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SASAKIAN GEOMETRY, HOLONOMY, AND SUPERSYMMETRY

CHARLES P. BOYER AND KRZYSZTOF GALICKI

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1. Introduction

Supersymmetry has emerged in physics as an attempt to unify the way physical theories deal with bosonic and fermionic particles. Since its birth around the early 70ties it has come to dominate theoretical high energy physics (for a historical perspective see [KS00] with the introduction by Kane and Shifman, and for a mathematical treatment see [Var04]). This dominance is still ongoing in spite of the fact that after almost 40 years there is no single experimental evidence that would directly and convincingly "prove" or "discover" the existence of supersymmetry in nature. On the other hand, especially in the last 20 years, supersymmetry has given birth to many beautiful mathematical theories. Gromov-Witten Theory, Seiberg-Witten Theory, Rozansky-Witten Theory as well as the Mirror Duality Conjecture are just a few of the more famous examples of important and deep mathematics having its origins in the physics of various supersymmetric theories.

Various supersymmetric field theories naturally include both Riemannian and pseudo-Riemannian manifolds. The latter is necessary in order to incorporate the

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physical space-time into the picture while the former typically describes the geometry associated with 'invisible' extra dimensions. It is mainly in such a context that Sasaki-Einstein manifolds appear in physics: they are compact Einstein manifolds of positive scalar curvature that occur in abundance in the physically interesting dimensions five and seven. Moreover, when they are simply connected they admit real Killing spinors. It is this last property that vitally connects them to Supergravity, Superstring, and M-Theory.

The main purpose of this review is to describe geometric properties of Sasaki-Einstein manifolds which make them interesting in modern theoretical physics. In spite of the fact that it is *supersymmetry* that connects Sasaki-Einstein spaces to physics, it is not the purpose of this article to describe what this concept really means to either physicists or mathematicians. There have been many recent attempts to frame these important notions of theoretical physics in precise mathematical terms. This enormous task is far beyond the scope of this article, so we refer the reader to recent monographs and references therein [DEF$^+$99, Var04, Jos01, AJPS97]. Here we content ourselves with providing the main theorems and results concerning Killing spinors.

It is most remarkable that, even though Sasaki-Einstein manifolds always have holonomy $SO(TM)$, *i.e.*, the holonomy of any generic Riemannian metric, they are far from being generic. In fact, the most interesting thing about this geometry is that it naturally relates to several different Riemannian geometries with reduced holonomies. It is this point that we will try to stress throughout this article. For more detailed exposition we refer the interested reader to our recent monograph on *Sasakian Geometry* [BG07].

The key to understanding the importance of Sasakian geometry is through its relation to Kählerian geometry. Before we define Sasakian manifolds and describe some of their elementary properties in Section 3 let us motivate things in the more familiar context of contact and symplectic manifolds. These two provide the mathematical foundations of Lagrangian and Hamiltonian Mechanics. Let (M,η,ξ) be a contact manifold where η is a contact form on M and ξ is its Reeb vector field. It is easy to see that the cone $(C(M)=\mathbb{R}_+\times M, \omega=d(t\eta))$ is symplectic. Likewise, the Reeb field defines a foliation of M and the transverse space $\mathcal{Z}$ is also symplectic. When the foliation is regular the transverse space is a smooth symplectic manifold giving a projection π called Boothby-Wang fibration, and $\pi^*\Omega = d\eta$ relates the contact and the symplectic structures as indicated by

$$\boxed{\begin{array}{ccc} (\mathcal{C}(M),\omega) & \hookleftarrow & (M,\eta,\xi) \\ & & \Big\downarrow{\scriptstyle\pi} \\ & & (\mathcal{Z},\Omega). \end{array}}$$

We do not have any Riemannian structure yet. It is quite reasonable to ask if there is a Riemannian metric g on M which "best fits" into the above diagram. As the preferred metrics adapted to symplectic forms are Kähler metrics one could ask for the Riemannian structure which would make the cone with the metric $\bar g = dt^2+t^2g$ together with the symplectic form ω into a Kähler manifold. Then $\bar g$

and ω define a complex structure $\bar{\Phi}$. Alternatively, one could ask for a Riemannian metric g on M which would define a Kähler metric h on $\mathcal{Z}$ via a Riemannian submersion. Surprisingly, in both cases the answer to these questions leads naturally and uniquely to **Sasakian Geometry**. Our diagram becomes

$$\boxed{\begin{array}{ccc} (\mathcal{C}(M),\omega,\bar{g},\bar{\Phi}) & \hookleftarrow & (M,\xi,\eta,g,\Phi) \\ & & \Big\downarrow{\scriptstyle\pi} \\ & & (\mathcal{Z},\Omega,h,J). \end{array}}$$

From this point of view it is quite clear that Kählerian and Sasakian geometries are inseparable, Sasakian Geometry being naturally *sandwiched* between two different types of Kählerian Geometry.

2. Cones, Holonomy, and Sasakian Geometry

As we have just described Sasakian manifolds can and will be (cf. Theorem-Definition 10) defined as bases of metric cones which are Kähler. Let us begin with the following more general

Definition 1. *For any Riemannian metric g_M on M,* **the warped product metric** *on $C(M)=\mathbb{R}^+\times M$ is the Riemannian metric defined by*

$$g=dr^2+\phi^2(r)g_M\ ,$$

where $r\in\mathbb{R}^+$ and $\phi=\phi(r)$ is a smooth function, called the **warping function**. *If $\phi(r)=r$ then $(C(M),g)$ is simply called the* **Riemannian cone** *or* **metric cone** *on M. If $\phi(r)=\sin r$ then $(C(M),g)$ is called the* **sine-cone** *on M.*

The relevance of sine-cones will become clear later while the importance of metric cones in relation to the Einstein metrics can be summarized by the following fundamental

Lemma 2. *Let (M,g) be a Riemannian manifold of dimension n, and consider $(C(M)=M\times\mathbb{R}^+,\bar{g})$ the cone on M with metric $\bar{g}=dr^2+r^2g$. Then if $\bar{g}$ is Einstein, it is Ricci-flat, and $\bar{g}$ is Ricci-flat if and only if g is Einstein with Einstein constant $n-1$.*

Interestingly, there is a similar lemma about sine-cone metrics.

Lemma 3. *Let (M^n,g) be an Einstein manifold with Einstein constant $n-1$ and consider $(C_s(M)=M\times(0,\pi),\bar{g}_s)$ the sine-cone on M with metric $\bar{g}_s=dr^2+(\sin^2 r)g$. Then $\bar{g}_s$ is Einstein with Einstein constant n.*

It is well-known that one characterization of Kählerian geometry is via the holonomy reduction. We now recall some basic facts about the holonomy groups of irreducible Riemannian manifolds. Let (M,g) be a Riemannian manifold and consider parallel translation defined by the Levi-Civita connection and its associated holonomy group which is a subgroup of the structure group $O(n,\mathbb{R})$ ($SO(n,\mathbb{R})$ in the oriented case). Since this connection ∇^g is uniquely associated to the metric g, we denote it by $\mathrm{Hol}(g)$, and refer to it as the *Riemannian holonomy* group or just the *holonomy group* when the context is clear. Indeed, it is precisely this Riemannian holonomy that plays an important role here. Now on a Riemannian manifold (M, g) there is a canonical epimorphism $\pi_1(M) \longrightarrow \mathrm{Hol}(g)/\mathrm{Hol}^0(g)$, in particular, if $\pi_1(M) = 0$ then $\mathrm{Hol}(g) = \mathrm{Hol}^0(g)$. In 1955 Berger proved the following theorem [Ber55] concerning Riemannian holonomy:

Theorem 4. *Let (M, g) be an oriented Riemannian manifold which is neither locally a Riemannian product nor locally symmetric. Then the restricted holonomy group $\mathrm{Hol}^0(g)$ is one of the following groups listed in Table 1.1.*

Table 1.1: Berger's Riemannian Holonomy Groups

$\mathrm{Hol}^0(g)$	$\dim(M)$	Geometry of M	Comments
$SO(n)$	n	orientable Riemannian	generic Riemannian
$U(n)$	$2n$	Kähler	generic Kähler
$SU(n)$	$2n$	Calabi-Yau	Ricci-flat Kähler
$Sp(n) \cdot Sp(1)$	$4n$	quaternionic Kähler	Einstein
$Sp(n)$	$4n$	hyperkähler	Ricci-flat
G_2	7	G_2-manifold	Ricci-flat
$Spin(7)$	8	$Spin(7)$-manifold	Ricci-flat

Originally Berger's list included $Spin(9)$, but Alekseevsky proved that any manifold with such holonomy must be symmetric [Ale68]. In the same paper Berger also claimed a classification of all holonomy groups of torsion-free affine (linear) connections that act irreducibly. He produced a list of possible holonomy representations up to what he claimed was a finite number of exceptions. But his classification had some gaps discovered 35 years later by Bryant [Bry91]. An infinite series of exotic holonomies was found in [CMS96] and finally the classification in the non-Riemannian affine case was completed by Merkulov and Schwachhöfer [MS99]. We refer the reader to [MS99] for the proof, references and the history of the general affine case. In the Riemannian case a new geometric proof of Berger's Theorem is now available [Olm05]. An excellent review of the subject just prior to the Merkulov and Schwachhöfer's classification can be found in [Bry96]. We should add that one of the first non-trivial results concerning manifolds with the exceptional holonomy groups of the last two rows of Table 1.1 is due to Bonan [Bon66] who established Ricci-flatness of manifolds with parallel spinors.

Manifolds with reduced holonomy have always been very important in physics. Partly because Calabi-Yau, hyperkähler, quaternionic Kähler, G_2 and $Spin(7)$ manifolds are automatically Einstein. In addition, all of these spaces appear as σ-model geometries in various supersymmetric models. What is perhaps less known is that all of these geometries are also related, often in more than one way, to Sasakian structures of various flavors. Let us list all such known relations.

- $SO(n)$**-holonomy.** As remarked this is holonomy group of a generic metric on an oriented Riemannian manifold (M^n, g). As we shall see Sasaki-Einstein metrics necessarily have maximal holonomy.
- $U(n)$**-holonomy and Kähler geometry.**
  (i) Metric cone on a *Sasakian* manifold is *Kähler.*
  (ii) Transverse geometry of a *Sasakian* manifold is *Kähler.*

(iii) Transverse geometry of a *positive Sasakian* manifold is *Fano*.
(iv) Transverse geometry of a *Sasaki-Einstein* manifold is *Fano* and *Kähler-Einstein* of *positive* scalar curvature.
(v) Transverse geometry of a *negative Sasakian* manifold is *canonical* in the sense that the transverse canonical bundle is ample.
(vi) Transverse geometry of a 3-*Sasakian* manifold is a *Kähler-Einstein* with a *complex contact structure*, *i.e., twistor geometry.*

- $SU(n)$**-holonomy and Calabi-Yau geometry.**
  (i) Metric cone on a *Sasaki-Einstein* manifold is *Calabi-Yau.*
  (ii) Transverse geometry of a *null Sasakian* manifold is *Calabi-Yau.*
- $Sp(n)Sp(1)$**-holonomy and Quaternionic Kähler geometry.**
  (i) Transverse geometry of the 3-dimensional foliation of a 3-*Sasakian* manifold is *quaternionic-Kähler* of *positive* scalar curvature.
  (ii) 3-*Sasakian* manifolds occur as conformal infinities of complete *quaternionic Kähler* manifolds of *negative* scalar curvature.
- $Sp(n)$**-holonomy and hyperähler geometry.**
  (i) Metric cone on a 3-*Sasakian* manifold is *hyperkähler.*
  (ii) Transverse geometry of a *null Sasakian* manifold with some additional structure is *hyperkähler.*
- G_2**-holonomy.**
  (i) The *'squashed' twistor space of a* 3-*Sasakian* 7-manifold is *nearly Kähler*; hence, the metric cone on it has holonomy inside G_2.
  (ii) *Sine-cone* on a Sasaki-Einstein 5-manifold is *nearly Kähler*; hence, its *metric cone* has holonomy inside G_2.
- $Spin(7)$**-holonomy.**
  (i) The *'squashed'* 3-*Sasakian* 7-manifold has a nearly parallel G_2-structure; hence, its metric cone has holonomy in $Spin(7)$.
  (ii) *Sine-cone* on a *'squashed' twistor space of a* 3-*Sasakian* 7-manifold has a *nearly parallel* G_2 *structure*; hence, its *metric cone* has holonomy inside $Spin(7)$.
  (iii) *Sine cone on a sine cone* on a 5-dimensional Sasaki-Einstein base has a *nearly parallel* G_2*-structure*; hence, its *metric cone* has holonomy inside $Spin(7)$.

Note that Sasakian manifolds are related to various other geometries in two very distinct ways. On one hand we can take a Sasakian (Sasaki-Einstein, 3-Sasakian, etc.) manifold and consider its metric or sine-cone. These cones frequently have interesting geometric properties and reduced holonomy. On the other hand, a Sasakian manifold is always naturally foliated by one-dimensional leaves (three-dimensional leaves in addition to the one-dimensional canonical foliation when the manifold is 3-Sasakian) and we can equally well consider the transverse geometries associated to such fundamental foliations. These too have remarkable geometric properties including reduced holonomy. In particular, Sasakian manifolds are not just related to all of the geometries on Berger's holonomy list, but more importantly, they provide a *bridge* between the different geometries listed there. We will investigate some of these bridges in the next two sections.

3. Sasakian and Kählerian geometry

Definition 5. *A $(2n+1)$-dimensional manifold M is a* **contact manifold** *if there exists a 1-form η, called a* **contact 1-form***, on M such that*

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A **contact structure** *on M is an equivalence class of such 1-forms, where $\eta' \sim \eta$ if there is a nowhere vanishing function f on M such that $\eta' = f\eta$.*

Lemma 6. *On a contact manifold (M, η) there is a unique vector field ξ, called the* **Reeb vector field***, satisfying the two conditions*

$$\xi \lrcorner \eta = 1, \qquad \xi \lrcorner d\eta = 0.$$

Definition 7. *An* **almost contact structure** *on a differentiable manifolds M is a triple (ξ, η, Φ), where Φ is a tensor field of type $(1,1)$ (i.e., an endomorphism of TM), ξ is a vector field, and η is a 1-form which satisfy*

$$\eta(\xi) = 1 \text{ and } \Phi \circ \Phi = -\mathbb{1} + \xi \otimes \eta,$$

where $\mathbb{1}$ is the identity endomorphism on TM. A smooth manifold with such a structure is called an **almost contact manifold***.*

Remark 8. The reader will notice from Definitions 5 and 7 that an almost contact structure actually has more structure than a contact structure! This is in stark contrast to the usual relationship between a structure and its 'almost structure'; however, we feel that the terminology is too well ensconced in the literature to be changed at this late stage.

