Einstein Manifolds via Contact Geometry

My talk is based on joint work with Krzysztof Galicki.

We consider a Riemannian manifold (M, g). On any manifold a linear connection ∇ gives a way of comparing tangent vectors at different points through the notion of parallel transport.

The First Fundamental Theorem of Riemannian geometry says that there exists a unique torsionfree connection that leaves the metric tensor g invariant, i.e. $\nabla g = 0$.

The Riemann curvature tensor R measures the "curvature" of (M, g). It is given by

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$$

where X, Y, Z are vector fields on M.

Two important irreducible components of the Riemann curvature tensor R are: the Ricci curvature

$$\mathrm{Ric}(X,Y) = \mathrm{Tr}\ (Z \mapsto R(X,Z)Y)$$

which is a symmetric tensor field and the scalar curvature

$$s_g = \mathrm{Tr}\ \mathrm{Ric}$$

which is a function on M.

Important Question (R. Thom): Is there a distinquished class or best class of Riemannian manifolds?

Answer: Perhaps the answer is the class of:

Einstein manifolds.

DEFINITION: *A Riemannian manifold* (M, g) *is called an Einstein manifold if there exists a smooth function* λ *on* M *such that*

$$Ric = \lambda g.$$

The Bianchi identities imply that λ is actually a constant, called the *Einstein constant.* So there are 3 cases

1. $\lambda > 0$.
2. $\lambda = 0$.
3. $\lambda < 0$.

There are two reasons why one might believe that the above answer is correct.

1. Einstein manifolds are the critical points of the "Hilbert Action" given by the total scalar curvature
$$S(g) = \int s_g \mu_g$$
where the measure μ_g is normalized to volume one, i.e. $\int \mu_g = 1$.
2. The class of Einstein manifolds is a natural generalization of the class of Riemannian manifolds of constant curvature. For dimension 2 the well known Uniformization Theorem says that every orientable surface admits a metric with curvature either $1, 0$, or -1. These are automatically Einstein. In higher dimensions the condition of constant curvature is too strong, and the Einstein condition is a natural generalization.

Here we shall be interested in the case $\lambda > 0$ in which case we refer to (M, g) as a *positive* Einstein manifold. Then we have the well-known Myers' Theorem which implies any complete positive Einstein manifold (M, g) is compact with finite fundamental group.

Our purpose is the construct explicit examples of positive Einstein manifolds and discuss their topology. In order to do this, we employ two constructions:

1. The method of symmetry reduction.
2. A topological monoid based on a construction of Wang and Ziller.

In order to get a handle on constructing explicit examples of Einstein manifolds, we introduce some additional structure, namely a contact structure.

DEFINITION: *A smooth orientable manifold M is called a contact manifold if there exists on M a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. A contact structure on M is then given by an equivalence class of such 1-forms, where $\eta' \sim \eta$ if there exists a nowhere vanishing function f on M such that $\eta' = f\eta$. Equivalently, a contact structure can be defined as a maximally nonintegrable codimension one subbundle $\mathcal{D}$ of the tangent bundle TM, known as the contact distribution.*

Contact manifolds are necessarily odd dimensional and η is called a *characteristic* or *contact* form. The relation between the two equivalent definitions is $\mathcal{D} = \ker\eta$ which is independent of the choice of 1-form η defining the contact structure. However, for our purposes it will be convenient to fix a contact 1-form η. When this is done there is a nowhere vanishing vector field ξ on M called the *characteristic* or *Reeb* vector field defined by the equations

$$\eta(\xi) = 1, \qquad \xi \rfloor d\eta = 0.$$

Since it is nowhere vanishing ξ defines a foliation $\mathcal{F}$ on M called the *characteristic foliation*. The nondegenerate 2-form $d\eta$ defines a symplectic structure on the subbundle $\mathcal{D}$. Contact geometry is the odd dimensional analogue of symplectic geometry.

In order to make the connection with Riemannian geometry, we put an almost complex structure on the vector bundle $\mathcal{D}$. More explicitly, let Φ be a smooth section of the endomorphism bundle End M such that

$$\Phi\xi = 0, \qquad \Phi^2 = -\mathbb{I} + \eta \otimes \xi.$$

This gives M the structure of an almost CR manifold. Then a Riemannian metric g on M is said to be *compatible* with the almost CR structure if

$$g \circ (\Phi \otimes \Phi) = g - \eta \otimes \eta.$$

Such compatible Riemannian metrics always exist. A case of particular importance is when the almost CR structure is integrable, that is Φ defines a CR structure on the subbundle $\mathcal{D}$. The condition for this integrability is:

$$[\Phi, \Phi] + 2d\eta \otimes \xi = 0$$

where

$$[\Phi, \Phi](X, Y) = [\Phi X, \Phi Y] + \Phi^2[X, Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y]$$

defines the Nijenhuis bracket. In the literature this condition is often referred to as *normality*.

DEFINITION: *A normal contact manifold with a compatible metric* (M, g, η, ξ, Φ) *is called a Sasakian manifold.*

We have the following analogies of geometries:

$$\text{CONTACT} \longleftrightarrow \text{SYMPLECTIC}$$

$$\text{SASAKIAN} \longleftrightarrow \text{KÄHLERIAN}$$

Sasakian geometry was defined by Sasaki and was further developed by the Japaneese school during the 1960's and 1970's. Actually there are many equivalent ways to characterize Sasakian manifolds and each has its own advantage. One particularly attractive characterization is:

PROPOSTION: *A Riemannian manifold (M, g) is Sasakian if and only if the holonomy group of the Riemannian cone on $\mathcal{S}$,*

$$(C(\mathcal{S}), \bar{g}) = (\mathbb{R}_+ \times \mathcal{S},\ dr^2 + r^2 g)$$

reduces to a subgroup of $U(n+1)$, where dim $\mathcal{S} = 2n+1$, that is, $(C(\mathcal{S}), \bar{g})$ is Kähler.

