$, and the weight vector is $\mathbf{w}_F=(\frac{d_f}{\gcd(k,d_f)},\frac{k}{\gcd(k,d_f)}\mathbf{w}_f)$. We shall always assume that $k\geq 2$, since the linear case $k=1$ is a hyperplane in a weighted projective space. The following is Theorem 7.1 of [BGN03b]:

Theorem 3. *Let $f(z_1,\cdots,z_m)$ be a weighted homogeneous polynomial of degree d and weights $\mathbf{w}=(w_1,\cdots,w_m)$ in $\mathbb{C}^m$ with an isolated singularity at the origin. Let $k\in\mathbb{Z}^+$ and consider the link L_g of the equation*

$$g=z_0^k+f(z_1,\cdots,z_m)=0.$$

Write the numbers $\frac{d}{w_i}$ in irreducible form $\frac{u_i}{v_i}$, and suppose that $\gcd(k,u_i)=1$ for each $i=1,\cdots,m$. Then the link L_g has weights $\frac{(d,k\mathbf{w})}{\gcd(k,d)}$ and degree $\mathrm{lcm}(k,d)$. Furthermore, L_g is a rational homology sphere with the order $|H_{m-1}(L_g,\mathbb{Z})|=k^{b_{m-2}(L_f)}$ where $b_{m-2}(L_f)$ is the $(m-2)$-nd Betti number of L_f.

When considering rational homology spheres we can assume without loss of generality that $\gcd(k,d)=1$. For if $\gcd(k,d)>1$ and $\gcd(k,u_i)=1$ for each $i=1,\cdots,m$, then any common factor of k and d must divide w_i for each i, and so will be an overall common factor which gives an equivalent link. So hereafter we shall assume that $\gcd(k,d)=1$.

The formula for computing the $(m-2)$-nd Betti number can be obtained from Milnor and Orlik [MO70], viz.

$$b_{m-2}(L_f)=\sum(-1)^{m-s}\frac{u_{i_1}\cdots u_{i_s}}{v_{i_1}\cdots v_{i_s}\mathrm{lcm}(u_{i_1},\cdots,u_{i_s})}, \tag{1}$$

where the sum is taken over all the 2^m subsets $\{i_1,\cdots,i_s\}$ of $\{1,\cdots,m\}$. This general form for b_{m-2} is suitable for computer computations.

3. Sasakian-Einstein links and Kähler-Einstein orbifolds

Let $X=X_{k,d,\mathbf{w}}$ denote the quotient orbifold by the natural S^1 action. By now it is well understood that a Sasakian-Einstein metric on the link L_F is equivalent to a Kähler-Einstein metric of positive scalar curvature on $X=X_{k,d,\mathbf{w}}$. See [BG01] for details. It is easy to see that $X_{k,d,\mathbf{w}}$ is Fano if and only if

$$k(|\mathbf{w}_f|-d_f)+d_f>0. \tag{2}$$

In particular, if $|\mathbf{w}_f|-d_f\geq 0$, then $X_{k,d_f,\mathbf{w}}$ is Fano for all positive integers k. There are three cases to consider. The case when the smaller link L_f is itself Fano, i.e. $|\mathbf{w}_f|-d_f>0$,. This will be ruled out by Proposition 4 below. The case when L_f is Calabi-Yau, i.e. $|\mathbf{w}_f|-d_f=0$. This case is interesting since we have good solutions for infinitely many k. Finally, there is the "canonical" case when $|\mathbf{w}_f|-d_f<0$, for which the link L_F is Fano for only finitely many k. Nevertheless, there could be infinitely many good solutions in this case by taking arbitrarily large degrees d_f.

A sufficient condition for the existence of a Kähler-Einstein metric on the orbifold $X=X_{k,d_f,\mathbf{w}}$ is that for every effective $\mathbb{Q}$-divisor D that is numerically equivalent to K_X^{-1}, the pair $(X,\frac{m-1+\epsilon}{m}D)$ is Kawamata log terminal or klt for some $\epsilon>0$. For a precise definition see [KM98], and for a more complete discussion in the present context see [DK01, BGN03a, BGK03]. In particular, this implies the condition

$$k(|\mathbf{w}_f|-d_f)+d_f<\frac{m}{m-1}\min\{d_f,kw_i\mid i=1,\cdots m\}. \tag{3}$$

This condition is necessary to satisfy the klt condition defined above, but it is far from sufficient, and we use it only to eliminate some cases. It can also be reiterated that while the klt condition on the pair $(X,\frac{m-1+\epsilon}{m}D)$ is sufficient to guarentee the existence of a Kähler-Einstein metric, it is far from necessary.