Let (M, η) be a contact manifold with a contact 1-form η and consider $\mathcal{D} = \ker \eta \subset TM$. The subbundle $\mathcal{D}$ is maximally non-integrable and it is called the *contact distribution.* The pair $(\mathcal{D}, \omega)$, where ω is the restriction of $d\eta$ to $\mathcal{D}$ gives $\mathcal{D}$ the structure of a symplectic vector bundle. We denote by $\mathcal{J}(\mathcal{D})$ the space of all almost complex structures J on $\mathcal{D}$ that are compatible with ω, that is the subspace of smooth sections J of the endomorphism bundle $\mathrm{End}(\mathcal{D})$ that satisfy

$$J^2 = -\mathbb{1}, \quad d\eta(JX, JY) = d\eta(X, Y), \quad d\eta(JX, X) > 0 \tag{1}$$

for any smooth sections X, Y of $\mathcal{D}$. Notice that each $J \in \mathcal{J}(\mathcal{D})$ defines a Riemannian metric $g_{\mathcal{D}}$ on $\mathcal{D}$ by setting

$$g_{\mathcal{D}}(X, Y) = d\eta(JX, Y). \tag{2}$$

One easily checks that $g_{\mathcal{D}}$ satisfies the compatibility condition $g_{\mathcal{D}}(JX, JY) = g_{\mathcal{D}}(X, Y)$. Furthermore, the map $J \mapsto g_{\mathcal{D}}$ is one-to-one, and the space $\mathcal{J}(\mathcal{D})$ is contractible. A choice of J gives M an almost CR structure. Moreover, by extending J to all of TM one obtains an almost contact structure. There are some choices of conventions to make here. We define the section Φ of $\mathrm{End}(TM)$ by $\Phi = J$ on $\mathcal{D}$ and $\Phi\xi = 0$, where ξ is the Reeb vector field associated to η. We can also extend the transverse metric $g_{\mathcal{D}}$ to a metric g on all of M by

$$g(X, Y) = g_{\mathcal{D}} + \eta(X)\eta(Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y) \tag{3}$$

for all vector fields X, Y on M. One easily sees that g satisfies the compatibility condition $g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y)$.

Definition 9. *A contact manifold M with a contact form η, a vector field ξ, a section Φ of* $\mathrm{End}(TM)$*, and a Riemannian metric g which satisfy the conditions*

$$\eta(\xi) = 1, \qquad \Phi^2 = -\mathbb{1} + \xi \otimes \eta,$$
$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

is known as a **metric contact structure** *on M.*

Definition–Theorem 10. *A Riemannian manifold (M, g) is called a* **Sasakian manifold** *if any one, hence all, of the following equivalent conditions hold:*

(i) *There exists a Killing vector field ξ of unit length on M so that the tensor field Φ of type $(1,1)$, defined by $\Phi(X) = -\nabla_X \xi$, satisfies the condition*
$$(\nabla_X \Phi)(Y) = g(X,Y)\xi - g(\xi, Y)X$$
for any pair of vector fields X and Y on M.

(ii) *There exists a Killing vector field ξ of unit length on M so that the Riemann curvature satisfies the condition*
$$R(X,\xi)Y = g(\xi,Y)X - g(X,Y)\xi,$$
for any pair of vector fields X and Y on M.

(iii) *The metric cone $(\mathcal{C}(M), \bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ is Kähler.*

We refer to the quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ as a *Sasakian structure* on M, where η is the 1-form dual vector field ξ. It is easy to see that η is a contact form whose Reeb vector field is ξ. In particular $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a special type of *metric contact structure*.

The vector field ξ is nowhere vanishing, so there is a 1-dimensional foliation $\mathcal{F}_\xi$ associated with every Sasakian structure, called the *characteristic foliation*. We will denote the space of leaves of this foliation by $\mathcal{Z}$. Each leaf of $\mathcal{F}_\xi$ has a holonomy group associated to it. The dimension of the closure of the leaves is called the *rank* of $\mathcal{S}$. We shall be interested in the case $\mathrm{rk}(\mathcal{S}) = 1$. We have

Definition 11. *The characteristic foliation $\mathcal{F}_\xi$ is said to be* **quasi-regular** *if there is a positive integer k such that each point has a foliated coordinate chart (U, x) such that each leaf of $\mathcal{F}_\xi$ passes through U at most k times. Otherwise $\mathcal{F}_\xi$ is called* **irregular***. If $k = 1$ then the foliation is called* **regular***, and we use the terminology* **non-regular** *to mean quasi-regular, but not regular.*

Let $(M, \mathcal{S})$ be a Sasakian manifold, and consider the contact subbundle $\mathcal{D} = \ker \eta$. There is an orthogonal splitting of the tangent bundle as

$$TM = \mathcal{D} \oplus L_\xi, \tag{4}$$

where L_ξ is the trivial line bundle generated by the Reeb vector field ξ. The contact subbundle $\mathcal{D}$ is just the normal bundle to the characteristic foliation $\mathcal{F}_\xi$ generated by ξ. It is naturally endowed with both a complex structure $J = \Phi|_{\mathcal{D}}$ and a symplectic structure $d\eta$. Hence, $(\mathcal{D}, J, d\eta)$ gives M a *transverse Kähler* structure with Kähler form $d\eta$ and metric $g_{\mathcal{D}}$ defined as in (2) which is related to the Sasakian metric g by $g = g_{\mathcal{D}} \oplus \eta \otimes \eta$ as in (3). We have [BG00] the following fundamental structure theorem:

Theorem 12. *Let (M, ξ, η, Φ, g) be a compact quasi-regular Sasakian manifold of dimension $2n+1$, and let $\mathcal{Z}$ denote the space of leaves of the characteristic foliation. Then the leaf space $\mathcal{Z}$ is a Hodge orbifold with Kähler metric h and Kähler form ω*

which defines an integral class $[\omega]$ in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$ so that $\pi:(M,g)\longrightarrow(\mathcal{Z},h)$ is an orbifold Riemannian submersion. The fibers of π are totally geodesic submanifolds of M diffeomorphic to S^1.

and its converse:

Theorem 13. *Let $(\mathcal{Z},h)$ be a Hodge orbifold. Let $\pi:M\longrightarrow\mathcal{Z}$ be the S^1 V-bundle whose first Chern class is $[\omega]$, and let η be a connection 1-form in M whose curvature is $2\pi^*\omega$, then M with the metric $\pi^*h+\eta\otimes\eta$ is a Sasakian orbifold. Furthermore, if all the local uniformizing groups inject into the group of the bundle S^1, the total space M is a smooth Sasakian manifold.*

Irregular structures can be understood by the following result of Rukimbira [Ruk95]:

Theorem 14. *Let (ξ,η,Φ,g) be a compact irregular Sasakian structure on a manifold M. Then the group $\mathfrak{Aut}(\xi,\eta,\Phi,g)$ of Sasakian automorphisms contains a torus T^k of dimension $k\geq 2$. Furthermore, there exists a sequence $(\xi_i,\eta_i,\Phi_i,g_i)$ of quasi-regular Sasakian structures that converge to (ξ,η,Φ,g) in the C^∞ compact-open topology.*

The orbifold cohomology groups $H^p_{orb}(\mathcal{Z},\mathbb{Z})$ were defined by Haefliger [Hae84]. In analogy with the smooth case a *Hodge orbifold* is then defined to be a compact Kähler orbifold whose Kähler class lies in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$. Alternatively, we can develop the concept of basic cohomology which works equally well in the irregular case, but only has coefficients in $\mathbb{R}$. It is nevertheless quite useful in trying to extend the notion of $\mathcal{Z}$ being Fano to both the quasi-regular and the irregular situation. This can be done in several ways. Here we will use the notion of basic Chern classes. Recall [Ton97] that a smooth p-form α on M is called *basic* if

$$\xi \lrcorner \alpha = 0, \qquad \mathcal{L}_\xi\alpha = 0\,, \tag{5}$$

and we let Λ^p_B denote the sheaf of germs of basic p-forms on M, and by Ω^p_B the set of global sections of Λ^p_B on M. The sheaf Λ^p_B is a module over the ring, Λ^0_B, of germs of smooth basic functions on M. We let $C^\infty_B(M)=\Omega^0_B$ denote global sections of Λ^0_B, i.e. the ring of smooth basic functions on M. Since exterior differentiation preserves basic forms we get a de Rham complex

$$\cdots\longrightarrow\Omega^p_B\xrightarrow{\ d\ }\Omega^{p+1}_B\longrightarrow\cdots \tag{6}$$

whose cohomology $H^*_B(\mathcal{F}_\xi)$ is called the *basic cohomology* of $(M,\mathcal{F}_\xi)$. The basic cohomology ring $H^*_B(\mathcal{F}_\xi)$ is an invariant of the foliation $\mathcal{F}_\xi$ and hence, of the Sasakian structure on M. It is related to the ordinary de Rham cohomology $H^*(M,\mathbb{R})$ by the long exact sequence [Ton97]

(7)
$$\cdots\longrightarrow H^p_B(\mathcal{F}_\xi)\longrightarrow H^p(M,\mathbb{R})\xrightarrow{\ j_p\ }H^{p-1}_B(\mathcal{F}_\xi)\xrightarrow{\ \delta\ }H^{p+1}_B(\mathcal{F}_\xi)\longrightarrow\cdots$$

where δ is the connecting homomorphism given by $\delta[\alpha]=[d\eta\wedge\alpha]=[d\eta]\cup[\alpha]$, and j_p is the composition of the map induced by $\xi\lrcorner$ with the well-known isomorphism $H^r(M,\mathbb{R})\approx H^r(M,\mathbb{R})^{S^1}$ where $H^r(M,\mathbb{R})^{S^1}$ is the S^1-invariant cohomology defined from the S^1-invariant r-forms $\Omega^r(M)^{S^1}$. We also note that $d\eta$ is basic even though η is not. Next we exploit the fact that the transverse geometry is Kähler. Let $\mathcal{D}_\mathbb{C}$ denote the complexification of $\mathcal{D}$, and decompose it into its eigenspaces with respect to J, that is, $\mathcal{D}_{\mathbb{C}} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. Similarly, we get a splitting of the complexification of the sheaf Λ^1_B of basic one forms on M, namely

$$\Lambda^1_B \otimes \mathbb{C} = \Lambda^{1,0}_B \oplus \Lambda^{0,1}_B\,.$$

We let $\mathcal{E}^{p,q}_B$ denote the sheaf of germs of basic forms of type (p,q), and we obtain a splitting

$$\Lambda^r_B \otimes \mathbb{C} = \bigoplus_{p+q=r} \mathcal{E}^{p,q}_B\,. \tag{8}$$

The basic cohomology groups $H^{p,q}_B(\mathcal{F}_\xi)$ are fundamental invariants of a Sasakian structure which enjoy many of the same properties as the ordinary Dolbeault cohomology of a Kähler structure.

Consider the complex vector bundle $\mathcal{D}$ on a Sasakian manifold (M,ξ,η,Φ,g). As such $\mathcal{D}$ has Chern classes $c_1(\mathcal{D}),\ldots,c_n(\mathcal{D})$ which can be computed by choosing a connection $\nabla^{\mathcal{D}}$ in $\mathcal{D}$ [Kob87]. Let us choose a local foliate unitary transverse frame $(X_1,\ldots,X_n)$, and denote by Ω^T the transverse curvature 2-form with respect to this frame. A simple calculation shows that Ω^T is a basic $(1,1)$-form. Since the curvature 2-form Ω^T has type $(1,1)$ it follows as in ordinary Chern-Weil theory that

Theorem 15. *The $k^{\rm th}$ Chern class $c_k(\mathcal{D})$ of the complex vector bundle $\mathcal{D}$ is represented by the basic (k,k)-form γ_k determined by the formula*

$$\det\Big(\mathbb{1}_n - \frac{1}{2\pi i}\Omega^T\Big) = 1 + \gamma_1 + \cdots + \gamma_k.$$

Since γ_k is a closed basic (k,k)-form it represents an element in $H^{k,k}_B(\mathcal{F}_\xi) \subset H^{2k}_B(\mathcal{F}_\xi)$ that is called the basic $k^{\rm th}$ Chern class and denoted by $c_k(\mathcal{F}_\xi)$.

We now concentrate on the first Chern classes $c_1(\mathcal{D})$ and $c_1(\mathcal{F}_\xi)$. We have

Definition 16. *A Sasakian structure $\mathcal{S} = (\xi,\eta,\Phi,g)$ is said to be* **positive (negative)** *if $c_1(\mathcal{F}_\xi)$ is represented by a positive (negative) definite $(1,1)$-form. If either of these two conditions is satisfied $\mathcal{S}$ is said to be* **definite***, and otherwise $\mathcal{S}$ is called* **indefinite**. *$\mathcal{S}$ is said to be* **null** *if $c_1(\mathcal{F}_\xi) = 0$.*

Notice that irregular structures cannot occur for negative or null Sasakian structures, since the dimension of $\mathfrak{Aut}(\xi,\eta,\Phi,g)$ is greater than one. In analogy with common terminology of smooth algebraic varieties we see that a positive Sasakian structure is a *transverse Fano structure*[1], while a null Sasakian structure is a *transverse Calabi-Yau structure*. The negative Sasakian case corresponds to the canonical bundle being ample.

4. Sasaki-Einstein and 3-Sasakian Geometry

Definition 17. *A Sasakian manifold $(M,\mathcal{S})$ is* **Sasaki-Einstein** *if the metric g is also Einstein.*

For any 2n+1-dimensional Sasakian manifold $\mathrm{Ric}(X,\xi) = 2n\eta(X)$ implying that any Sasaki-Einstein metric must have positive scalar curvature. Thus any complete Sasaki-Einstein manifold must have a finite fundamental group. Furthermore the

[1]For a more algebro-geometric approach to positivity and fundamentals on log Fano orbifolds see [BG07].

metric cone $(\mathcal{C}(M),\bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ on M is Kähler Ricci-flat (Calabi-Yau).

The following theorem [BG00] is an orbifold version of the famous Kobayashi bundle construction of Einstein metrics on bundles over positive Kähler-Einstein manifolds [Bes87, Kob56].