This is completely analogous to the weaker case of contact geometry. A manifold M is a contact manifold with contact form η if and only if the cone $C(M) = M \times \mathbb{R}^+$ is symplectic with symplectic form $dr \wedge \eta + r d\eta$, where r is the coordinate on $\mathbb{R}^+$.

DEFINITION: *A Sasakian manifold $(\mathcal{S}, g)$ is called Sasakian-Einstein if g is also Einstein.*

Proposition: *A Riemannian manifold of dimension $2n+1$ is Sasakian-Einstein if and only if the restricted holonomy group of the Riemannian cone is contained in $SU(n+1)$, that is. $(C(\mathcal{S}), dr^2 + r^2 g)$ is Calabi-Yau.*

On a Sasakian manifold the Ricci tensor satisfies

$$Ric(X, \xi) = 2n\eta(X).$$

It follows immediately that an Sasakian-Einstein manifold necessarily has positive scalar curvature equal to $2n(2n+1)$ and that any complete Sasakian-Einstein manifold is compact with finite fundamental group.

Let $(\mathcal{S}, g, \eta)$ be a Sasakian manifold, and consider the characteristic foliation $\mathcal{F}$ on $\mathcal{S}$. We are interested in the case that all the leaves of $\mathcal{F}$ are compact. The assumption that all leaves are compact is equivalent to the assumption that $\mathcal{F}$ is *quasi-regular*, i.e. each point $p \in \mathcal{S}$ has a cubical neighborhood U such that any leaf $\mathcal{L}$ of $\mathcal{F}$ intersects a transversal through p at most a finite number of times $N(p)$. Furthermore, $\mathcal{S}$ is called *regular* if $N(p) = 1$ for all $p \in \mathcal{S}$. In this case, the foliation $\mathcal{F}$ is simple, and defines a global submersion. In the quasi-regular case it is well-known that ξ generates a locally free circle action on $\mathcal{S}$, and that the space of leaves is a compact *orbifold* (or V-manifold). We shall denote the space of leaves of the foliation $\mathcal{F}$ on $\mathcal{S}$ by $\mathcal{Z}$. Then the natural projection $\pi : \mathcal{S} \longrightarrow \mathcal{Z}$ is an orbifold submersion and a Siefert fibration. Actually, much more is true as we show below.

Recall that an orbifold or V-manifold (Satake) is very similar to a manifold except that the local Euclidean condition is replaced by neighborhoods U that are homeomorphic to $\mathbb{R}^n/\Gamma$ where Γ is a finite group that can always be taken to be a subgroup of $O(n)$. Furthermore, Γ can be taken to be the identity on a dense open set. Then (U, Γ) together with the homeomorphism is called a *local uniformizing system*. Much of the standard differential geometry on manifolds carries over to the orbifold category.

Structure Theorem (B-Galicki): *Let $(\mathcal{S}, g)$ be a complete quasi-regular Sasakian-Einstein manifold of dimension $2n+1$. Then*

(i) The leaf space $\mathcal{Z}$ is a compact complex orbifold with a Kähler-Einstein metric h of positive scalar curvature $4n(n+1)$ in such a way that $\pi : (\mathcal{S}, g) \longrightarrow (\mathcal{Z}, h)$ is an orbifold Riemannian submersion. The fibres of π are totally geodesic submanifolds of $\mathcal{S}$ diffeomorphic to S^1.

(ii) $\mathcal{Z}$ is also a simply connected normal projective algebraic variety. In fact, it is a $\mathbb{Q}$-factorial Fano variety. Hence, it is uniruled with Kodaira dimension $\kappa(\mathcal{Z}) = -\infty$.

Recall that a *Fano variety* means that the anti-canonical sheaf K^{-1} is ample, $\mathbb{Q}$-*factorial* means that for every Weil divisor D on $\mathcal{Z}$ there is an integer m such that mD is a Cartier divisor.

Since an orbifold fibration is not a fibration in the usual sense, the usual techniques in topology for fibrations do not apply directly. However, Haefliger [Hae] has defined orbifold homology, cohomology, and homotopy groups which do have an analogue in the standard theory. Let X be an orbifold of dimension n, and let P denote the bundle of orthonormal frames on X. It is a smooth manifold on which the orthogonal group $O(n)$ acts locally freely with the quotient X. Let $EO(n) \longrightarrow BO(n)$ denote the universal $O(n)$ bundle. Consider the diagonal action of $O(n)$ on $EO(n) \times P$ and denote the quotient by BX. Now there is a natural projection $p : BX \longrightarrow X$ with generic fiber the contractible space $EO(n)$.

Haefliger defines the orbifold cohomology, homology, and homotopy groups by

$$H^i_{orb}(X, \mathbb{Z}) = H^i(BX, \mathbb{Z}), \quad H_i^{orb}(X, \mathbb{Z}) = H_i(BX, \mathbb{Z}),$$

$$\pi_i^{orb}(X) = \pi_i(BX).$$

This definition of π_1^{orb} is equivalent to Thurston's better known definition in terms of orbifold deck transformations, and when X is a smooth manifold these orbifold groups coincide with the usual groups. Using the Leray spectral sequence one easily sees that the map $p : BX \longrightarrow X$ induces an isomorphism $H^i_{orb}(\mathcal{S}, \mathbb{Z}) \otimes \mathbb{Q} \simeq H^i(\mathcal{S}, \mathbb{Z}) \otimes \mathbb{Q}$.