Proposition 4. *If $\mathbf{w}_f-d>0$ then $X_{k,d,\mathbf{w}}$ does not satisfy the klt condition 3.*

Proof. If $\mathbf{w}_f-d>0$ then the left hand side of 3 is at least $k+d$. So for 3 to hold we must have

$$(m-1)k<d,\ \text{and}\qquad (m-1)d<k.$$

This gives a contradiction. □

Next we look at the case $\mathbf{w}_f-d=0$. Then the klt condition 3 becomes

$$(m-1)d<mk\min\{w_i\}. \tag{4}$$

This is clearly satisfied for k large enough, namely for $k>\frac{m-1}{m}d\max\frac{1}{w_i}$.

We are particularly interested in the case of perturbations of Brieskorn-Pham (BP) singularities. In [BGK03] János Kollár and the authors gave sufficient conditions for the existence of Kähler-Einstein metrics on Fano orbifolds arising as perturbations of BP singularities. However, these conditions, although better than 3 above, are still not optimal. We consider weighted homogeneous polynomials of the form

$$P(z_0,\cdots,z_m)=\sum_{i=0}^{m}z_i^{a_i}+tp(z_0,\dots,z_m),$$

where $t \in \mathbb{C}$, and $w(p) = \mathrm{lcm}(a_0, \cdots, a_m)$ which is the degree of the polynomial. We impose a conditon on the zero set $Y(\mathbf{a}, p) := P^{-1}(0)$, namely the genericity condition, which is always satisfied by $p \equiv 0$,

(GC) The intersections of $Y(\mathbf{a}, p)$ with any number of hyperplanes $(z_i = 0)$ are all smooth outside the origin.

Any polynomial or singularity satisfying condition GC is referred to as a (weighted homogeneous) *perturbation* of a Brieskorn-Pham polynomial or singularity. Furthermore, we define

$$C^j = \mathrm{lcm}(a_i : i \neq j), \quad b_j = \gcd(a_j, C^j).$$

The theorem proved in [BGK03] is;

Theorem 5. *The orbifold $Y(\mathbf{a}, p)/\mathbb{C}^*$ is Fano and has a Kähler-Einstein metric if it satisfies condition (GC) and*

$$1 < \sum_{i=0}^{m} \frac{1}{a_i} < 1 + \frac{m}{m-1} \min_{i,j} \Big\{ \frac{1}{a_i}, \frac{1}{b_i b_j} \Big\}.$$

4. Rational Homology 5-Spheres

We now specialize to the case of dimension 5, i.e. $m = 3$. Our 5-manifolds are constructed as k-fold branched covers of S^5 branched over certain Seifert manifolds that are in turn S^1 orbifold V-bundles over a compact Riemann surface of genus g. Our construction is similar to that in [Sav79]. Let $f_3(z_1, z_2, z_3)$ be a weighted homogeneous polynomial of an isolated hypersurface singularity in $\mathbb{C}^3$ with weights $\mathbf{w} = (w_1, w_2, w_3)$ and degree d. The link $L_{\mathbf{w}}$ defined by $L_{\mathbf{w}} = \{f_3 = 0\} \cap S^5$ is a Seifert fibration over an algebraic curve $C_{\mathbf{w}}$ in the weighted projective space $\mathbb{P}(\mathbf{w})$. Let $g = g(\mathbf{w})$ denote the genus of the curve $C_{\mathbf{w}}$. In the case of rational homology 5-spheres, Theorem 3 specializes to $|H_2(L_f, \mathbb{Z})| = k^{2g}$, while the Betti number formula 1 specializes to the genus formula of Orlik and Weigreich [OW71]

$$g(C_{\mathbf{w}}) = \frac{1}{2} \Big(\frac{d^2}{w_1 w_2 w_3} - d \sum_{i<j} \frac{\gcd(w_i, w_j)}{w_i w_j} + \sum_i \frac{\gcd(d, w_i)}{w_i} - 1 \Big).$$

The following lemma whose proof is clear from the genus formula will prove to be useful.