Theorem 18. *Let $(\mathcal{Z},h)$ be a compact Fano orbifold with $\pi_1^{orb}(\mathcal{Z}) = 0$ and Kähler-Einstein metric h. Let $\pi : M \longrightarrow \mathcal{Z}$ be the S^1 V-bundle whose first Chern class is $\frac{c_1(\mathcal{Z})}{Ind(\mathcal{Z})}$. Suppose further that the local uniformizing groups of $\mathcal{Z}$ inject into S^1. Then with the metric $g = \pi^* h + \eta \otimes \eta$, M is a compact simply connected Sasaki-Einstein manifold.*

Here $\mathrm{Ind}(\mathcal{Z})$ is the *orbifold Fano index* [BG00] defined to be the largest positive integer such that $\frac{c_1(\mathcal{Z})}{\mathrm{Ind}(\mathcal{Z})}$ defines a class in the orbifold cohomology group $H^2_{orb}(\mathcal{Z},\mathbb{Z})$. A very special class of Sasaki-Einstein spaces is naturally related to several quaternionic geometries.

Definition 19. *Let (M,g) be a Riemannian manifold of dimension m. We say that (M,g) is* **3-Sasakian** *if the metric cone $(\mathcal{C}(M),\bar{g}) = (\mathbb{R}_+ \times M,\ dr^2 + r^2 g)$ on M is hyperkähler.*

We emphasize the important observation of Kashiwada [Kas71] that a 3-Sasakian manifold is automatically Einstein. We denote a Sasakian manifold with a 3-Sasakian structure by $(M,\boldsymbol{\mathcal{S}})$, where $\boldsymbol{\mathcal{S}} = (\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3)$ is a triple or a 2-sphere of Sasakian structures $\mathcal{S}_i = (\eta_i,\xi_i,\Phi_i,g)$.

Remark 20. In the 3-Sasakian case there is an extra structure, *i.e.*, the transverse geometry $\mathcal{O}$ of the 3-dimensional foliation which is quaternionic-Kähler. In this case, the transverse space $\mathcal{Z}$ is the twistor space of $\mathcal{O}$ and the natural map $\mathcal{Z} \longrightarrow \mathcal{O}$ is the orbifold twistor fibration [Sal82]. We get the following diagram which we denote by $\diamondsuit(M,\boldsymbol{\mathcal{S}})$ [BGM93, BGM94]:

$$\begin{tikzcd} & & \boxed{\textbf{Hyperkähler Geometry}} & & \\ & & \mathcal{C}(M) \arrow[dl] \arrow[dr] & & \\ \boxed{\textbf{Twistor Geometry}} & \mathcal{Z} \arrow[dr] & & M \arrow[ll] \arrow[dl] & \boxed{\textbf{3-Sasakian Geometry}} \\ & & \mathcal{O} & & \\ & & \boxed{\textbf{Quaternion Kähler Geometry}} & & \end{tikzcd} \tag{9}$$

Remark 21. The table below summarizes properties of the cone and transverse geometries associated to various metric contact structures.

Cone Geometry of $\mathcal{C}(M)$	M	Transverse Geometry of $\mathcal{F}_\xi$
Symplectic	Contact	Symplectic
Kähler	Sasakian	Kähler
Kähler	positive Sasakian	Fano, $c_1(\mathcal{Z}) > 0$
Kähler	null Sasakian	Calabi-Yau, $c_1(\mathcal{Z}) = 0$
Kähler	negative Sasakian	ample canonical bundle, $c_1(\mathcal{Z}) < 0$
Calabi-Yau	Sasaki-Einstein	Fano, Kähler-Einstein
Hyperkähler	3-Sasakian	$\mathbb{C}$-contact, Fano, Kähler-Einstein

For numerous examples and constructions of Sasaki-Einstein and 3-Sasakian manifolds see [BG07]. We finish this section with a remark that both the 3-Sasakian metric on M and the twistor space metric on $\mathcal{Z}$ admit 'squashings' which are again Einstein. More generally, let $\pi : M \longrightarrow B$ be an orbifold Riemannian submersion with g the Riemannian metric on M. Let $\mathcal{V}$ and $\mathcal{H}$ denote the vertical and horizontal subbundles of the tangent bundle TM. For each real number $t > 0$ we construct a one parameter family g_t of Riemannian metrics on M by defining

$$g_t|_{\mathcal{V}} = tg|_{\mathcal{V}}\,, \qquad g_t|_{\mathcal{H}} = g|_{\mathcal{H}}\,, \qquad g_t(\mathcal{V},\mathcal{H}) = 0\,. \tag{10}$$

So for each $t > 0$ we have an orbifold Riemannian submersion with the same base space. Furthermore, if the fibers of g are totally geodesic, so are the fibers of g_t . We apply the canonical variation to the orbifold Riemannian submersion $\pi : M \longrightarrow \mathcal{O}$ and $\pi : \mathcal{Z} \longrightarrow \mathcal{O}$

Theorem 22. *Every 3-Sasakian manifold M admits a second Einstein metric of positive scalar curvature. Furthermore, the twistor space $\mathcal{Z}$ also admits a second orbifold Einstein metric which is Hermitian-Einstein, but not Kähler-Einstein.*

5. Toric Sasaki-Einstein 5-Manifolds

Examples of Sasaki-Einstein manifolds are plentiful and we refer the interested reader to our monograph for a detailed exposition [BG07]. Here we would like to consider the toric Sasaki-Einstein structures in dimension 5 again referring to [BG07] for all necessary details. Toric Sasaki-Einstein 5-manifolds recently emerged from physics in the context of *supersymmetry* and the so-called AdS/CFT duality conjecture which we will discuss in the last section. It is known that, in dimension 5, toric Sasaki-Einstein structures can only occur on the k-fold connected sums $k(S^2 \times S^3)$ [BG07]. The first inhomogeneous toric Sasaki-Einstein structures on $S^2 \times S^3$ were constructed by Gauntlett, Martelli, Sparks, and Waldram. It follows that $S^2 \times S^3$ admits infinitely many distinct quasi-regular and irregular toric Sasaki-Einstein structures [GMSW04b]. Toric geometry of these examples was further explored in [MS05, MSY06a, MSY06b]. We will now describe a slightly different approach to a more general problem.

Consider the symplectic reduction of $\mathbb{C}^n$ (or equivalently the Sasakian reduction of S^{2n-1}) by a k-dimensional torus T^k. Every complex representation of a T^k on $\mathbb{C}^n$ can be described by an exact sequence

$$0 \longrightarrow T^k \xrightarrow{f_\Omega} T^n \longrightarrow T^{n-k} \longrightarrow 0\,.$$

The monomorphism f_Ω can be represented by the diagonal matrix

$$f_\Omega(\tau_1,\ldots,\tau_k)=\mathrm{diag}\left(\prod_{i=1}^k \tau_i^{a_1^i},\ldots,\prod_{i=1}^k \tau_i^{a_n^i}\right),$$

where $(\tau_1,..,\tau_k)\in S^1\times\cdots\times S^1=T^k$ are the complex coordinates on T^k, and $a^i_\alpha\in\mathbb{Z}$ are the coefficients of a $k\times n$ integral *weight* matrix $\Omega\in\mathcal{M}_{k,n}(\mathbb{Z})$. We have [BG07]

Proposition 23. *Let $X(\Omega)=(\mathbb{C}^n\setminus\mathbf{0})/\!\!/T^k(\Omega)$ denote the Kähler quotient of the standard flat Kähler structure on $(\mathbb{C}^n\setminus\mathbf{0})$ by the weighted Hamiltonian T^k-action with an integer weight matrix Ω. Consider the Kähler moment map*

$$\mu^i_\Omega(\mathbf{z})=\sum_{\alpha=1}^n a^i_\alpha|z_\alpha|^2,\qquad i=1,\ldots,k\,. \tag{11}$$

If all minor $k\times k$ determinants of Ω are non-zero then $X(\Omega)=C(Y(\Omega))$ is a cone on a compact Sasakian orbifold $Y(\Omega)$ of dimension $2(n-k)-1$ which is the Sasakian reduction of the standard Sasakian structure on S^{2n-1}. In addition, the projectivization of $X(\Omega)$ defined by $\mathcal{Z}(\Omega)=X(\Omega)/\mathbb{C}^*$ is a Kähler reduction of the complex projective space $\mathbb{CP}^{n-1}$ by a Hamiltonian T^k-action defined by Ω and it is the transverse space of the Sasakian structure on $Y(\Omega)$ induced by the quotient. If

$$\sum_\alpha a^i_\alpha=0,\qquad \forall\ i=1,\ldots,k \tag{12}$$

then $c_1(X(\Omega))=c_1(\mathcal{D})=0$. In particular, the orbibundle $Y(\Omega)\longrightarrow\mathcal{Z}(\Omega)$ is anti-canonical. Moreover, the cone $C(Y(\Omega))$, its Sasakian base $Y(\Omega)$, and the transverse space $\mathcal{Z}(\Omega)$ are all toric orbifolds.

Remark 24. The conditions on the matrix Ω that assure that $Y(\Omega)$ is a smooth manifold are straightforward to work out. They involve gcd conditions on certain minor determinants of Ω.

This proposition is nicely summarized by the 'reduction' diagram

$$\begin{array}{ccccc}\mathbb{CP}^{n-1} & \longleftarrow & S^{2n-1} & \longleftarrow & \mathbb{C}^n\setminus(\mathbf{0})\\ \Downarrow & & \Downarrow & & \Downarrow\\ \mathcal{Z}(\Omega) & \longleftarrow & Y(\Omega) & \longleftarrow & C(Y(\Omega)).\end{array} \tag{13}$$

Both the toric geometry and the topology of $Y(\Omega)$ depend on Ω. Furthermore, $Y(\Omega)$ comes equipped with a family of Sasakian structures. When $n-k=3$, assuming that $Y(\Omega)$ is simply connected (which is an additional condition on Ω), we must have $m(S^2\times S^3)$ for some $m\le k$. We will be mostly interested in the case when $m=k$.

Gauntlett, Martelli, Sparks, and Waldram [GMSW04b] gave an explicit construction of a Sasaki-Einstein metric for $\Omega=(p,p,-p+q,-p-q)$, where p and q are relatively prime nonnegative integers with $p>q$. (The general case for $k=1$ was treated later in [CLPP05, MS05], see Remark 27 below). To connect with the original notation we write $Y(\Omega)=Y_{p,q}$. Then we get:

One can check that $Y_{1,0}$ is just the homogeneous metric on $S^2\times S^3$ which is both toric and regular. The next simplest example is $Y_{2,1}$ which, as a toric contact (Sasakian) manifold, is a circle bundle over the blow up of $\mathbb{CP}^2$ at one point $F_1=\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ [MS06]. As F_1 cannot admit any Kähler-Einstein metric, Kobayashi's bundle construction cannot give a compatible Sasaki-Einstein structure. But there is a choice of a Reeb vector field in the torus which makes it possible to give $Y_{2,1}$ a Sasaki-Einstein metric. The Sasaki-Einstein structure on $Y_{2,1}$ is not quasi-regular and this was the first such example in the literature. Hence, $S^2 \times S^3$ admits infinitely many toric quasi-regular Sasaki-Einstein structures and infinitely many toric irregular Sasaki-Einstein structures of rank 2. We have the following generalization of the $Y_{p,q}$ metrics due to [FOW06, CFO07]:

Theorem 25. *Let $Y(\Omega)$ be as in Proposition 23. Then $Y(\Omega)$ admits a toric Sasaki-Einstein structure which is unique up to a transverse biholomorphism.*

This existence of a Sasaki-Einstein metric is proved in [FOW06] although the authors do not draw all the conclusions regarding possible toric Sasaki-Einstein manifolds that can be obtained. They give one interesting example of an irregular Sasaki-Einstein structure which generalizes the $Y_{2,1}$ example of [MS05] in the following sense: One considers a regular positive Sasakian structure on the anti-canonical circle bundle over the del Pezzo surface $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ which gives a toric Sasakian structure on $2(S^2 \times S^3)$. The regular Sasakian structure on $2(S^2 \times S^3)$ cannot have any Sasaki-Einstein metric. However, as it is with $Y_{2,1}$ Futaki, Ono and Wang [FOW06] show that one can deform the regular structure to a unique irregular Sasaki-Einstein structure. A slightly different version of the Theorem 25 is proved in [CFO07] where uniqueness is also established. Cho, Futaki and Ono work with toric diagrams rather than with Kähler (Sasakian) quotients which amounts to the same thing by Delzant's construction. We should add that the results of [CFO07] apply to the toric Sasaki-Einstein manifolds in general dimension and not just in dimension 5.

Corollary 26. *The manifolds $k(S^2 \times S^3)$ admit infinite families of toric Sasaki-Einstein structures for each $k \geq 1$.*

As in the $k = 1$ case one would expect infinitely many quasi-regular and infinitely many irregular such Sasaki-Einstein structures for each Ω satisfying all the condition.

Remark 27. The general anticanonical circle reduction was considered independently in two recent papers, [CLPP05, MS05]. There it was shown that for $\Omega = \mathbf{p} = (p_1, p_2, -q_1, -q_2)$, with $p_i, q_i \in \mathbb{Z}^+$, $p_1 + p_2 = q_1 + q_2$, and $\gcd(p_i, q_j) = 1$ for all $i, j = 1, 2$, the 5-manifold $Y(\Omega) \approx S^2 \times S^3$ admits a Sasaki-Einstein structure which coincides with that on $Y_{p,q}$ when $p_1 = p_2 = p$ and $q_1 = p - q, q_2 = p + q$. In [CLPP05] this family is denoted by $L^5(a, b, c)$, where $\mathbf{p} = (a, b, -c, -a - b + c)$ and they write the metric explicitly. However, in this case it appears to be harder (though, in principle, possible) to write down the condition under which the Sasaki-Einstein Reeb vector field $\xi = \xi(a, b, c)$ is quasi-regular. A priori, it is not even clear whether the quasi-regularity condition has any additional solutions beyond those obtained for the subfamily $Y_{p,q}$. Moreover, it follows from [CFO07] that the metrics of [CLPP05, MS05] describe all possible toric Sasaki-Einstein structures on $S^2 \times S^3$.

There have been similar constructions of a two-parameter family $X_{p,q}$ of toric Sasaki-Einstein metrics on $2(S^2 \times S^3)$ [HKW05], and another two-parameter family, called $Z_{p,q}$, on $3(S^2 \times S^3)$ [OY06c]. All these examples, and many more, can be obtained as special cases of Theorem 25 as they are all $Y(\Omega)$ for some choice of Ω. The $Y_{p,q}$, $L^5(a, b, c)$, $X_{p,q}$ and $Z_{p,q}$ metrics have received a lot of attention because of the role such Sasaki-Einstein manifolds play in the AdS/CFT Duality Conjecture. They created an avalanche of papers studying the properties of these metrics from the physics perspective [ABCC06, OY06c, OY06b, OY06a, KSY05, HEK05, BZ05, BBC05, BFZ05, SZ05, HKW05, BLMPZ05, BHK05, BFH+05, Pal05, HSY04]. The AdS/CFT duality will be discussed in the last section.