Definition: *Let $c_1(\mathcal{L})$ denote the first Chern class of a line V-bundle $\mathcal{L}$. The index of a Fano orbifold $\mathcal{Z}$ is the largest positive integer m such that $\frac{c_1(K_{\mathcal{Z}}^{-1})}{m}$ is an element of $H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. The index of $\mathcal{Z}$ is denoted by Ind$(\mathcal{Z})$.*

Definition: *The index of a complete Sasakian-Einstein manifold $\mathcal{S}$ is defined to be the index of $\mathcal{Z}$ if $\mathcal{S}$ is quasi-regular, and 0 if $\mathcal{S}$ is not quasi-regular. The index is denoted by Ind$(\mathcal{S})$.*

The index $\mathrm{Ind}(\mathcal{S})$ is an invariant of the Sasakian structure.

We now have:

Inversion Theorem (B-Galicki): *Let $\mathcal{Z}$ be a compact Fano orbifold of complex dimension n with $\pi_1^{orb}(\mathcal{Z}) = 0$ and a Kähler-Einstein metric h of scalar curvature $4n(n+1)$. Let $\mathcal{S}$ denote the total space of the S^1 V-bundle on $\mathcal{Z}$ whose first Chern class is $\frac{c_1(\mathcal{Z})}{\mathrm{Ind}(\mathcal{Z})}$. Then $(\mathcal{S}, g)$ is a Sasakian-Einstein orbifold with $g = \pi^* h + \eta \otimes \eta$ and where η is the connection 1-form in the V-bundle $\pi : \mathcal{S} \longrightarrow \mathcal{Z}$ whose curvature is the Kähler 2-form on $\mathcal{Z}$. Furthermore, if the local uniformizing groups Γ_i inject into the group S^1 of the bundle $\mathcal{S}$ is a simply connected smooth manifold.*

Remark: It is the orbifold structure of $\mathcal{Z}$ that is important here. There are examples of distinct Sasakian structures which are S^1 V-bundles over distinct Fano orbifolds which are isomorphic as (singular) projective algebraic varieties.

In the regular case when $\mathcal{Z}$ is a smooth manifold this theorem is due to Baum, Friedrich, Grunewald, and Kath. Using these results we can easily determine the homogeneous Sasakian-Einstein manifolds:

Theorem (B-Galicki): *Let $\mathcal{S}$ be a quasi-regular homogeneous Sasakian-Einstein manifold. Then $\mathcal{S}$ is an S^1-bundle over a generalized flag manifold G/P. Conversely, given any generalized flag manifold G/P there is a circle bundle $\pi : \mathcal{S}{\longrightarrow}G/P$ whose total space $\mathcal{S}$ is a homogeneous Sasakian-Einstein manifold.*

An important special case is what is known as a *3-Sasakian structure.*

Definition: *Let $(\mathcal{S}, g)$ be a Riemannian manifold of real dimension m. We say that $(\mathcal{S}, g)$ is 3-Sasakian if the holonomy group of the Riemannian cone on $\mathcal{S}$ reduces to a subgroup of $Sp(\frac{m+1}{4})$. In particular, $m = 4n + 3, n \geq 1$ and $(C(\mathcal{S}), \bar{g})$ is hyperkähler.*

3-Sasakian manifolds are manifolds with a certain metric quaternionic contact structure. They were first defined independently by Kuo and Udriste around 1970 using tensorial methods. Essentially there is a triple of Sasakian structures which satisfy certain quaternionic-like relations. Shortly thereafter, Kashiwada proved that every 3-Sasakian manifold is Einstein with positive scalar curvature. Other important properties of 3-Sasakian manifolds were obtained early to mid 1970's by Ishihara, Konishi, Sasaki, Tachibana, Tanno, and Yu. Nevertheless, the only examples known until fairly recently were spheres and their quotients. Recently the authors together with B. Mann and E. Rees discovered many infinite sequences of 3-Sasakian manifolds with highly non-trivial topology.

Proposition: *Let $(\mathcal{S}, g)$ be a Riemannian manifold. Then $\mathcal{S}$ is 3-Sasakian if and only if it admits three characteristic vector fields $\{\xi^1, \xi^2, \xi^3\}$ of 3 different Sasakian structures such that $g(\xi^a, \xi^b) = \delta_{ab}$ and $[\xi^a, \xi^b] = 2\epsilon_{abc}\xi^c$.*

From the point of view of the holonomy group of the cone we see that since $Sp(n+1) \subset SU(2n+2)$ a 3-Sasakian manifold is automatically Sasakian-Einstein. A 3-Sasakian manifold necessarily has relatively high index due to the existence of a complex contact structure on the space of leaves $\mathcal{Z}$ of the characteristic foliation. (Here there are 3 characteristic foliations, but they are easily seen to be equivalent).

Theorem (B-Galicki): *The index $Ind(\mathcal{S})$ of a 3-Sasakian manifold of dimension $4n+3$ is either $n+1$ or $2n+2$ and equals $2n+2$ if and only if the universal cover $\tilde{\mathcal{S}}$ of $\mathcal{S}$ is a sphere S^{4n+3}.*

There are topological obstructions for a manifold of dimension $4n+3$ to admit a 3-Sasakian structure, namely the odd Betti numbers below the middle dimension must vanish (Galicki-Salamon). Furthermore, there are examples of Sasakian-Einstein manifolds of dimension $4n+3$ that cannot admit any 3-Sasakian structure (B-Galicki). Examples in dimension 7 are given by certain circle bundles over Fermat hypersurfaces of degrees 3 or 4 in $\mathbb{CP}^4$. These have nonvanishing b_3.