Lemma 6. *If $|\mathbf{w}| = d, \gcd(w_i, d) = w_i$, and $\gcd(w_i, w_j) = 1$ for all $i \neq j$, then* $g = 1$.

In view of Proposition 4 we restrict ourselves to the case $\mathbf{w}_f - d \leq 0$. In the case of Brieskorn 3-manifolds L_f Milnor [Mil75] refers to the cases $\mathbf{w}_f - d > 0, \mathbf{w}_f - d = 0$, and $\mathbf{w}_f - d < 0$ as spherical, Euclidean, and hyperbolic, respectively. In [OW71] (with an erratum in [OW77]) Orlik and Wagreich give a classification of all weighted homogeneous polynomials with only an isolated singularity at the origin up to $\mathbb{C}^*$-equivariant diffeomorphism. They gave a list of six classes, but there was an error and two more were reported in [OW77] which completes their classification. Here we analyze the possible k-fold branched covers when the Siefert manifold is Euclidean, i.e. $\mathbf{w}_f = d$, which is straightforward. It turns out that the singularities arising here are precisely the unimodal parabolic singularities of [AGnZV85]. In any case it is easy to prove:

Proposition 7. *Let L_f be the link of a weighted homogeneous polynomial $f = f(z_1, z_2, z_3)$ of degree d in three complex variables. Suppose further that $|\mathbf{w}| = d$. Then f is a weighted homogeneous polynomial with weight vector $\mathbf{w}$, degree d, and number of monomials n of degree d given by one of the three cases:*

$\mathbf{w}$	d	n
$(1,2,3)$	6	7
$(1,1,2)$	4	9
$(1,1,1)$	3	10

Let L_F be a k-fold branched cover of a 3-dimensional Euclidean link, then by Proposition 7 L_F is the link of an isolated hypersurface singularity of the form

$$z_0^k + f_d(z_1, z_2, z_3) = 0$$

with weights $(d, k\mathbf{w}_f)$ where d and $\mathbf{w}_f$ are given in the table above, and f_d is an arbitrary polynomial polynomial of the given weights subject to the condition that the singularitiy is isolated. There is a dense open subset of the parameter space where this is a perturbation of a Brieskorn-Pham singularity. It is easy to see that the bounds given in Theorem 5 are satisfied if we choose $k \geq 3$ in the $d = 4$ and $d = 3$ cases, and $k \geq 5$ in the $d = 6$ cases. Hence, in these cases we get Sasakian-Einstein metrics on the links L_F. Furthermore, in the $d = 3$ and $d = 6$ cases there is a two real parameter family of solutions. This proves Theorem 2. □

Proposition 7 gives a classification of all the possible Euclidean cases. We can also consider hyperbolic cases. Since the weights are unrestricted there are many, so we only mention one which is a special case of Example 9 below. We consider links associated to perturbations of the BP singularity

$$z_0^k + z_1^l + z_2^l + z_3^l = 0.$$

The Betti number formula 7 below simplifies to $b_1 = (l-2)(l-1)$. Thus, by Theorem 3 we get rational homology 5-spheres $M^5_{k,l}$ with $H_2(M^5_{k,l}, \mathbb{Z})$ of order $k^{(l-2)(l-1)}$. From equation 6 below we see that l must be either 4 or 5. The Fano condition and klt inequalities of Theorem 5 are satisfied only when $k = 3$ and $l = 4$. By Equation 9 the number of effective real parameters is 12. Thus, there is a rational homology 5-sphere $M^5_{3,4}$ with $|H_2(M^5_{3,4}, \mathbb{Z})| = 3^6$ that admits a Sasakian-Einstein metric depending on 12 real parameters.