6. The Dirac Operator and Killing Spinors

We begin with a definition of spinor bundles and the bundle of Clifford algebras of a vector bundle [LM89, Fri00]. Recall that the *Clifford algebra* $Cl(\mathbb{R}^n)$ over $\mathbb{R}^n$ can be defined as the quotient algebra of the tensor algebra $\mathcal{T}(\mathbb{R}^n)$ by the two-sided ideal $\mathcal{I}$ generated by elements of the form $v\otimes v+q(v)$ where q is a quadratic form on $\mathbb{R}^n$.

Definition 28. *Let E be a vector bundle with inner product $\langle\cdot,\cdot\rangle$ on a smooth manifold M, and let $\mathcal{T}(E)$ denote the tensor bundle over E. The* **Clifford bundle** *of E is the quotient bundle $Cl(E)=\mathcal{T}(E)/\mathcal{I}(E)$ where $\mathcal{I}$ is the bundle of ideals (two-sided) generated pointwise by elements of the form $v\otimes v+\langle v,v\rangle$ with $v\in E_x$. A* **real spinor bundle** *$S(E)$ of E is a bundle of modules over the Clifford bundle $Cl(E)$. Similarly, a* **complex spinor bundle** *is a bundle of complex modules over the complexification $Cl(E)\otimes\mathbb{C}$.*

As vector bundles $Cl(E)$ is isomorphic to the exterior bundle $\Lambda(E)$, but their algebraic structures are different. The importance of $Cl(E)$ is that it contains the spin group $Spin(n)$, the universal (double) covering group of the orthogonal group $SO(n)$, so one obtains all the representations of $Spin(n)$ by studying representations of $Cl(E)$. We assume that the vector bundle E admits a spin structure, so $w_2(E)=0$. We are interested mainly in the case when (M,g) is a Riemannian spin manifold and $E=TM$ in which case we write $S(M)$ instead of $S(TM)$. The Levi-Civita connection ∇ on TM induces a connection, also denoted ∇, on any of the spinor bundles $S(M)$, or more appropriately on the sections $\Gamma(S(M))$.

Definition 29. *Let (M^n,g) be a Riemannian spin manifold and let $S(M)$ be any spinor bundle. The* **Dirac operator** *is the first order differential operator $D:\Gamma(S(M))\longrightarrow\Gamma(S(M))$ defined by*

$$D\psi=\sum_{j=1}^n E_j\cdot\nabla_{E_j}\psi\,,$$

where $\{E_j\}$ is a local orthonormal frame and $\cdot$ denotes Clifford multiplication.

The Dirac operator, of course originating with the famous Dirac equation describing fermions in theoretical physics, was brought into mathematics by Atiyah and Singer in [AS63]. Then Lichnerowicz [Lic63] proved his famous result that a Riemannian spin manifold with positive scalar curvature must have vanishing $\hat{A}$-genus. An interesting question on any spin manifold is: what are the eigenvectors of the Dirac operator. In this regard the main objects of interest consists of special sections of certain spinor bundles called *Killing spinor fields* or just *Killing spinors* for short. Specifically, (cf. [BFGK91, Fri00])

Definition 30. *Let (M, g) be a complete n-dimensional Riemannian spin manifold, and let $S(M)$ be a spin bundle (real or complex) on M and ψ a smooth section of $S(M)$. We say that ψ is a* **Killing spinor** *if for every vector field X there is $\alpha \in \mathbb{C}$, called* **Killing number**, *such that*

$$\nabla_X \psi = \alpha X \cdot \psi \,.$$

Here $X \cdot \psi$ denotes the Clifford product of X and ψ. We say that ψ is **imaginary** *when $\alpha \in \mathrm{Im}(\mathbb{C}^*)$, ψ is* **parallel** *if $\alpha = 0$ and ψ is* **real**[2] *if $\alpha \in \mathrm{Re}(\mathbb{C}^*)$.*

We shall see shortly that the three possibilities for the Killing number α : real, imaginary, or 0, are the only possibilities. The name Killing spinor derives from the fact that if ψ is a non-trivial Killing spinor and α is real, the vector field

$$X_\psi = \sum_{j=1}^{n} g(\psi, E_j \cdot \psi) E_j \tag{14}$$

is a Killing vector field for the metric g (which, of course, can be zero). If ψ is a Killing spinor on an n-dimensional spin manifold, then

$$D\psi = \sum_{j=1}^{n} E_j \cdot \nabla_{E_j} \psi = \sum_{j=1}^{n} \alpha E_j \cdot E_j \cdot \psi = -n\alpha\psi \,. \tag{15}$$

So Killing spinors are eigenvectors of the Dirac operator with eigenvalue $-n\alpha$. In 1980 Friedrich [Fri80] proved the following remarkable theorem:

Theorem 31. *Let (M^n, g) be a Riemannian spin manifold which admits a non-trivial Killing spinor ψ with Killing number α. Then (M^n, g) is Einstein with scalar curvature $s = 4n(n-1)\alpha^2$.*

A proof of this is a straightforward curvature computation which can be found in either of the books [BFGK91, Fri00]. It also uses the fact that a non-trivial Killing spinor vanishes nowhere. It follows immediately from Theorem 31 that α must be one of the three types mentioned in Definition 30. So if the Killing number is real then (M, g) must be a positive Einstein manifold. In particular, if M is complete, then it is compact. On the other hand if the Killing number is pure imaginary, Friedrich shows that M must be non-compact.

The existence of Killing spinors not only puts restrictions on the Ricci curvature, but also on both the Riemannian and the Weyl curvature operators [BFGK91].

Proposition 32. *Let (M^n, g) be a Riemannian spin manifold. Let ψ be a Killing spinor on M with Killing number α and let $\mathcal{R}, \mathcal{W} : \Lambda^2 M \longrightarrow \Lambda^2 M$ be the Riemann and Weyl curvature operators, respectively. Then for any vector field X and any 2-form β we have*

$$\mathcal{W}(\beta) \cdot \psi = 0\,; \tag{16}$$

$$(\nabla_X \mathcal{W})(\beta) \cdot \psi = -2\alpha\big(X \lrcorner \mathcal{W}(\beta)\big) \cdot \psi\,; \tag{17}$$

$$(\mathcal{R}(\beta) + 4\alpha^2 \beta) \cdot \psi = 0\,; \tag{18}$$

$$(\nabla_X \mathcal{R})(\beta) \cdot \psi = -2\alpha\big(X \lrcorner \mathcal{R}(\beta) + 4\alpha^2 \beta(X)\big) \cdot \psi\,. \tag{19}$$

[2]Here the standard terminology real and imaginary Killing spinors can be somewhat misleading. The Killing spinor ψ is usually a section of a complex spinor bundle. So a real Killing spinor just means that α is real.

These curvature equations can be used to prove (see [BFGK91] or [Fri00])

Theorem 33. *Let (M^n, g) be a connected Riemannian spin manifold admitting a non-trivial Killing spinor with $\alpha \neq 0$. Then (M, g) is locally irreducible. Furthermore, if M is locally symmetric, or $n \leq 4$, then M is a space of constant sectional curvature equal to $4\alpha^2$.*

Friedrich's main objective in [Fri80] was an improvement of Lichnerowicz's estimate in [Lic63] for the eigenvalues of the Dirac operator. Indeed, Friedrich proves that the eigenvalues λ of the Dirac operator on any compact manifold satisfy the estimate

$$\lambda^2 \geq \frac{1}{4}\frac{ns_0}{n-1}\,, \tag{20}$$

where s_0 is the minimum of the scalar curvature on M. Thus, Killing spinors ψ are eigenvectors that realize equality in equation (20). Friedrich also proves the converse that any eigenvector of D realizing the equality must be a Killing spinor with

$$\alpha = \pm\frac{1}{2}\sqrt{\frac{s_0}{n(n-1)}}\,. \tag{21}$$

Example 34. [Spheres] In the case of the round sphere (S^n, g_0) equality in equation (20) is always attained. So normalizing such that $s_0 = n(n-1)$, and using Bär's Correspondence Theorem 38 below the number of corresponding real Killing spinors equals the number of constant spinors on $\mathbb{R}^{n+1}$ with the flat metric. The latter is well known (see the appendix of [PR88]) to be $2^{\lfloor n/2 \rfloor}$ for each of the values $\alpha = \pm\frac{1}{2}$, where $\lfloor n/2 \rfloor$ is the largest integer less than or equal to $n/2$.

Remark 35. Actually (without making the connection to Sasakian geometry) already in [Fri80] Friedrich gives a non-spherical example of a compact 5-manifold with a real Killing spinor: $M = SO(4)/SO(2)$ with its homogeneous Kobayashi-Tanno Sasaki-Einstein structure.

We now wish to relate Killing spinors to the main theme of this article, Sasakian geometry. First notice that if a Sasakian manifold M^{2n+1} admits a Killing spinor, Theorem 31 says it must be Sasaki-Einstein, so the scalar curvature $s_0 = 2n(2n+1)$, and equation (21) implies that $\alpha = \pm\frac{1}{2}$. We have the following result of Friedrich and Kath [FK90]

Theorem 36. *Every simply connected Sasaki-Einstein manifold admits non-trivial real Killing spinors. Furthermore,*

(i) *if M has dimension $4m+1$ then (M, g) admits exactly one Killing spinor for each of the values $\alpha = \pm\frac{1}{2}$,*
(ii) *if M has dimension $4m+3$ then (M, g) admits at least two Killing spinors for one of the values $\alpha = \pm\frac{1}{2}$.*

Outline of Proof. (Details can be found in [FK90] or the book [BFGK91].) Every simply connected Sasaki-Einstein manifold is known to be spin, so M has a spin bundle $S(M)$. Given a fixed Sasakian structure $\mathcal{S} = (\xi, \eta, \Phi, g)$ we consider two subbundles $\mathcal{E}_\pm(\mathcal{S})$ of $S(M)$ defined by

$$\mathcal{E}_\pm(\mathcal{S}) = \{\psi \in S(M) \ \mid \ (\pm 2\Phi X + \pounds_\xi X)\cdot\psi = 0, \quad \forall X \in \Gamma(TM)\}\,. \tag{22}$$

Set $\nabla^{\pm}_X = \nabla_X \pm \frac{1}{2}X\cdot$. A straightforward computation shows that $\nabla^{\pm}$ preserves the subbundles $\mathcal{E}_{\pm}$ and defines a connection there. Moreover, by standard curvature computations it can be shown that the connection $\nabla^{\pm}$ is flat in $\mathcal{E}_{\pm}(\mathcal{S})$. So it has covariantly constant sections which are precisely the Killing spinors. One then uses some representation theory of $\mathrm{Spin}(2n+1)$ to compute the dimensions of $\mathcal{E}_{+}(\mathcal{S})$ and $\mathcal{E}_{-}(\mathcal{S})$ proving the result. □

We have the following:

Corollary 37. *Let (M, g) be a Sasaki-Einstein manifold of dimension $2m+1$. Then (M, g) is locally symmetric if and only if (M, g) is of constant curvature. Moreover, $\mathrm{Hol}(g) = SO(2m+1)$ and (M, g) is locally irreducible as a Riemannian manifold.*

Proof. If necessary, go to the universal cover $\tilde{M}$. This is a compact simply connected Sasaki-Einstein manifold; hence, it admits a non-trivial Killing spinor by Theorem 36. The first statement then follows from the Theorem 33. The second statement follows from the Berger Theorem 4. Since M has dimension $2m+1$ the only possibilities for $\mathrm{Hol}(g)$ are $SO(2m+1)$ and G_2. But the latter is Ricci flat, so it cannot be Sasaki-Einstein. □

Friedrich and Kath began their investigation in dimension 5 [FK89] where they showed that a simply-connected compact 5-manifold which admits a Killing spinor must be Sasaki-Einstein. In dimension 7 they showed that there are exactly three possibilities: weak G_2-manifolds, Sasaki-Einstein manifolds which are not 3-Sasakian, and 3-Sasakian manifolds [FK90]. Later Grunewald gave a description of 6-manifolds admitting Killing spinors [Gru90]. We should add an earlier result of Hijazi who showed that the only 8-dimensional manifold with Killing spinors must be the round sphere [Hij86]. By 1990 a decade of research by many people slowly identified all the ingredients of a classification of such manifolds in terms of their underlying geometric structures. The pieces of the puzzle consisting of round spheres in any dimension, Sasaki-Einstein manifolds in odd dimensions, nearly Kähler manifolds in dimension 6, and weak G_2-holonomy manifolds in dimension 7 were all in place with plenty of interesting examples to go around [BFGK91]. What remained at that stage was to show that in even dimensions greater than 8 there is nothing else but the round spheres, while in odd dimensions greater than 7 the only such examples must be Sasaki-Einstein. The missing piece of the puzzle was finally uncovered by Bär: real Killing spinors on M correspond to parallel spinors on the cone $C(M)$ [Bär93]. A bit earlier Wang [Wan89] had shown that on a simply connected complete Riemannian spin manifold the existence of parallel spinors corresponds to reduced holonomy. This led Bär to an elegant description of the geometry of manifolds admitting real Killing spinors (in any dimension) in terms of special holonomies of the associated cones. We refer to the correspondence between real Killing spinors on M and parallel spinors on the cone $C(M)$ (equivalently reduced holonomy) as *Bär's correspondence.* In particular, this correspondence not only answered the last remaining open questions, but also allowed for simple unified proofs of most of the theorems obtained earlier.

7. Real Killing Spinors, Holonomy and Bär's Correspondence

As mentioned the Bär correspondence relates real Killing spinors on a compact Riemannian spin manifold (M,g) to parallel spinors on the Riemannian cone $(C(M),\bar{g})$. We now make this statement precise.