The Quotient Construction

The quotient construction originated in symplectic geometry, but there is a contact version as well. Let $\mathfrak{G}$ be a Lie group which acts effectively on a contact manifold (M, η) and leaves the contact 1-form η invariant. Then there is a moment map

$$\mu : M \longrightarrow \mathfrak{g}^*$$

defined by

$$< \mu, \tau >= \eta(X^\tau)$$

where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$, and X^τ is the vector field on M corresponding to $\tau \in \mathfrak{g}$. Then we have

THEOREM: *Let $M, \eta, \mathfrak{G}, \mu$ be as above. Then if* 0 *is a regular value of the moment map μ and $\mathfrak{G}$ acts freely on the submanifold $\mu^{-1}(0) \stackrel{\iota}{\longrightarrow} M$ the quotient $\bar{M} = \mu^{-1}(0)/\mathfrak{G}$ is a contact manifold with a fixed contact 1-form $\bar{\eta}$ which satisfies $\iota^*\eta = \pi^*\bar{\eta}$ where π is the natural projection. Furthermore, if (M, η, g) is Sasakian then $(\bar{M}, \bar{\eta}, \bar{g})$ is Sasakian where $\bar{g}$ satisfies $\iota^* g = \pi^* \bar{g}$.*

The famous convexity theorem of Atiyah, Guillemin-Sternberg also generalizes to the contact case.

THEOREM [B-G]: *Let (M, η) be a compact contact manifold, and let μ be the moment map of a torus $\mathfrak{T}$ acting on M which preserves the contact form η. Then the image $\mu(M)$ is a convex compact polytope lying in the characteristic hyperplane defined by $\eta(\xi) = 1$.*

The Einstein condition is harder to gaurentee. It is not true in general that the moment map reduction of a Sasakian-Einstein manifold is an Sasakian-Einstein manifold. There is however one case where the Einstein condition is automatically satisfied under reduction, the 3-Sasakian reduction. This is related to the hyperkähler quotient of

Hitchin, Karlhede, Linström, and Roček. However, it has a distinct advantage over the hyperkähler case, in that we often obtain smooth compact quotients. Let $(\mathcal{S}, \eta^a, g)$ be a 3-Sasakian manifold with a nontrivial group $\mathfrak{G}$ of automorphisms of the 3-Sasakian structure. In this case the moment map

$$\mu : \mathcal{S} \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

is defined by

$$< \mu, \tau >= \sum_{a=1}^{3} \eta^a(X^\tau) e_a$$

where e_1, e_2, e_3 denotes the standard basis of $\mathbb{R}^3$.

Theorem (B-G-M): *Under the above conditions assume that* 0 *is a regular value of* μ *and that* $\mathfrak{G}$ *acts freely on the submanifold* $\mu_{\mathcal{S}}^{-1}(0) \xrightarrow{\iota} \mathcal{S}$. *Then the quotient* $(\bar{\mathcal{S}}, \bar{\eta^a}, \bar{g})$ *is a 3-Sasakian manifold where as previously* $\iota^*\eta^a = \pi^*\bar{\eta^a}$ *and* $\iota^* g = \pi^* \bar{g}$.

As we shall see this theorem is a powerful tool for constructing smooth compact Einstein manifolds of dimension $4n + 3$.

3-Sasakian Toric Manifolds

We can use the 3-Sasakian reduction theorem to obtain new 3-Sasakian manifolds by toral reductions from the 3-Sasakian sphere S^{4n+3}. On S^{4n+3} the group $I_0(\mathcal{S}, g)$ of 3-Sasakian automorphisms is $Sp(n+1)$. We choose a k-dimensional subtorus T^k in the maximal torus T^n. Every quaternionic representation of a k-torus T^k on $\mathbb{H}^{n+1}$ can be described via a homomorphism $f_\Omega : T^k \longrightarrow T^{n+1}$

$$f_\Omega(\tau_1, \ldots, \tau_k) = \begin{pmatrix} \prod_{i=1}^k \tau_i^{a_1^i} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \prod_{i=1}^k \tau_i^{a_{n+1}^i} \end{pmatrix},$$

where $(\tau_1, .., \tau_k) \in S^1 \times \cdots \times S^1 = T^k$ are the complex coordinates on T^k, and $a_j^i \in \mathbb{Z}$ are the coefficients of a $k \times (n+1)$ integral *weight* matrix $\Omega = (a_\alpha^l)_{\alpha=1,\ldots,n+1}^{l=1,\ldots,k} \in \mathcal{M}_{k,n+1}(\mathbb{Z})$.

Let $\{e_l\}_{l=1}^k$ denote the standard basis for $\mathfrak{t}_k^* \simeq \mathbb{R}^k$. Then the 3-Sasakian moment map $\mu_\Omega : S^{4n+3} \longrightarrow \mathfrak{t}_k^* \otimes \mathbb{R}^3$ of the k-torus action defined by $\varphi_{(\tau_1,\ldots,\tau_k)}(\mathbf{u}) = f_\Omega(\tau_1, \ldots, \tau_k)\mathbf{u}$, is given by $\mu_\Omega = \sum_l \mu_\Omega^l e_l$ with

$$\mu_\Omega^l(\mathbf{u}) = \sum_\alpha \bar{u}_\alpha i a_\alpha^l u_\alpha.$$

Let us denote further denote the triple $(T^k, f_\Omega, \varphi_{(\tau_1,\ldots,\tau_k)})$ by $T^k(\Omega)$. We want to study the reduced space

$$\mathcal{S}(\Omega) = S^{4n+3} /\!/\!/ T^k(\Omega) = \mu_\Omega^{-1}(0)/T^k(\Omega)$$

.