5. Examples in Dimension $2m - 1$

Example 8. [Branched Covers of Calabi-Yau hypersurfaces] First we consider k-fold branched covers of Fermat-Calabi-Yau hypersurfaces

$$z_0^k + z_1^m + \cdots z_m^m = 0 \tag{5}$$

with $\gcd(k, m) = 1$. For $m \geq 3$ the link M_k^{2m-1} is a simply connected nontrivial rational homology sphere. The $(m-2)$-nd Betti number of the Calabi-Yau link is

$$b_{m-2} = (-1)^m \Big(1 + \frac{(1-m)^m - 1}{m}\Big).$$

So by Theorem 3 $|H_{m-1}(M_k^{2m-1}, \mathbb{Z})| = k^{b_{m-2}}$. For example, in dimension 7 ($m = 4$) we get $|H_2(M_k^7, \mathbb{Z})| = k^{21}$ whereas, in dimension 9 we have $|H_2(M_k^9, \mathbb{Z})| = k^{204}$.

It is easy to see that the klt condition in Theorem 3 is satisfied if $k > m(m-1)$ in which case the rational homology spheres M_k^{2m-1} admits a family of Sasakian-Einstein metrics. The number of effective complex parameters μ is determined by (cf. [BGK03])

$$\mu = h^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(d)) - \sum_i h^0(\mathbb{P}(\mathbf{w}), \mathcal{O}(w_i))$$

where $\mathbf{w} = (m, k, \cdots, k)$ and $d = mk$. Thus, we find

$$\mu = \binom{2m-1}{m} - m^2.$$

By Sterling's formula one sees that μ grows exponentially with m. In dimension 7 we get 38 real parameters, while in dimension 9 we have 202 effective real parameters. This example is enough to prove Theorem 1. □

In the orbifold category there are many Calabi-Yau hypersurfaces in weighted projective spaces $\mathbb{P}(\mathbf{w})$. For example, for general m, one can consider $\mathbf{w} = (1, \cdots, 1, m-1)$ of degree $d = 2(m-1)$, or $\mathbf{w} = (1, \cdots, 1, m-2, m-2)$ of degree $d = 3(m-2)$, etc. For rational homology spheres of dimension 7, we obtain examples from branched covers of Reid's list (cf. [IF00]) of 95 log K3 surfaces, and in dimension 9 from branched covers of the over 6000 Calabi-Yau orbifolds in complex dimension 3 [CLS90].

Example 9 (Branched Covers in the Canonical Case)**.** We consider links L_F of branched covers of a canonical Fermat hypersurface of the form

$$F = z_0^k + z_1^l + \cdots + z_m^l = 0$$

with $l \geq m+1$ and $\gcd(k, l) = 1$. Here the quotient X will be Fano if and only if

$$k < \frac{l}{l-m}.$$

Also for any branched cover we are only interested in $k \geq 2$. Combining this with the Fano condition gives an upper bound on l for fixed m, namely $l < 2m$. Thus, we get the range for l

$$m + 1 \leq l \leq 2m - 1. \tag{6}$$

The link $L_F = M_k^{2m-1}$ is a rational homology sphere with $|H_{m-1}(M_k^{2m-1}, \mathbb{Z})| = k^{b_{m-2}}$ where

$$b_{m-2} = (-1)^m \Big(1 + \frac{(1-l)^m - 1}{l}\Big). \tag{7}$$

We now look at the klt condition, that is, the right hand inequality of Theorem 5. We see that L_F will have Sasakian-Einstein metrics if we choose k to satisfy

$$\frac{(m-1)l^2}{(m-1)l(l-m) + m} < k < \frac{l}{l-m}. \tag{8}$$

As above we can compute the number of effective complex parameters μ to be

$$\mu = \binom{m+l-1}{l} - m^2. \tag{9}$$

As a special case we consider $l = m + 1$. In this case the only solution to the inequality 8 is $k = m$. This gives Sasakian-Einstein metrics on rational homology spheres depending on $2(\binom{2m}{m+1} - m^2)$ effective real parameters.

Other solutions can be worked out, for example, the singularity

$$z_0^k + z_1^{2m} + \cdots + z_{m-1}^{2m} + z_m^2 = 0.$$

This satisfies the inequalities of Theorem 5 if we choose $k = 2m - 1$.

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