Theorem 38. *Let (M^n,g) be a complete Riemannian spin manifold and $(C(M^n),\bar{g})$ be its Riemannian cone. Then there is a one to one correspondence between real Killing spinors on (M^n,g) with $\alpha=\pm\frac{1}{2}$ and parallel spinors on $(C(M^n),\bar{g})$.*

Proof. The existence of a parallel spinor on $(C(M^n),\bar{g})$ implies that $\bar{g}$ is Ricci flat by Theorem 31. Then by Lemma 2 (M^n,g) is Einstein with scalar curvature $s=n(n-1)$. So any Killing spinors must have $\alpha=\pm\frac{1}{2}$ by equation (21). As in the proof of Theorem 36, $\nabla^{\pm}_X=\nabla_X\pm\frac{1}{2}X\cdot$ defines a connection in the spin bundle $S(M)$. The connection 1-forms $\omega^{\pm}$ of $\nabla^{\pm}$ are related to the connection 1-form ω of the Levi-Civita connection by $\omega^{\pm}=\omega\pm\frac{1}{2}\beta$, where β is a 1-form called the *soldering form.* This can be interpreted as a connection with values in the Lie algebra $\mathfrak{spin}(n+1)=\mathfrak{spin}(n)\oplus\mathbb{R}^n$, and pulls back to the Levi-Civita connection in the spin bundles on the cone $(C(M^n),\bar{g})$. So parallel spinors on the cone correspond to parallel spinors on (M,g) with respect to the connection $\nabla^{\pm}$ which correspond precisely to real Killing spinors with respect to the Levi-Civita connection. □

Now we have the following definition:

Definition 39. *We say that a Riemannian spin manifold (M,g) is of* **type** (p,q) *if it carries exactly p linearly independent real Killing spinors with $\alpha>0$ and exactly q linearly independent real Killing spinors with $\alpha<0$.*

The following theorem has an interesting history. As mentioned above it was Bär [Bär93] who recognized the correspondence between real Killing spinors on (M,g) and parallel spinors on the Riemannian cone $(C(M),\bar{g})$. The relation between parallel spinors and reduced holonomy was anticipated in the work of Hitchin [Hit74] and Bonan [Bon66], but was formalized in the 1989 paper of Wang [Wan89]. It has also been generalized to the non-simply connected case in [Wan95, MS00].

Theorem 40. *Let (M^n,g) be a complete simply connected Riemannian spin manifold, and let $\mathrm{Hol}(\bar{g})$ be the holonomy group of the Riemannian cone $(C(M),\bar{g})$. Then (M^n,g) admits a non-trivial real Killing spinor with (M^n,g) of type (p,q) if and only if $\big(\dim M,\mathrm{Hol}(\bar{g}),(p,q)\big)$ is one of the 6 possible triples listed in the table below:*

$\dim(M)$	$\mathrm{Hol}(\bar{g})$	type (p,q)
n	id	$(2^{\lfloor n/2\rfloor},2^{\lfloor n/2\rfloor})$
$4m+1$	$SU(2m+1)$	$(1,1)$
$4m+3$	$SU(2m+2)$	$(2,0)$
$4m+3$	$Sp(m+1)$	$(m+2,0)$
7	$Spin(7)$	$(1,0)$
6	G_2	$(1,1)$

Here $m\geq 1$, and $n>1$.

Outline of Proof. Since (M,g) is complete and has a non-trivial real Killing spinor, it is compact by Theorem 31. It then follows from a theorem of Gallot [Gal79] that if the Riemannian cone $(C(M),\bar{g})$ has reducible holonomy it must be flat. So we can apply Berger's Theorem 4. Now Wang [Wan89] used the spinor representations of the possible irreducible holonomy groups on Berger's list to give the correspondence between these holonomy groups and the existence of parallel spinors. First he showed that the groups listed in Table 2 that are not on the above table do not admit parallel spinors. Then upon decomposing the spin representation of the group in question into irreducible pieces, the number of parallel spinors corresponds to the multiplicity of the trivial representation. Wang computes this in all but the first line of the table when $(C(M),\bar{g})$ is flat. In this case (M,g) is a round sphere as discussed in Example 34, so the number of linearly independent constant spinors is $(2^{\lfloor n/2\rfloor},2^{\lfloor n/2\rfloor})$. By Bär's Correspondence Theorem 38 real Killing spinors on (M,g) correspond precisely to parallel spinors on $(C(M),\bar{g})$. Note that the hypothesis of completeness in Wang's theorem [Wan89] is not necessary, so that the correspondence between the holonomy groups and parallel spinors holds equally well on Riemannian cones. However, the completeness assumption on (M,g) guarantees the irreducibility of the cone $(C(M),\bar{g})$ as mentioned above. □

Let us briefly discuss the types of geometry involved in each case of this theorem. As mentioned in the above proof the first line of the table corresponds to the round spheres. The next three lines correspond to Sasaki-Einstein geometry, so Theorem 40 generalizes the Friedrich-Kath Theorem 36 in this case. The last of these three lines corresponds precisely to 3-Sasakian geometry by Definition 19. Finally the two cases whose cones have exceptional holonomy will be discussed in more detail in Section 8.1 below. Suffice it here to mention that it was observed by Bryant and Salamon [BS89] that a cone on a nearly parallel G_2 manifold has its own holonomy in $Spin(7)$. It is interesting to note that Theorem 40 generalizes the result of Hijazi in dimension eight mentioned earlier as well as part of the last statement in Theorem 33, namely

Corollary 41. *Let (M^{2n},g) be a complete simply connected Riemannian spin manifold of dimension $2n$ with $n\neq 3$ admitting a non-trivial real Killing spinor. Then M is isometric to the round sphere.*

We end this section with a brief discussion of the non-simply connected case. Here we consider two additional cases for $\mathrm{Hol}(\bar{g})$, namely $SU(2m+2)\rtimes\mathbb{Z}_2$ and $Sp(2)\times\mathbb{Z}_d$. See [Wan95, MS00] for the list of possibilities.

Example 42. $\mathrm{Hol}(\bar{g})=SU(2m)\rtimes\mathbb{Z}_2$. Consider the $(4m-1)$-dimensional Stiefel manifold $V_2(\mathbb{R}^{2m+1})$ with its homogeneous Sasaki-Einstein metric. The quotient manifold M^{4m-1}_σ of $V_2(\mathbb{R}^{2m+1})$ by the free involution σ induced from complex conjugation has an Einstein metric which is "locally Sasakian". The cone $C(M^{4m-1}_\sigma)$ is not Kähler and its holonomy is $\mathrm{Hol}(\bar{g})=SU(2m+2)\rtimes\mathbb{Z}_2$. According to Wang [Wan95] $C(M^{4m-1}_\sigma)$ admits a spin structure with precisely one parallel spinor if and only if m is even, and according to Moroianu and Semmelmann [MS00] $C(M^{4m-1}_\sigma)$ admits exactly two spin structures each with precisely one parallel spinor if m is even. Thus, by Theorem 38 M^{4m-1}_σ admits exactly two spin structures each with exactly one Killing spinor if and only if m is even.

Example 43. Consider a 3-Sasakian manifold $(M^{4n-1},\boldsymbol{\mathcal{S}})$ and choose a Reeb vector field $\xi(\boldsymbol{\tau})$. Let C_m be the cyclic subgroup of order $m>2$ of the circle group generated by $\xi(\boldsymbol{\tau})$. Assume that m is relatively prime to the order $\upsilon(\boldsymbol{\mathcal{S}})$ of $\boldsymbol{\mathcal{S}}$ and that the generic fibre of the fundamental 3-dimensional foliation $\mathcal{F}_Q$ is $SO(3)$, so that C_m acts freely on M^{4n-1}. This last condition on the generic fibre is easy to satisfy; for example, it holds for any of the 3-Sasakian homogeneous spaces other than the standard round sphere, as well as the bi-quotients described in [BGM94]. (To handle the case when the generic fibre is $Sp(1)$ we simply need to divide m by two when it is even). Since C_m is not in the center of $SO(3)$, the quotient M^{4n-1}/C_m is not 3-Sasakian. However, C_m does preserve the Sasakian structure determined by $\xi(\boldsymbol{\tau})$, so M^{4n-1}/C_m is Sasaki-Einstein. The cone $C(M^{4n-1}/C_m)$ has holonomy $Sp(n)\times\mathbb{Z}_m$, and admits precisely $\frac{n+1}{m}$ parallel spinors if and only if m divides $n+1$ [Wan95, MS00]. Thus, by Theorem 38 M^{4n-1}/C_m admits precisely $\frac{n+1}{m}$ Killing spinors when m divides $n+1$.

8. Geometries Associated with 3-Sasakian 7-manifolds

It is most remarkable that to each $4n$-dimensional positive QK metric $(\mathcal{O},g_{\mathcal{O}})$ (even just locally) one can associate *nine* other Einstein metrics in dimensions $4n+k$, $k=1,2,3,4$. Alternatively, one could say that each 3-Sasakian metric (M,g) canonically defines an additional nine Einstein metrics in various dimensions. We have already encountered all of these metrics. First there are the four geometries of the diamond diagram $\diamondsuit(M,\boldsymbol{\mathcal{S}})$. Then M and $\mathcal{Z}$ admit additional "squashed" Einstein metrics discussed in Theorem 22. Thus we get five Einstein metrics with positive Einstein constants: $(\mathcal{O},g_{\mathcal{O}}),(M,g),(M',g'),(\mathcal{Z},h),(\mathcal{Z}',h')$. Of course $M\simeq M'$ and $\mathcal{Z}\simeq\mathcal{Z}'$ as smooth manifolds (orbifolds) but they are different as Riemannian manifolds (orbifolds), hence, the notation. Let us scale all these metrics so that the Einstein constant equals the dimension of the total space minus 1. Note that any 3-Sasakian metric already has this property. In the other four cases this is a choice of scale which is quite natural due to Lemma 2. However, note that this is not the scale one gets for $(\mathcal{Z},h)$, and $(\mathcal{O},g_{\mathcal{O}})$ via the Riemannian submersion from (M,g). Now, in each case one can consider its Riemannian cone which will be Ricci-flat by Lemma 2. We thus obtain five Ricci-flat metrics on the corresponding Riemannian cones. In addition, one can also take (iterated) sine-cone metrics defined in (1) on the same five bases. These metrics are all Einstein of positive scalar curvature (cf. Lemma 3). Let us summarize all this with the following extension of $\diamondsuit(M,\boldsymbol{\mathcal{S}})$:

(23)
$$
\begin{tikzcd}
C(\mathcal{Z}') & \mathcal{Z}' \arrow[l, hook'] \arrow[dr] & & M' \arrow[dl] \arrow[r, hook] & C(M') \\
& & \mathcal{O} \arrow[r, hook] & C(\mathcal{O}) & \\
C(\mathcal{Z}) & \mathcal{Z} \arrow[l, hook'] \arrow[ur] & & M \arrow[ul] \arrow[ll] \arrow[r, hook] & C(M)
\end{tikzcd}
$$

There would perhaps be nothing special about all these 10 (and many more by iterating sine-cone construction) geometries beyond what has already been discussed in the previous sections. This is indeed true when $\dim(M)>7$. However, when $\dim(M)=7$, or, alternatively, when $\mathcal{O}$ is a positive self-dual Einstein orbifold metric (more generally, just a local metric of this type) some of the metrics occurring in diagram (23) have additional properties. We shall list all of them first. For the moment, let us assume that (M, g) is a compact 3-Sasakian 7-manifold, then the following hold:

(1) $(\mathcal{O}, g_{\mathcal{O}})$ is a positive self-dual Einstein manifold (orbifold). We will think of it as the *source* of all the other geometries.
(2) $(C(\mathcal{O}), dt^2 + t^2 g_{\mathcal{O}})$ is a 5-dimensional Ricci-flat cone with base $\mathcal{O}$.
(3) $(\mathcal{Z}, h)$ is the orbifold twistor space of $\mathcal{O}$.
(4) $(\mathcal{Z}', h')$ is a nearly-Kähler manifold (orbifold).
(5) (M, g) is the 3-Sasakian manifold.
(6) (M', g') is a 7-manifold with weak G_2 structure.
(7) $(C(\mathcal{Z}'), dt^2 + t^2 h')$ is a 7-manifold with holonomy inside G_2.
(8) $(C_s(\mathcal{Z}'), dt^2 + (\sin^2 t)h')$ is a 7-manifold with weak G_2 structure.
(9) $(C(\mathcal{Z}), dt^2 + t^2 h)$ is a 7-dimensional Ricci-flat cone with base $\mathcal{Z}$.
(10) $(C(M), dt^2 + t^2 g)$ is hyperkähler with holonomy contained in $Sp(2)$.
(11) $(C(M'), dt^2 + t^2 g')$ has holonomy contained in $Spin(7)$.

The cases (2) and (8) do not appear to have any special properties other than Ricci-flatness. The cases (1), (3), (5), and (10) are the four geometries of $\diamondsuit(M, \boldsymbol{\mathcal{S}})$. The five remaining cases are all very interesting from the point of view of the classification of Theorem 40. Indeed $\mathcal{Z}'$ and $C(\mathcal{Z}')$ are examples of the structures listed in the last row of the table while $C_2(\mathcal{Z}')$, M' and $C(M')$ give examples of the structures listed in the fifth row. In particular, our diagram (23) provides for a cornucopia of the orbifold examples in the first case and smooth manifolds in the latter.

8.1. **Nearly Parallel G_2-Structures and $Spin(7)$ Holonomy Cones.** Recall, that geometrically G_2 is defined to be the Lie group acting on the imaginary octonions $\mathbb{R}^7$ and preserving the 3-form

$$\begin{aligned}\varphi = {} & \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge (\alpha_4 \wedge \alpha_5 - \alpha_6 \wedge \alpha_7) \\ & + \alpha_2 \wedge (\alpha_4 \wedge \alpha_6 - \alpha_7 \wedge \alpha_5) + \alpha_3 \wedge (\alpha_4 \wedge \alpha_7 - \alpha_5 \wedge \alpha_6),\end{aligned} \tag{24}$$

where $\{\alpha_i\}_{i=1}^7$ is a fixed orthonormal basis of the dual of $\mathbb{R}^7$. A G_2 structure on a 7-manifold M is, by definition, a reduction of the structure group of the tangent bundle to G_2. This is equivalent to the existence of a global 3-form $\varphi \in \Omega^3(M)$ which may be written locally as 24. Such a 3-form defines an associated Riemannian metric, an orientation class, and a spinor field of constant length.

Definition 44. *Let (M, g) be a complete 7-dimensional Riemannian manifold. We say that (M, g) is a* **nearly parallel**[3] *G_2 structure if there exist a global 3-form $\varphi \in \Omega^3(M)$ which locally can be written in terms of a local orthonormal basis as in 24, and $d\varphi = c \star \varphi$, where $\star$ is the Hodge star operator associated to g and $c \neq 0$ is a constant whose sign is fixed by an orientation convention.*

The case $c = 0$ in Definition 44 is somewhat special. In particular, it is known [Sal89] that the condition $d\varphi = 0 = d \star \varphi$ is equivalent to the condition that φ be parallel, *i.e.*, $\nabla\varphi = 0$ which is equivalent to the condition that the metric g has

[3] It had become customary to refer to this notion as 'weak holonomy G_2', a terminology introduced by Gray [Gra71]. However, it was pointed out to us by the anonomous referee that this terminology is misleading due to the fact that Gray's paper contains errors rendering the concept of weak holonomy useless as discovered by Alexandrov [Ale05]. Hence, the term 'nearly parallel' used in [FKMS97] is preferred.

holonomy group contained in G_2. The following theorem provides the connection with the previous discussion on Killing spinors [Bär93]

Theorem 45. *Let (M, g) be a complete 7-dimensional Riemannian manifold with a nearly parallel G_2 structure. Then the holonomy $\mathrm{Hol}(\bar{g})$ of the metric cone $(C(M), \bar{g})$ is contained in $Spin(7)$. In particular, $C(M)$ is Ricci-flat and M is Einstein with positive Einstein constant $\lambda = 6$.*

Remark 46. The sphere S^7 with its constant curvature metric is isometric to the isotropy irreducible space $Spin(7)/G_2$. The fact that G_2 leaves invariant (up to constants) a unique 3-form and a unique 4-form on $\mathbb{R}^7$ implies immediately that this space has a nearly parallel G_2 structure.