Consider the $\binom{n}{k}$ minor determinants

$$\Delta_{\alpha_1 \ldots \alpha_k} = \det \begin{pmatrix} a_{\alpha_1}^1 & \ldots & a_{\alpha_k}^1 \\ \vdots & & \vdots \\ a_{\alpha_1}^k & \ldots & a_{\alpha_k}^k \end{pmatrix}$$

obtained by deleting $n+1-k$ columns of Ω.

DEFINITION: *Let $\Omega \in \mathcal{M}_{k,n+1}(\mathbb{Z})$ be the weight matrix.*

(i) If $\Delta_{\alpha_1\cdots\cdots\alpha_k} \neq 0, \forall \quad 1 \leq \alpha_1 < \cdots < \alpha_k \leq n+1$, then we say that Ω is non-degenerate.

Suppose Ω is non-degenerate and let g be the kth determinantal divisor, i.e., the gcd of all the k by k minor determinants $\Delta_{\alpha_1\cdots\cdots\alpha_k}$. Then Ω is said to be admissible if in addition we have

(ii) $\gcd(\Delta_{\alpha_2\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\hat{\alpha}_s\cdots\alpha_{k+1}}, \ldots, \Delta_{\alpha_1\cdots\alpha_k}) = g$ for all sequences of length (k+1) such that $1 \leq \alpha_1 < \cdots < \alpha_s < \cdots < \alpha_{k+1} \leq n+1$.

THEOREM (B-G-M): *Let $\mathcal{S}(\Omega)$ be a 3-Sasakian toral quotient space. Then*

(i) if Ω is non-degenerate, $\mathcal{S}(\Omega)$ is an orbifold.

(ii) If Ω is degenerate, then either $\mathcal{S}(\Omega)$ is a singular stratified manifold which is not an orbifold or it is an orbifold obtained by reduction of a lower dimensional sphere $S^{4n-4r-1}$ by a torus $T^{k-r}(\Omega')$ or a finite quotient of such, where $1 \leq r \leq k$ and Ω' is non-degenerate. (When $r = k$ the quotient is the sphere $S^{4n-4k-1}$).

(iii) Assuming that Ω is non-degenerate $\mathcal{S}(\Omega)$ is a smooth manifold if and only if Ω is admissible.

When can we find admissible Ω?

Generally this question is non-trivial. Admissible Ω clearly exist for $k = 1$ for any n, i.e. any dimension by simply choosing the elements of Ω to be pairwise relatively prime. They also exist in dimension 7 for any k as the following Ω shows:

$$\Omega = \begin{pmatrix} 1 & 0 & \dots & 0 & a_1 & b_1 \\ 0 & 1 & \dots & 0 & a_2 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_k & b_k \end{pmatrix}.$$

We simply choose (a_i, b_i) to be relatively prime, and if for some pair i, j $a_i = \pm a_j$ or $b_i = \pm b_j$ then we must have $b_i \neq \pm b_j$ or $a_i \neq \pm a_j$, respectively.

However, there are mod 2 obstructions to smoothness which show somewhat surprisingly:

THEOREM (B-G-M): *Let Ω be admissible and $k > 1$. Then dim $\mathcal{S}(\Omega) \leq 15$. Furthermore, if $k > 4$, then dim $\mathcal{S}(\Omega) = 7$.*

This says that there are **smooth** toral quotients $\mathcal{S}(\Omega)$ only in the cases indicated, and we shall see shortly that k is precisely the second Betti number $b_2(\mathcal{S}(\Omega))$. Moreover, a theorem of R. Bielawski says that any 3-Sasakian manifold $(\mathcal{S}, g)$ whose automorphism group $I_0(\mathcal{S}, g)$ contains a torus T^{n+1-k} is homeomorphic to a $\mathcal{S}(\Omega)$.

Furthermore, we have an existence theorem:

THEOREM (B-G-M-Rees,B-G-M): *There exist smooth toric quotients $\mathcal{S}(\Omega)$ in dimension 7 with any second Betti number $b_2(\mathcal{S}(\Omega)) = k$, and in dimensions 11 and 15 with second Betti numbers $b_2(\mathcal{S}(\Omega)) = 2, 3,$ and 4.*

The Topology of $\mathcal{S}(\Omega)$

The full isometry group of $\mathcal{S}(\Omega)$ is $T^{n+1-k} \times Sp(1)$ and in the case dim $\mathcal{S}(\Omega) = 7$ one can obtain a stratification of $\mathcal{S}$ by analysing the quotient $Q(\Omega) = \mathcal{S}(\Omega)/(T^2 \times Sp(1)$. For general dimension Bielawski has obtained the rational homology of $\mathcal{S}(\Omega)$ by viewing $\mathcal{S}(\Omega)$ as a boundary at infinity of a hyperkähler orbifold that is the limit of certain toric hyperkähler orbifolds studied by Bielawski and Dancer.