Definition 47. *Let (M, g) be a complete 7-dimensional Riemannian manifold. We say that g is a* **proper G_2-metric** *if $\mathrm{Hol}(\bar{g}) = \mathrm{Spin}(7)$.*

We emphasize here that G_2 is the structure group of M, not the Riemannian holonomy group. Specializing Theorem 40 to dimension 7 gives the following theorem due to Friedrich and Kath [FK90].

Theorem 48. *Let (M^7, g) be a complete simply-connected Riemannian spin manifold of dimension 7 admitting a non-trivial real Killing spinor with $\alpha > 0$ or $\alpha < 0$. Then there are four possibilities:*

(i) *(M^7, g) is of type $(1, 0)$ and it is a proper G_2-manifold,*
(ii) *(M^7, g) is of type $(2, 0)$ and it is a Sasaki-Einstein manifold, but (M^7, g) is not 3-Sasakian,*
(iii) *(M^7, g) is of type $(3, 0)$ and it is 3-Sasakian,*
(iv) *$(M^7, g) = (S^7, g_{can})$ and is of type $(8, 8)$.*

Conversely, if (M^7, g) is a compact simply-connected proper G_2-manifold then it carries precisely one Killing spinor with $\alpha > 0$. If (M^7, g) is a compact simply-connected Sasaki-Einstein 7-manifold which is not 3-Sasakian then M carries precisely 2 linearly independent Killing spinors with $\alpha > 0$. Finally, if (M^7, g) is a 3-Sasakian 7-manifold, which is not of constant curvature, then M carries precisely 3 linearly independent Killing spinors with $\alpha > 0$.

Remark 49. The four possibilities of the Theorem 48 correspond to the sequence of inclusions
$$\mathrm{Spin}(7) \supset SU(4) \supset Sp(2) \supset \mathbb{1}.$$
All of the corresponding cases are examples of nearly parallel G_2 metrics. If we exclude the trivial case when the associated cone is flat, we have three types of nearly parallel G_2 geometries. Following [FKMS97] we use the number of linearly independent Killing spinors to classify these geometries, and call them type I, II, and III corresponding to cases (i), (ii), and (iii) of Theorem 48, respectively.

We are now ready to describe the G_2 geometry of the $M' \hookrightarrow C(M')$ part of Diagram 23 [GS96, FKMS97]:

Theorem 50. *Let $(M, \boldsymbol{\mathcal{S}})$ be a 7-dimensional 3-Sasakian manifold. Then the 3-Sasakian metric g is a nearly parallel G_2 metric. Moreover, the second Einstein metric g' given by Theorem 22 and scaled so that the Einstein constant $\lambda = 6$ is a nearly parallel G_2 metric; in fact, it is a proper G_2 metric.*

Proof. For the second Einstein metric g' we have three mutually orthonormal 1-forms $\alpha^1 = \sqrt{t}\eta^1, \quad \alpha^2 = \sqrt{t}\eta^2, \quad \alpha^3 = \sqrt{t}\eta^3$, where t is the parameter of the canonical variation. Let $\{\alpha^4, \alpha^5, \alpha^6, \alpha^7\}$ be local 1-forms spanning the annihilator of the vertical subbundle $\mathcal{V}_3$ in $T^*\mathcal{S}$ such that

$$\begin{aligned} \bar{\Phi}^1 &= 2(\alpha^4 \wedge \alpha^5 - \alpha^6 \wedge \alpha^7)\,, \\ \bar{\Phi}^2 &= 2(\alpha^4 \wedge \alpha^6 - \alpha^7 \wedge \alpha^5)\,, \\ \bar{\Phi}^3 &= 2(\alpha^4 \wedge \alpha^7 - \alpha^5 \wedge \alpha^6)\,. \end{aligned}$$

Then the set $\{\alpha^1, \dots, \alpha^7\}$ forms a local orthonormal coframe for the metric g'. Let

$$\Upsilon = \eta_1 \wedge \eta_2 \wedge \eta_3\,, \qquad \Theta = \sum_a \eta_a \wedge \bar{\Phi}_a = \sum_a \eta_a \wedge d\eta_a + 6\Upsilon \tag{25}$$

In terms of the 3-forms Υ and Θ we have $\varphi = \frac{1}{2}\sqrt{t}\Theta + \sqrt{t}^3\Upsilon$. One easily sees that this is of the type of equation (24) and, therefore, defines a compatible G_2-structure. Moreover, a straightforward computation gives

$$d\varphi = \frac{1}{2}\sqrt{t}\Omega + \sqrt{t}(t+1)d\Upsilon, \qquad \star\varphi = -\frac{1}{2}td\Upsilon - \frac{1}{24}\Omega\,.$$

Thus, $d\varphi = c \star \varphi$ is solved with $\sqrt{t} = 1/\sqrt{5}$, and $c = -12/\sqrt{5}$. So g' is nearly parallel. That g' is a proper G_2 metric is due to [FKMS97]. The idea is to use Theorem 48. Looking at the four possibilities given in that theorem, we see that it suffices to show that g' is not Sasaki-Einstein. The details are in [FKMS97]. □

Example 51. 3-Sasakian 7-manifolds are plentiful [BG07]. All of them give, by Theorem 50, examples of type I and type III geometries. Examples of simply connected type I geometries that do not arise via Theorem 50 are the homogeneous Aloff-Wallach spaces $M^7_{m,n}$, $(m,n) \neq (1,1)$ which, as special cases of Eschenburg bi-quotients [CMS96, BFGK91], are together with an isotropy irreducible homogeneous space defined as follows: Consider the space $\mathcal{H}_2$ of homogeneous polynomials of degree 2 in three real variables (x_1, x_2, x_3). As $\dim(\mathcal{H}_2) = 5$ it gives rise to the embedding $SO(3) \subset SO(5)$. We take $M = SO(5)/SO(3)$. This example was used by Bryant to get the first 8-dimensional metric with holonomy $Spin(7)$ [Bry87]. Examples of type II geometries (Sasaki-Einstein) are equally rich [BG07]. In particular, there are hundreds of examples of type II nearly parallel G_2 metrics on each of the 28 homotopy spheres in dimension 7.

Remark 52. According to [CMS96] the Aloff-Wallach manifold $M^7_{1,1}$ has three Einstein metrics. One is the homogeneous 3-Sasakian metric. The second is the proper G_2 metric of Theorem 50. The third Einstein metric is also nearly parallel most likely being of type I, but we could not positively exclude type II as a possibility.

Open Problem 53. Classify all compact 7-manifolds with nearly parallel G_2 structures of type I, II, or III, respectively.

The classification of type III consists of the classification of all compact 3-Sasakian 7-manifolds. This is probably very hard. The case of 3-Sasakian 7-manifolds with vanishing $\mathfrak{aut}(M, \boldsymbol{\mathcal{S}})$ appears quite difficult. The type II classification (7-dimensional Sasaki-Einstein manifolds which are not 3-Sasakian) is clearly completely out of reach at the moment. A classification of proper nearly parallel

G_2 structures on a compact manifold that do not arise via Theorem 50 would be very interesting and it is not clear how hard this problem really is.

Remark 54. The holonomy $Spin(7)$ cone metrics are plentiful but never complete. However, some of these metrics can be deformed to complete holonomy $Spin(7)$ ones on non compact manifolds. The first example was obtained by Bryant and Salamon who observed that the spin bundle over S^4 with its canonical metric carries a complete metric with holonomy $Spin(7)$ [BS89]. Locally the metric was later considered also in [GPP90]. More generally, spin orbibundles over positive QK orbifolds also carry such complete orbifold metrics as observed by Bryant and Salamon in [BS89]. Other complete examples were constructed later by physicists [CGLP02, CGLP04, KY02a, KY02b]. Finally, the first compact examples were obtained in 1996 by Joyce [Joy96a, Joy99]. See Joyce's book [Joy00] for an excellent detailed exposition of the methods and the discussion of examples.

Open Problem 55. [Complete metrics on cones] Let $(M^7, \boldsymbol{\mathcal{S}})$ be any 3-Sasakian 7-manifold and let (M^7, g') be the associated proper nearly parallel G_2 squashed metric. Consider the two Riemannian cones for these metrics.

(i) When does the metric cone $(C(M), dt^2 + t^2 g')$ admit complete holonomy $Spin(7)$ deformations?
(ii) When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete holonomy $Sp(2)$ (hyperkähler) deformations?

In other dimensions one also could ask the following more general questions:

(iii) Let $(M^{4n+3}, \boldsymbol{\mathcal{S}})$ be a compact 3-Sasakian manifold. When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete hyperkähler (or just Calabi-Yau) deformations?
(iv) Let $(M^{2n+1}, \mathcal{S})$ be a compact Sasaki-Einstein manifold. When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete Calabi-Yau deformations?
(v) Let (M^7, g) be a compact nearly parallel G_2-manifold. When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete holonomy $Spin(7)$ deformations?
(vi) Let (M^6, g) be a compact strict nearly Kähler manifold. When does the metric cone $(C(M), dt^2 + t^2 g)$ admit complete holonomy G_2 deformations?

The metric on the spin bundle $S(S^4)$ by Bryant and Salamon is a deformation of the $Spin(7)$ holonomy metric on the cone over the squashed metric on S^7 [CGLP02, CGLP04], so there are examples of such deformations regarding question (i). Regarding (ii), we recall that every compact 3-Sasakian 3-manifold is isometric to S^3/Γ and the metric cone is the flat cone $\mathbb{C}^2/\Gamma$. Hence, one could think of (ii) as a 7-dimensional analogue of a similar problem whose complete solution was given by Kronheimer [Kro89a]. There are non-trivial examples also in the higher dimensional cases. The metric cone on the homogeneous 3-Sasakian manifold $\mathcal{S}(1,1,1)$ of [BGM94] admits complete hyperkähler deformations, namely the Calabi metric on $T^*\mathbb{CP}^2$. We do not know of any other examples at the moment. In case (iv) of the Calabi-Yau cones on Sasaki-Einstein manifolds, however, there are many such examples. Futaki very recently proved that such a complete Calabi-Yau metric exists for all the regular toric Sasaki-Einstein manifolds of Section 5 [Fut07]. In such cases the metric can be thought of as a complete Ricci-flat Kähler metric on the canonical bundle over a toric Fano manifold. Futaki's result should generalize to the case of toric log Fano orbifolds.

8.2. **Nearly Kähler 6-Manifolds and G_2 Holonomy Cones.** In this section we explain the geometry of the $\mathcal{Z}' \hookrightarrow C(\mathcal{Z}')$ part of the diagram 23. Before we specialize to dimension 6 we begin with a more general introduction. Nearly Kähler manifolds were first studied by Tachibana in [Tac59] and they appear under the name of almost Tachibana spaces in Chapter VIII of the book [Yan65]. They were then rediscovered by Gray [Gra70] and given the name nearly Kähler manifolds which by now is the accepted name.

Definition 56. *A* **nearly Kähler manifold** *is an almost Hermitian manifold (M, g, J, ω) such that $(\nabla_X J)X = 0$ for all tangent vectors X, where ∇ is the Levi-Civita connection and J is the almost complex structure. One says that a nearly Kähler manifold is* **strict** *if it is not Kähler.*

This definition is equivalent to the condition

$$(\nabla_X J)Y + (\nabla_Y J)X = 0 \tag{26}$$

for all vector fields X, Y, which is to say that J is a *Killing tensor* field. An alternative characterization of nearly Kähler manifolds is given by

Proposition 57. *An almost Hermitian manifold (M, g, J, ω) is nearly Kähler if and only if*

$$\nabla\omega = \frac{1}{3} d\omega \, .$$

In particular, a strict nearly Kähler structure is never integrable.

Any nearly Kähler manifold can be locally decomposed as the product of a Kähler manifold and a strict nearly Kähler manifold. Such a decomposition is global in the simply connected case [Nag02b]. Hence, the study of nearly Kähler manifolds reduces to the case of strict ones. In addition every nearly Kähler manifold in dimension 4 must be Kähler so that the first interesting dimension is six.

The following theorem establishes relationship between the twistor space $\mathcal{Z} {\longrightarrow} \mathcal{O}$ of a quaternionic Kähler manifold (orbifold) and nearly Kähler geometry.

Theorem 58. *Let $\pi : (\mathcal{Z}, h) {\longrightarrow} (\mathcal{O}, g_{\mathcal{O}})$ be the twistor space of a positive QK manifold with its Kähler structure (J, h, ω_h). Then $\mathcal{Z}$ admits a strict nearly Kähler structure (J_1, h_1, ω_{h_1}). If $TM = \mathcal{V} \oplus \mathcal{H}$ is the natural splitting induced by π then*

$$h|_{\mathcal{V}} = 2h_1|_{\mathcal{V}} \, , \quad h|_{\mathcal{H}} = h_1|_{\mathcal{H}} = \pi^*(g_{\mathcal{O}}) \, , \tag{27}$$

$$J|_{\mathcal{V}} = -J_1|_{\mathcal{V}} \, , \quad J|_{\mathcal{H}} = J_1|_{\mathcal{H}} \ . \tag{28}$$

Theorem 58 is due to Eells and Salamon [ES83] when $\mathcal{O}$ is 4-dimensional. The higher dimensional analogue was established in [AGI98] (see also [Nag02b]).

Remark 59. Observe that the metric of the nearly Kähler structure of Theorem 58, in general, is *not* Einstein. In particular, h_1 is not the squashed metric h' introduced in the diagram 23, unless $\dim(\mathcal{Z}) = 6$. In six dimensions, we can scale h_1 so that it has scalar curvature $s = 30$ and then indeed $h_1 = h'$ as one can easily check.