The orbit space $Q(\Omega)$ can be stratified as follows:

$$Q(\Omega) = Q_0(\Omega) \sqcup Q_1(\Omega) \sqcup Q_2(\Omega)$$

and:

LEMMA (B-G-M-R): *Let dim $\mathcal{S}(\Omega) = 7$. Then*

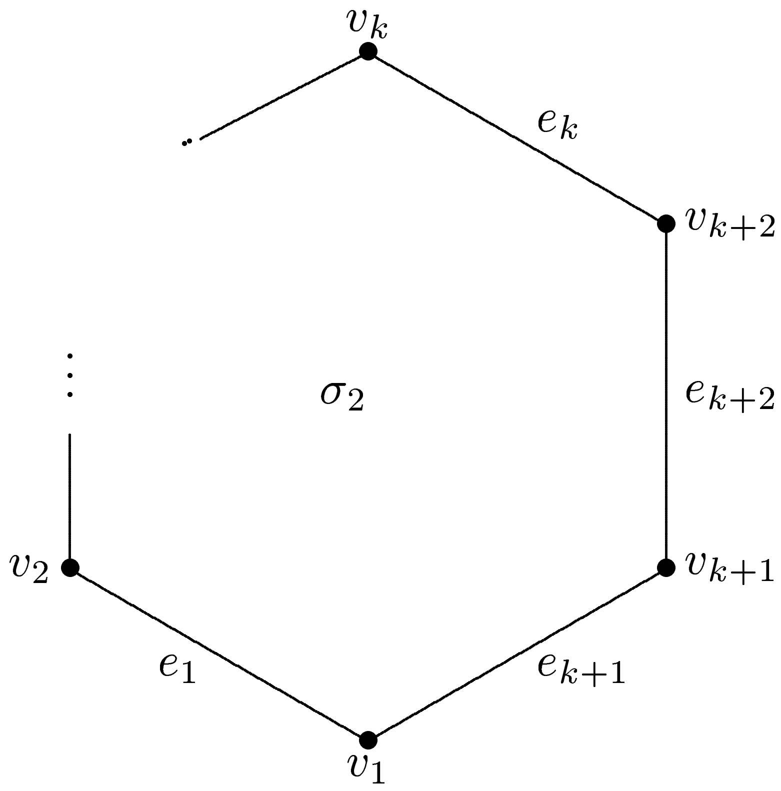
(i) The orbit space $Q(\Omega)$ is homeomorphic to the closed disc $\bar{D}^2$, and the subset of singular orbits $Q_1(\Omega) \sqcup Q_0(\Omega)$ is homeomorphic to the boundary $\partial\bar{D}^2 \simeq S^1$.

(ii) $Q_2(\Omega)$ is homeomorphic to the open disc D^2.

(iii) $Q_1(\Omega)$ is homeomorphic to the disjoint union of $k+2$ copies of the open unit interval.

(iv) $Q_0(\Omega)$ is a set of $k+2$ points.

Hence, the quotient $Q(\Omega)$ can be depicted by the polygon:



The orbit space $Q(\Omega)$

The topology of $\mathcal{S}(\Omega)$ can then be analysed by using a Leray spectral sequence on the induced stratification of $\mathcal{S}(\Omega)$. This gives

THEOREM (B-G-M-R,B-G-M): *Let $\mathcal{S}(\Omega)$ be a smooth 3-Sasakian toric quotient of dimension 7. Then $\mathcal{S}(\Omega)$ is simply connected and $H_2(\mathcal{S}(\Omega),\mathbb{Z}) \simeq \mathbb{Z}^k$ where $k = b_2(\mathcal{S}(\Omega))$. Furthermore, the rational homology is given by*

$$b_i(\mathcal{S}(\Omega)) = \begin{cases} 1 & \text{if } i = 0,7; \\ 0 & \text{if } i = 1,3,4,6; \\ k & \text{if } i = 2,5. \end{cases}$$

There is certainly torsion in $H_3(\mathcal{S}(\Omega),\mathbb{Z})$, but we have only been able to determine this in the case $b_2(\mathcal{S}(\Omega))=k=1$. In this case we have determined the complete cohomology ring in all dimensions. Here, we have $\Omega=\mathbf{p}=(p_1,\cdots,p_n)$ with the p_i's relatively prime. Then

Theorem (B-G-M): *Let $\mathcal{S}(\mathbf{p})$ be a 3-Sasakian manifold with $b_2=k=1$ obtained by a circle reduction from S^{4n+3}. Then, as rings,*

$$H^*\big(\mathcal{S}(\mathbf{p}),\mathbb{Z}\big) \;\cong\; \left(\frac{\mathbb{Z}[e_2]}{[e_2^{n+1}=0]}\otimes E[f_{2n+1}]\right)/\mathcal{R}(\mathbf{p}).$$

Here e_2 and f_{2n+1} are generators with the subscripts indicating the cohomological dimension of each generator. Furthermore, the relations $\mathcal{R}(\mathbf{p})$ are generated by $\sigma_n(\mathbf{p})e_2^n=0$ and $f_{2n+1}e_2^n=0$, where $\sigma_n(\mathbf{p})=\sum_{j=1}^{n+1}p_1\cdots\hat{p_j}\cdots p_{n+1}$ is the $(n)^{th}$ elementary symmetric polynomial in the entries of $\mathbf{p}$.

Notice this theorem shows that $H^{2n}(\mathcal{S}(\mathbf{p});\mathbb{Z}) \;=\; \mathbb{Z}_{\sigma_n(\mathbf{p})}$.