Definition 60. *Let $M = G/H$ be a homogeneous space. We say that M is* **3-symmetric** *if G has an automorphism σ of order 3 such that $G^\sigma_0 \subset H \subset G^\sigma$, where G^σ is the fixed point set of σ and G^σ_0 is the identity component in G^σ_0.*

We have the following two theorems concerning nearly Kähler homogeneous Riemannian manifolds. The first is due to Wolf and Gray in all dimensions but six [WG68a, WG68b]. They also conjectured that the result is true for strict nearly Kähler 6-manifolds. The Wolf-Gray conjecture was proved quite recently by Butruille [But05, But06] which is the second theorem below.

Theorem 61. *Every compact homogeneous strict nearly Kähler manifold M of dimension different than 6 is 3-symmetric.*

Theorem 62. *Let (M,g) be a strict nearly Kähler 6-dimensional Riemannian homogeneous manifold. Then M is isomorphic as a homogeneous space to a finite quotient of G/H, where G and H are one of the following:*

1. $G = SU(2)\times SU(2)$ *and* $H=\{\mathrm{id}\}$;
2. $G=G_2$ *and* $H=SU(3)$, *where metrically* $G/H=S^6$ *the round sphere;*
3. $G=Sp(2)$ *and* $H=SU(2)U(1)$, *where* $G/H=\mathbb{CP}^3$ *with its nearly Kähler metric determined by Theorem 58;*
4. $G=SU(3)$ *and* $H=T^2$, *where* G/H *is the flag manifold with its nearly Kähler metric determined by Theorem 58.*

Each of these manifolds carries a unique invariant nearly Kähler structure, up to homothety.

In every dimension, the only known compact examples of nearly Kähler manifolds are 3-symmetric. On the other hand, Theorem 58 can be easily generalized to the case of orbifolds so that there are plenty examples of compact inhomogeneous strict nearly Kähler orbifolds in every dimension.

Theorem 63. *Let M be a compact simply-connected strict nearly Kähler manifold. Then, in all dimensions, as a Riemannian manifold M decomposes as a product of*

1. 3-*symmetric spaces,*
2. *twistor spaces of positive QK manifolds $\mathcal{Q}$ such that $\mathcal{Q}$ is not symmetric,*
3. 6-*dimensional strict nearly Kähler manifold other than the ones listed in Theorem 62.*

This theorem is due to Nagy [Nag02a], but our formulation uses the result of Butruille together with the fact that the twistor spaces of all symmetric positive QK manifolds are 3-symmetric. The LeBrun-Salamon conjecture can now be phrased as follows

Conjecture 64. *Any compact simply connected strict irreducible nearly Kähler manifold (M,g) of dimension greater than 6 must be a 3-symmetric space.*

In particular, the Conjecture 64 is automatically true in dimensions $4n$ because of Nagy's classification theorem and also true in dimensions 10 and 14 because all positive QK manifolds in dimension 8 and 12 are known. The third case leads to an important

Open Problem 65. Classify all compact strict nearly Kähler manifolds in dimension 6.

Dimension six is special not just because of the rôle it plays in Theorem 63. They have several remarkable properties which we summarize in the following theorem.

Theorem 66. *Let (M,J,g,ω_g) be a compact strict nearly Kähler 6-manifold. Then*

(i) *The metric g is Einstein of positive scalar curvature.*
(ii) $c_1(M) = 0$ *and* $w_2(M) = 0$.
(iii) *If g is scaled so it has Einstein constant $\lambda = 5$ then the metric cone $(C(M), dt^2t + t^2g)$ has holonomy contained in G_2. In particular, $C(M)$ is Ricci-flat.*

The first property is due to Matsumoto [Mat72] while the second is due to Gray [Gra76]. The last part is due to Bär [Bär93]. In fact, nearly Kähler 6-manifolds is the geometry of the last row of the table of Theorem 40. More precisely we have the following theorem proved by Grunewald [Gru90]:

Theorem 67. *Let (M^6, g) be a complete simply connected Riemannian spin manifold of dimension 6 admitting a non-trivial Killing spinor with $\alpha > 0$ or $\alpha < 0$. Then there are two possibilities:*
(i) *(M, g) is of type $(1, 1)$ and it is a strict nearly Kähler manifold,*
(ii) *$(M, g) = (S^6, g_{can})$ and is of type $(8, 8)$.*

Conversely, if (M, g) is a compact simply-connected strict nearly Kähler 6-manifold of non-constant curvature then M is of type $(1, 1)$.

Compact strict nearly Kähler manifolds with isometries were investigated in [MNS05] where it was shown that

Theorem 68. *Let (M, J, g, ω_g) be a compact strict nearly Kähler 6-manifold. If M admits a unit Killing vector field, then up to finite cover M is isometric to $S^3 \times S^3$ with its standard nearly Kähler structure.*

Remark 69. The first example of a non-trivial G_2 holonomy metric was found by Bryant [Bry87], who observed that a cone on the complex flag manifold $U(3)/T^3$ carries an incomplete metric with G_2-holonomy. The flag $\mathrm{U}(3)/T^3$ is the twistor space of the complex projective plane $\mathbb{CP}^2$ and as such it also has a strict nearly Kähler structure. As explained in this section, this therefore is just one possible example. One gets such non-trivial metrics also for the cones with bases $\mathbb{CP}^3$ and $S^3 \times S^3$ with their homogeneous strict nearly Kähler structures. Interestingly, in some cases there exist complete metrics with G_2 holonomy which are smooth deformations of the asymptotically conical ones. This fact was noticed by Bryant and Salamon [BS89] who constructed complete examples of G_2 holonomy metrics on bundles of self-dual 2-forms over $\mathbb{CP}^2$ and S^4. Replacing the base with any positive QK orbifold $\mathcal{O}$ gives complete (in the orbifold sense) metrics on orbibundles of self-dual 2-forms over $\mathcal{O}$. Locally some of these metrics were considered in [San03]. More complete examples of explicit G_2 holonomy metrics on non-compact manifolds were obtained by Salamon [Sal04]. G_2 holonomy manifolds with isometric circle actions were investigated by Apostolov and Salamon [AS04]. The first compact examples are due to the ground breaking work of Joyce [Joy96b].

9. Geometries Associated with Sasaki-Einstein 5-manifolds

Like 3-Sasakian manifolds Sasaki-Einstein 5-manifolds are naturally associated to other geometries introduced in the previous section. Of course, each such space $(M^5, \mathcal{S})$ comes with its Calabi-Yau cone $(C(M), \bar{g})$ and, if the Sasaki-Einstein structure $\mathcal{S}$ is quasi-regular, with its quotient log del Pezzo surface $(\mathcal{Z}, h)$. But as it turns out, there are *two* more Einstein metrics associated to g. The examples of this section also illustrate how the Theorem 40 and Bär's correspondence break down when (M,g) is a manifold with Killing spinors which is, however, *not complete*.

We begin by describing a relation between 5-dimensional Sasaki-Einstein structures and six-dimensional nearly Kähler structures which was uncovered recently in [FIMU06]. This relation involves the sine-cones of Definition 1. We use the notation $\bar{g}_s$ to distinguish the sine-cone metric from the usual Riemannian cone metric $\bar{g}$. Of course this metric is not complete, but one can compactify M obtaining a very tractable stratified space $\bar{M} = N\times[0,\pi]$ with conical singularities at $t=0$ and $t=\pi$. Observe the following simple fact which shows that the Riemannian cone on a sine cone is always a Riemannian product.

Lemma 70. *Let (M,g) be a Riemannian manifold. Then the product metric $ds^2 = dx^2+dy^2+y^2g$ on $\mathbb{R}\times C(M)$ can be identified with the iterated cone metric on $C(C_s(M))$.*

Proof. Consider the map $\mathbb{R}^+\times(0,\pi)\longrightarrow\mathbb{R}\times\mathbb{R}^+$ given by polar coordinate change $(r,t)\mapsto(x,y)=(r\cos t, r\sin t)$, where $r>0$ and $t\in(0,\pi)$. We get

$$ds^2 = dx^2+dy^2+y^2g = dr^2+r^2dt^2+r^2\sin^2 tg = dr^2+r^2(dt^2+\sin^2 tg)\,. \quad \square$$

So the iterated Riemannian cone $(C(C_s(M)), ds^2)$ has reducible holonomy $1\times \mathrm{Hol}(C(M))$. This leads to

Corollary 71. *Let (N,g) be a Sasaki-Einstein manifold of dimension $2n+1$. Then the sine-cone $C_s(N)$ with the metric $\bar{g}_s = dr^2+(\sin^2 r)g$ is Einstein with Einstein constant $2n+1$.*

We are particularly interested in the case $n=2$. Compare Lemma 70 with the following result in [Joy00], Propositions 11.1.1-2:

Proposition 72. *Let (M^4,g_4) and (M^6,g_6) be Calabi-Yau manifolds. Let $(\mathbb{R}^3, ds^2 = dx^2+dy^2+dz^2)$ and $(\mathbb{R}, ds^2=dx^2)$ be the Euclidean spaces. Then*

1. *$(\mathbb{R}^3\times M^4, g=ds^2+g_4)$ has a natural G_2 structure and g has holonomy $\mathrm{Hol}(g)\subset \mathbb{1}_3\times SU(2)\subset G_2$,*
2. *$(\mathbb{R}\times M^6, g=ds^2+g_6)$ has a natural G_2 structure and g has holonomy $\mathrm{Hol}(g)\subset 1\times SU(3)\subset G_2$.*

As long as (M^4,g_4) and (M^6,g_6) are simply connected then the products $\mathbb{R}^3\times M^4$ and $\mathbb{R}\times M^6$ are simply connected G_2-holonomy manifolds with reducible holonomy groups and parallel Killing spinors. Note that this does not violate Theorem 40 as these spaces are not Riemannian cones over complete Riemannian manifolds. Using (ii) of Proposition 72 we obtain the following corollary of Theorem 3 first obtained in [FIMU06]

Corollary 73. *Let (N^5,g) be a Sasaki-Einstein manifold. Then the sine cone $C_s(N^5) = N^5\times(0,\pi)$ with metric $\bar{g}_s$ is nearly Kähler of Einstein constant $\lambda=5$. Furthermore $\bar{g}_s$ approximates pure $SU(3)$ holonomy metric near the cone points.*

Using Corollary 73 we obtain a host of examples of nearly Kähler 6-manifolds with conical singularities by choosing N^5 to be any of the Sasaki-Einstein manifolds constructed in [BGN03, BGN02, BG03, Kol07, Kol05, GMSW04b, GMSW04a, CLPP05, FOW06, CFO07]. For example, in this way we obtain nearly-Kähler metrics on $N\times(0,\pi)$ where N is any Smale manifold with a Sasaki-Einstein metric such as S^5 or $k(S^2 \times S^3)$, etc. Note that every simply connected strict nearly Kähler manifold has exactly two real Killing spinors. So as long as N^5 is simply connected $C_s(N^5)$ will have two real Killing spinors. Using Theorem 3 the Sasaki-Einstein metrics constructed in [BG00, BGK05, BGKT05, GK05, BG06] in all odd dimensions also give new Einstein metrics on $C_s(N^{2n+1})$. For example, one obtains many positive Einstein metrics on $\Sigma^{2n+1} \times (0, \pi)$ where Σ^{2n+1} is any odd dimensional homotopy sphere bounding a parallelizable manifold. Of course, there are no Killing spinors unless $n = 2$. Returning to the case of dimension 6, a somewhat more general converse has been obtained in [FIMU06], namely

Theorem 74. *Any totally geodesic hypersurface N^5 of a nearly Kähler 6-manifold M^6 admits a Sasaki-Einstein structure.*

The method in [FIMU06] uses the recently developed notion of hypo $SU(2)$ structure due to Conti and Salamon [CS06]. The study of sine cones appears to have originated in the physics literature [BM03, ADHL03], but in one dimension higher. Now recall the following result of Joyce (cf. [Joy00], Propositions 13.1.2-3)

Proposition 75. *Let (M^6, g_6) and (M^7, g_7) be Calabi-Yau and G_2-holonomy manifolds, respectively. Let $(\mathbb{R}^2, ds^2 = dx^2 + dy^2)$ and $(\mathbb{R}, ds^2 = dx^2)$ be Euclidean spaces. Then*

(1) $(\mathbb{R}^2 \times M^6, g = ds^2 + g_6)$ *has a natural $Spin(7)$ structure and g has holonomy* $\mathrm{Hol}(g) \subset \mathbb{1}_2 \times SU(3) \subset Spin(7)$,
(2) $(\mathbb{R} \times M^7, g = ds^2 + g_7)$ *has a natural $Spin(7)$ structure and g has holonomy* $\mathrm{Hol}(g) \subset 1 \times G_2 \subset Spin(7)$.

Again, if (M^6, g_6) and (M^7, g_7) are simply connected so are the $Spin(7)$-manifolds $\mathbb{R}^2 \times M^6$ and $\mathbb{R} \times M^7$ so that they have parallel spinors. Not surprisingly, in view of Lemma 70 and Proposition 75, the sine cone construction now relates strict nearly Kähler geometry in dimension 6 to nearly parallel G_2 geometry in dimension 7. More precisely [BM03]

Theorem 76. *Let (N^6, g) be a strict nearly Kähler 6-manifold such that g has Einstein constant $\lambda_6 = 5$. Then the manifold $C_s(N) = N^6 \times (0, \pi)$ with its sine cone metric $\bar{g}_s$ has a nearly parallel G_2 structure with Einstein constant $\lambda_7 = 6$ and it approximates pure G_2 holonomy metric near the cone points.*

Proof. Just as before, starting with (N^6, g_6) we consider its metric cone $C(N^6)$ with the metric $\bar{g} = dy^2 + y^2 g_6$ and the product metric g_8 on $\mathbb{R} \times C(N^6)$. With the above choice of the Einstein constant we see that $g_8 = dx^2 + dy^2 + y^2 g_6$ must have holonomy $\mathrm{Hol}(g_8) \subset 1 \times G_2 \subset Spin(7)$. By Lemma 70 g_8 is a metric cone on the metric $g_7 = dt^2 + \sin^2 t g_6$, which must, therefore, have weak G_2 holonomy and the Einstein constant $\lambda_7 = 6$. □

Again, any simply connected weak G_2-manifold has at least one Killing spinor. That real Killing spinor on $C_s(N^6)$ will lift to a parallel spinor on $C(C_s(N^6)) = \mathbb{R} \times C(N^6)$ which is a non-complete $Spin(7)$-manifold of holonomy inside $1 \times G_2$. One can iterate the two cases by starting with a compact Sasaki-Einstein 5-manifold N^5 and construct either the cone on the sine cone of N^5 or the sine cone on the sine cone of N^5 to obtain a nearly parallel G_2 manifold. We list the Riemannian manifolds coming from this construction that are irreducible.