In more general dimension there is:

Theorem (Bielawski): *Let $\mathcal{S}(\Omega)$ is a toric 3-Sasakian orbifold of dimension $4(n-k)+3$. Then we have*

$$b_{2i}(\mathcal{S}(\Omega))=\binom{k+i-1}{k},\qquad i<n+1-k.$$

The Sasakian-Einstein Monoid

Let $\mathcal{S}_1, \mathcal{S}_2$ be two compact simply connected (i.e. $\pi_1^{orb} = 0$) quasi-regular Sasakian-Einstein orbifolds, and let $\mathcal{Z}_1, \mathcal{Z}_2$ be the associated leaf spaces of their characteristic foliations. Now $\mathcal{Z}_i$ are complex Fano orbifolds with Kähler-Einstein metrics h_i having integral Kähler forms in $H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. The orbifold product $(\mathcal{Z}_1 \times \mathcal{Z}_2, h_1 + h_2)$ is Kähler, but not Einstein. However, one can rescale the metrics to obtain a Kähler-Einstein metric h on $\mathcal{Z}_1 \times \mathcal{Z}_2$ with scalar curvature equal to $4n(n+1)$ where $n = n_1 + n_2$, the sum of dimensions of $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Thus, by our Inversion Theorem the total space of the V-bundle on $\mathcal{Z}_1 \times \mathcal{Z}_2$ whose first Chern class is

$$\frac{c_1(\mathcal{Z}_1) + c_1(\mathcal{Z}_2)}{\gcd(\mathrm{Ind}(\mathcal{Z}_1), \mathrm{Ind}(\mathcal{Z}_2))}$$

is a simply connected Sasakian-Einstein orbifold. We are interested in obtaining smooth Sasakian-Einstein manifolds. For this we define the *relative indices* of the pair $(\mathcal{S}_1, \mathcal{S}_2)$ by

$$l_i = \frac{\mathrm{Ind}(\mathcal{S}_i)}{\gcd(\mathrm{Ind}(\mathcal{S}_1), \mathrm{Ind}(\mathcal{S}_2))},$$

and the *order* of $\mathcal{S}_i$ to be the order of $\mathcal{Z}_i$ as an orbifold, that is the least common multiple of the orders of all the local uniformizing groups of $\mathcal{Z}_i$. Summarizing we have

Theorem [B-G]: *Let $\mathcal{S}_1, \mathcal{S}_2$ be two simply connected quasi-regular Sasakian-Einstein orbifolds of dimensions $2n_1+1$ and $2n_2+1$, respectively. Let $\mathcal{S}_i$ have orders m_i and relative indices l_i. Then there exists a multiplication called the join and denoted by $\star$ such that $\mathcal{S}_1 \star \mathcal{S}_2$ is a simply connected quasi-regular Sasakian-Einstein orbifold of dimension $2(n_1+n_2)+1$. If both $\mathcal{S}_1$ and $\mathcal{S}_2$ are smooth Sasakian-Einstein manifolds then $\mathcal{S}_1 \star \mathcal{S}_2$ is a smooth manifold if and only if* $\gcd(m_1 l_2, m_2 l_1) = 1$. *Furthermore, to each such pair there is a two-dimensional lattice of Einstein orbifolds each having the same rational cohomology as $\mathcal{S}_1 \star \mathcal{S}_2$.*

Using a simple and elegant spectral sequence argument employed by Wang and Ziller we are able to compute the cohomology rings of many examples of the joins of Sasakian-Einstein manifolds. Let $\mathcal{S}_1, \mathcal{S}_2$ be compact simply connected Sasakian-Einstein manifolds and consider the commutative diagram of fibrations

$$\begin{tikzcd}
\mathcal{S}_1 \times \mathcal{S}_2 \arrow[r] \arrow[d, "="] & B(\mathcal{S}_1 \star \mathcal{S}_2) \arrow[r] \arrow[d] & BS^1 \arrow[d, "\psi"] \\
\mathcal{S}_1 \times \mathcal{S}_2 \arrow[r] & B\mathcal{Z}_1 \times B\mathcal{Z}_2 \arrow[r] & BS^1 \times BS^1.
\end{tikzcd}$$

The maps are all the obvious ones. In particular, ψ is determined by the S^1 action $\psi(\tau) = (\tau^{l_2}, \tau^{-l_1})$. The point is that the differentials in the Serre spectral sequence of the top fibration are determined through naturality by the differentials in the Serre spectral sequence of the bottom fibration. We give several examples:

Theorem [B-G]: *Let $\mathcal{S}$ be any simply connected 3-Sasakian*

7-manifold which is not the 7-sphere. Then

$$H^q(S^{2m+1}\star\mathcal{S},\mathbb{Q})\approx\begin{cases} H^q(S^{2m+1}\times\mathcal{Z},\mathbb{Q}) & \text{if } m>2;\\ H^q(S^2\times\mathcal{S},\mathbb{Q}) & \text{if } m=1;\\ \begin{cases}\mathbb{Q} & \text{if } q=0,11;\\ \mathbb{Q}^{k+1} & \text{if } q=2,4,7,9;\\ 0 & \text{if } q=1,10,3,8;\end{cases} & \text{if } m=2\end{cases}$$

where $k=b_2(\mathcal{S})$. Moreover, for the case when $m=2$ if $k>1$ then $H^5(S^5\star\mathcal{S},\mathbb{Q})\approx H^6(S^5\star\mathcal{S},\mathbb{Q})\approx 0$; whereas if $k=1$ there are two possibilities; either $H^5(S^5\star\mathcal{S},\mathbb{Q})\approx H^6(S^5\star\mathcal{S},\mathbb{Q})\approx 0$, or $H^5(S^5\star\mathcal{S},\mathbb{Q})\approx H^6(S^5\star\mathcal{S},\mathbb{Q})\approx\mathbb{Q}$.

In certain cases we can obtain integral information.

THEOREM [B-G]: *Let $\mathcal{S}$ be a simply connected Sasakian-Einstein 7-manifold of index 2 with $H^3(\mathcal{S},\mathbb{Z})\approx 0$ and second Betti number $b_2(\mathcal{S})=k$. Then we have*

$$H^2(S^3\star\mathcal{S},\mathbb{Z})\approx\mathbb{Z}^{k+1},\qquad H^3(S^3\star\mathcal{S},\mathbb{Z})\approx 0.$$

More specifically, if $\mathcal{S}(p_1,p_2,p_3)$ is one of the simply connected 3-Sasakian 7-manifolds described previously with the p_i's pairwise relatively prime. Then there is an isomorphism

$$H^q(S^3\star\mathcal{S}(p_1,p_2,p_3),\mathbb{Z})\approx\begin{cases}\mathbb{Z} & \text{if } q=0,9,5;\\ \mathbb{Z}^2 & \text{if } q=2,7;\\ \mathbb{Z}\oplus\mathbb{Z}_{\sigma_2} & \text{if } q=4;\\ \mathbb{Z}_{\sigma_2} & \text{if } q=6;\\ 0 & \text{if otherwise,}\end{cases}$$

where $\sigma_2=p_1p_2+p_1p_3+p_2p_3$. In fact there is a ring isomorphism $H^*(S^3\star\mathcal{S}(p_1,p_2,p_3),\mathbb{Z})\approx H^*(S^2\times\mathcal{S}(p_1,p_2,p_3),\mathbb{Z})$.

The procedure can also be iterated.

THEOREM [B-G]: *Let $\mathcal{S}$ be a simply connected Sasakian-Einstein 7-manifold of index 2 with $H^3(\mathcal{S},\mathbb{Z}) = 0$. Then there are group isomorphisms*

$$H^2(S^3 \star \cdots \star S^3 \star \mathcal{S},\mathbb{Z}) \approx H^2(S^2 \times \cdots \times S^2 \times \mathcal{S},\mathbb{Z}),$$

$$H^3(S^3 \star \cdots \star S^3 \star \mathcal{S},\mathbb{Q}) \approx 0,$$

and ring isomorphisms

$$H^*(S^3 \star \cdots \star S^3 \star \mathcal{S},\mathbb{Q}) \approx H^*(S^2 \times \cdots \times S^2 \times \mathcal{S},\mathbb{Q}),$$

and

$$H^*(S^3 \star \cdots \star S^3 \star \mathcal{S}(p_1,p_2,p_3),\mathbb{Z}) \approx H^*(S^2 \times \cdots \times S^2 \times \mathcal{S}(p_1,p_2,p_3),\mathbb{Z}). \blacksquare$$

It is interesting to speculate as to whether $S^3 \star \cdots \star S^3 \star \mathcal{S}$ is actually homeomorphic (or even diffeomorphic) to $S^2 \times \cdots \times S^2 \times \mathcal{S}$ in this theorem. But even if $S^3 \star \cdots \star S^3 \star \mathcal{S}$ were diffeomorphic to $S^2 \times \cdots \times S^2 \times \mathcal{S}$ the Sasakian-Einstein metric could not be the product metric.

The join construction provides examples of what we call *$\mathcal{SE}$-reducible.* A Sasakian-Einstein manifold $\mathcal{S}$ is *$\mathcal{SE}$-irreducible* if writing $\mathcal{S} \simeq \mathcal{S}_1 \star \mathcal{S}_2$ implies that either $\mathcal{S}_1$ or $\mathcal{S}_2$ is the circle S^1 with its flat structure (the identity in the monoid). For example, a homogeneous Sasakian-Einstein manifold $\mathcal{S} = G/H$ is irreducible if and only if G is simple. Generally, G is a compact semi-simple Lie group.

Some Corollaries

We give a list of some of the consequences that follow from our work together with other known results.

Corollary 1: *There are infinite families of simply connected Sasakian-Einstein manifolds in all odd dimensions* ≥ 7 *with arbitrary second Betti number.*

A remarkable theorem of Gromov says that if there is a negative number κ and a Riemannian metric with bounded diameter such that all sectional curvatures are $\geq \kappa$, then the total Betti number of the manifold must be bounded. This together with our results give

Corollary 2: *For any negative real number* κ *there are infinitely many Sasakian-Einstein manifolds in all odd dimensions* ≥ 7*, which do not admit metrics whose sectional curvatures are all greater than or equal to* κ.

Corollary 3: *In every odd dimension greater than 5, there are infinitely many distinct homotopy types of simply connected compact Sasakian-Einstein manifolds with the same rational cohomology groups. In particular, in each such dimension, there are infinitely many Sasakian-Einstein manifolds with arbitrarily small injectivity radii.*

Corollary 4: *In each dimension of the form* $4n+3$ *with* $n \geq 1$ *there are compact Sasakian-Einstein manifolds that do not admit a 3-Sasakian structure, and for* $n > 2$ *there are such manifolds having arbitrary second Betti number.*

Corollary 5: *In every odd dimension greater than 3 there are compact simply connected manifolds that admit continuous families of Sasakian-Einstein structures.*

An Open Problem

A well-known result of Smale says that every simply connected 5-manifold with free second homology group is diffeomorphic to $S^5 \#_k S^2 \times S^3$ for some k. All regular simply connected Sasakian-Einstein manifolds of dimension 5 are circle bundles over del Pezzo surfaces with a positive Kähler-Einstein metric. All such del Pezzo surfaces are $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ blown up at k points in general position with $0 \leq k \leq 8$. By Matsushima's well-known theorem the cases $k = 1$ and 2 cannot admit a Kähler-Einstein metric, and by results of Tian and Yau the cases $k = 3, \cdots, 8$ do admit Kähler-Einstein metrics. We know of no non-regular Sasakian-Einstein manifolds of dimension 5 that are not quotients of regular Sasakian-Einstein manifolds by a finite group.

Open Problem: Classify all simply connected Sasakian-Einstein manifolds of dimension 5.

The study of this problems entails the study of del Pezzo orbifolds admitting a Kähler-Einstein metric.