Proposition 77. *Let (N^5, g_5) be a compact Sasaki-Einstein manifold which is not of constant curvature. Then the following have irreducible holonomy groups:*

(1) *the manifold $C(N^5)$ with the metric $g_6 = dt^2 + t^2 g_5$ has holonomy $SU(3)$;*
(2) *the manifold $C_s(N^5) = N^5 \times (0, \pi)$ with metric $g_6 = dt^2 + \sin^2 t\, g_5$ is strict nearly Kähler;*
(3) *the manifold $C_s(C_s(N^5)) = N^5 \times (0, \pi) \times (0, \pi)$ with the metric $g_7 = d\alpha^2 + \sin^2\alpha(dt^2 + \sin^2 t\, g_5)$ has a nearly parallel G_2 structure.*

In addition we have the reducible cone metrics: $C(C_s(N^5)) = \mathbb{R} \times C(N^5)$ has holonomy in $1 \times SU(3) \subset G_2$ and $C(C_s(C_s(N^5))) = \mathbb{R}\times C(C_s(N^5)) = \mathbb{R}\times\mathbb{R}\times C(N^5)$ has holonomy $1\!\!1_2 \times SU(3) \subset 1 \times G_2 \subset Spin(7)$. If N^5 is simply connected then G_5, g_6 and g_7 admit two Killing spinors. For a generalization involving conformal factors see [MO07].

Remark 78. Recall Remark 49. Note that when a nearly parallel G_2 metric is not complete then the type I-III classification is no longer valid. The group $Spin(7)$ has other subgroups than the ones listed there and we can consider the following inclusions of (reducible) holonomies

$$\mathrm{Spin}(7) \supset G_2 \times 1 \supset SU(3) \times 1\!\!1_2 \times \supset SU(2) \times 1\!\!1_3 \supset 1\!\!1_8\,.$$

According to the Friedrich-Kath Theorem 48 the middle three cannot occur as holonomies of Riemannian cones of complete 7-manifolds with Killing spinors. But as the discussion of this section shows, they most certainly can occur as holonomy groups of Riemannian cones of incomplete nearly parallel G_2 metrics. These metrics can be still separated into three types depending on the holonomy reduction: say the ones that come from strict nearly Kähler manifolds are generically of type I_s while the ones that come from Sasaki-Einstein 5-manifolds via the iterated sine cone construction are of type II_s and of type III_s when $H \subset SU(3)$ is some proper non-trivial subgroup. On the other hand, it is not clear what is the relation between the holonomy reduction and the actual number of Killing spinors one gets in each case.

10. Geometric Structures on Manifolds and Supersymmetry

The intricate relationship between supersymmetry and geometric structures on manifolds was recognized along the way the physics of supersymmetry slowly evolved from its origins: first globally supersymmetric field theories (70ties) arose, later came supergravity theory (80ties), which evolved into superstring theory and conformal field theory (late 80ties and 90ties), and finally into M-Theory and the supersymmetric branes of today. At every step the "first" theory would quickly led to various generalizations creating many different new ones: so it is as if after discovering plain vanilla ice cream one would quickly find oneself in an Italian ice cream parlor confused and unable to decide which flavor was the right choice for the hot afternoon. This is a confusion that is possibly good for one's sense of taste, but many physicists believe that there should be just one theory, the Grand Unified Theory which describes our world at any level.[4] An interesting way out of this conundrum is to suggest that even if two theories appear to be completely different,

[4]Actually, string theory of today appears to offer a rather vast range of vacua (or possible universes). Such possible predictions have been nicknamed the *string landscape* [Sus03]. This fact has been seen as a drawback by some, but not all, physicists (see more recent discussion on

if both are consistent and admissible, they actually *do* describe the same physical world and, therefore, *they should be dual* to one another in a certain sense. This gave rise to various duality conjectures such as the Mirror Symmetry Conjecture or the AdS/CFT Duality Conjecture.

The first observation of how supersymmetry can restrict the underlying geometry was due to Zumino [Zum79] who discovered that globally $N = 1$ supersymmetric σ-models in $d = 4$ dimensions require that the bosonic fields (particles) of the theory are local coordinates on a Kähler manifold. Later Alvarez-Gaumé and Friedman observed that $N = 2$ supersymmetry requires that the σ-model manifold be not just Kähler but hyperkähler [AGF81]. This relation between globally supersymmetric σ-models and complex manifolds was used by Lindström and Roček to discover the hyperkähler quotient construction in [LR83, HKLR87].

The late seventies witnessed a series of attempts to incorporate gravity into the picture which quickly led to the discovery of various supergravity theories. Again the $N = 1$ supergravity-matter couplings in $d = 4$ dimensions require bosonic matter fields to be coordinates on a Kähler manifold with some special properties [WB82] while $N = 2$ supergravity demands that the σ-model manifold be quaternionic Kähler [BW83]. The quaternionic underpinnings of the matter couplings in supergravity theories lead to the discovery of quaternionic Kähler reduction in [Gal87, GL88].

At the same time manifolds with Killing spinors emerged as important players in the physics of the supergravity theory which in $D = 11$ dimensions was first predicted by Nahm [Nah78] and later constructed by Cremmer, Julia and Scherk [CJS79]. The well-known Kaluza-Klein trick applied to a $D = 11$ supergravity model is a way of constructing various limiting *compactifications* which would better describe the apparently four-dimensional physical world we observe. The geometry of such a compactification is simply a Cartesian product $\mathbb{R}^{3,1} \times M^7$, where $\mathbb{R}^{3,1}$ is the Minkowski space-time (or some other Lorentzian 4-manifold) and M^7 is a compact manifold with so small a radius that its presence can only be felt and observed at the quantum level. Many various models for M^7 were studied in the late seventies which by the eighties had already accrued into a vast physics literature (cf. the extensive three-volume monograph by Castellani, D'Auria and Fré [CDF91]). Most of the models assumed a homogeneous space structure on $M^7 = G/H$ (see Chapter V.6 in [CDF91], for examples). Two things were of key importance in terms of the required physical properties of the compactified theory. First, the compact space M^7, as a Riemannian manifold, had to be Einstein of positive scalar curvature. Second, although one could consider any compact Einstein space for the compactification, the new theory would no longer be supersymmetric unless (M^7, g) admitted Killing spinor fields, and the number of them would be exactly the number of *residual* supersymmetries of the compactified theory. For that reason compactification models involving (S^7, g_0) were quite special as they gave the maximally supersymmetric model. However, early on it was realized that there are other, even homogeneous, 7-manifolds of interest. The $Sp(2)$-invariant Jensen metric on S^7, or as physicists correctly nicknamed it, the *squashed 7-sphere*

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landscape and *swampland* in [Vaf05, OV06]). The insistence that the universe we experience, and this on such a limited scale at best, *is the only Universe*, is largely a matter of 'philosophical attitude' towards science. See the recent book of Leonard Susskind on the anthropic principle, string theory and the cosmic landscape [Sus05].

is one of the examples. Indeed, Jensen's metric admits exactly one Killing spinor field since it has a nearly parallel G_2 structure. Of course, any of the Einstein geometries in the table of Theorem 40 can be used to obtain such supersymmetric models.

The $D = 11$ supergravity theory only briefly looked liked it was the Grand Theory of Einstein's dream. It was soon realized that there are difficulties with getting from $D = 11$ supergravity to the standard model. The theory which was to solve these and other problems was Superstring Theory and later M-Theory (which is yet to be constructed). With the arrival of superstring theory and M-theory, supersymmetry continues its truly remarkable influence on many different areas of mathematics and physics: from geometry to analysis and number theory. For instance, once again five, six, and seven-dimensional manifolds admitting real Killing spinors have become of interest because of the so called AdS/CFT Duality. Such manifolds have emerged naturally in the context of p-brane solutions in superstring theory. These so-called p-branes, "near the horizon" are modelled by the pseudo-Riemannian geometry of the product $\mathrm{AdS}_{p+2} \times M$, where AdS_{p+2} is the $(p+2)$-dimensional anti-de-Sitter space (a Lorentzian version of a space of constant sectional curvature) and (M, g) is a Riemannian manifold of dimension $d = D - p - 2$. Here D is the dimension of the original supersymmetric theory. In the most interesting cases of M2-branes, M5-branes, and D3-branes D equals either 11 (Mp-branes of M-theory) or 10 (Dp-branes in type IIA or type IIB string theory). String theorists are particularly interested in those vacua of the form $\mathrm{AdS}_{p+2} \times M$ that preserve some residual supersymmetry. It turns out that this requirement imposes constraints on the geometry of the Einstein manifold M which is forced to admit real Killing spinors. Depending on the dimension d, the possible geometries of M are as follows:

d	Geometry of M	$(\mu, \bar{\mu})$
any	round sphere	$(1, 1)$
7	nearly parallel G_2	$(\frac{1}{8}, 0)$
	Sasaki–Einstein	$(\frac{1}{4}, 0)$
	3-Sasakian	$(\frac{3}{8}, 0)$
6	nearly Kähler	$(\frac{1}{8}, \frac{1}{8})$
5	Sasaki–Einstein	$(\frac{1}{4}, \frac{1}{4})$

where the notation $(\mu, \bar{\mu})$, which is common in the physics literature, represents the ratio of the number of real Killing spinors of type (p, q) to the maximal number of real Killing spinors that can occur in the given dimension. This maximum is, of course, realized by the round sphere of that dimension. So this table is just a translation of the table of Theorem 40 for the special dimensions that occur in the models used by the physicists.

Furthermore, given a p-brane solution of the above type, the interpolation between $\mathrm{AdS}_{p+2} \times M$ and $\mathbb{R}^{p,1} \times \mathcal{C}(M)$ leads to a conjectured duality between the supersymmetric background of the form $\mathrm{AdS}_{p+2} \times M$ and a $(p+1)$-dimensional superconformal field theory of n coincident p-branes located at the conical singularity of the $\mathbb{R}^{p,1} \times \mathcal{C}(M)$ vacuum. This is a generalized version of the Maldacena or AdS/CFT Conjecture [Mal99]. In the case of D3-branes of string theory the relevant near horizon geometry is that of $\mathrm{AdS}_5 \times M$, where M is a Sasaki-Einstein

5-manifold. The D3-brane solution interpolates between $\mathrm{AdS}_5\times M$ and $\mathbb{R}^{3,1}\times\mathcal{C}(M)$, where the cone $\mathcal{C}(M)$ is a Calabi-Yau threefold. In its original version the Maldacena conjecture (also known as AdS/CFT duality) states that the 't Hooft large n limit of $N=4$ supersymmetric Yang-Mills theory with gauge group $SU(n)$ is dual to type IIB superstring theory on $\mathrm{AdS}_5\times S^5$ [Mal99]. This conjecture was further examined by Klebanov and Witten [KW99] for the type IIB theory on $\mathrm{AdS}_5\times T^{1,1}$, where $T^{1,1}$ is the other homogeneous Sasaki-Einstein 5-manifold $T^{1,1}=S^2\times S^3$ and the Calabi-Yau 3-fold $\mathcal{C}(T^{1,1})$ is simply the quadric cone in $\mathbb{C}^4$. Using the well-known fact that $\mathcal{C}(T^{1,1})$ is a Kähler quotient of $\mathbb{C}^4$ (or, equivalently, that $S^2\times S^3$ is a Sasaki-Einstein quotient of S^7), a dual super Yang-Mills theory was proposed, representing D3-branes at the conical singularities. In the framework of D3-branes and the AdS/CFT duality the question of what are all the possible near horizon geometries M and $\mathcal{C}(M)$ might be of importance. Much of the interest in Sasaki-Einstein manifolds is precisely due to the fact that each such explicit metric, among other things, provides a useful model to test the AdS/CFT duality. We refer the reader interested in the mathematics and physics of the AdS/CFT duality to the recent book in the same series [Biq05]. In particular, in this context, Sasaki-Einstein geometry is discussed in one of the articles there [GMSW05].

Remark 79. [**G_2 holonomy manifolds unification scale and proton decay**] Until quite recently the interest in 7-manifolds with G_2 holonomy as a source of possible physical models was tempered by the fact the Kaluza-Klein compactifications on smooth and complete manifolds of this type led to models with no charged particles. All this has dramatically changed in the last few years largely because of some new developments in M-theory. Perhaps the most compelling reasons for reconsidering such 7-manifolds was offered by Atiyah and Witten who considered the dynamics on manifolds with G_2 holonomy which are asymptotically conical [AW02]. The three models of cones on the homogeneous nearly Kähler manifolds mentioned earlier are of particular interest, but Atiyah and Witten consider other cases which include orbifold (quotient) singularities. Among other things they point to a very interesting connection between Kronheimer's quotient construction of the ALE metrics [Kro89a, Kro89b] and asymptotically conical manifolds with G_2-holonomy. To explain the connection, consider Kronheimer's construction for $\Gamma=\mathbb{Z}_{n+1}$. Suppose one chooses a circle $S^1_{k,l}\simeq \mathrm{U}(1)\in K(\mathbb{Z}_{n+1})=\mathrm{U}(1)^n$ and then one considers a 7-manifold obtained by performing Kronheimer's HK quotient construction with zero momentum level ($\boldsymbol{\xi}=0$) while "forgetting" the three moment map equations corresponding to this particular circle. An equivalent way of looking at this situation is to take the Kronheimer quotient with nonzero momentum $\boldsymbol{\xi}=a\in\mathfrak{sp}(1)$ but only for the moment map of the chosen circle $S^1_{k,l}$ (such $\boldsymbol{\xi}$ is never in the "good set") and then consider the fibration of singular Kronheimer quotients over a 3-dimensional base parameter space. Algebraically this corresponds to a partial resolution of the quotient singularity and this resolution depends on the choice of $S^1_{k,l}$, hence $\boldsymbol{\xi}$. This example was first introduced in [AW02]. It can be shown that the 7-manifold is actually a cone on the complex weighted projective 3-space with weights (k,k,l,l), where $k+l=n+1$. It then follows from the physical model considered that such a cone should admit a metric with G_2 holonomy. However, unlike the homogeneous cones over the four homogeneous strict nearly Kähler manifolds of Theorem 62, the metric in this case is not known explicitly. This construction appears to differ from all previous geometric constructions of metrics with G_2 holonomy. One can consider similar constructions for other choices of $S^1 \subset K(\Gamma)$ [BB02].

In [FW03] using a specific models of M-theory compactifications on manifolds with G_2 holonomy, Friedman and Witten address the fundamental questions concerning the unification scale (*i.e.*, the scale at which the Standard Model of $SU(3) \times SU(2) \times U(1)$ unifies in a single gauge group) and proton decay. The authors point out that the results obtained are model dependent, but some of the calculations and conclusions apply to a variety of different models.

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Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131.
E-mail address: cboyer@math.unm.edu
E-mail address: galicki@math.unm.edu