5. Appendix

Algebra Preliminaries - Tensor algebra

In this section we will develop not only the algebraic prerequisites for a full treatment ot the exterior differential calculus but also some tensor calculus necessary for the study of tensor fields on manifolds. This subject is usually known as multilinear algebra. Not all proofs will be given here, so the reader should review a book on modern algebra.

We first recall some notation, Let V and W be a finite dimensional vector spaces. The space of all linear maps $\phi: V \to W$ will be denoted by Hom (V, W). It is not difficult to show that Hom (V, W) is a vector space of dimension equal to (dim V)(dim W). The space Hom (V, V) is usually denoted as End V, the space of endomorphisms of V. Notice that since $\phi: V \to W$ is linear it is a morphism of vector spaces, i.e. a vector space homomorphism. The subspace of End V consisting of all those ϕ that are isomorphisms onto is denoted by Aut V, the vector space of automorphisms of V. It can be shown that End V has a natural Lie algebra structure while Aut V has a natural group structure. Indeed by considering matrix representations we can identify Aut $V = GL(V)$ - the general linear group of V, and End $V = gl(v)$- the Lie algebra of $GL(V)$. We also mention that the dual V^* can be written as $V^* = \mathrm{Hom}(V,\mathbb{R})$ (R = real numbers). We shall oftern write $v^*(u)$ as the pairing $\langle u, v^*\rangle$ for $u \varepsilon V$, $v^* \varepsilon V^*$.

Let us now consider the direct product of K vector spaces of dimension n_k, $V_1 \times \ldots \times V_k$. An element of $V_1 \times \ldots \times V_k$ is written as a k-tuple $(v_1, \ldots, v_k)$. Given another vector space W, a mapping $\phi = V_1 \times \ldots \times V_k \longrightarrow W$ is called <u>n-linear</u>, (or simply <u>multilinear</u>) if for every i $(1 \leq i \leq k)$

$$\phi(v_1, \ldots, v_{i-1}, \alpha v_i + \beta w_i, v_{i+1}, \ldots, v_k) = \alpha\phi(v_1, \ldots, v_i, \ldots, v_k) \tag{5.1}$$
$$+ \beta\phi(v_1, \ldots, w_i, \ldots, v_k), \quad v_i, w_i \in V_i, \alpha, \beta \in R.$$

The set of all k-linear mapping $\phi : V_i \times \ldots \times V_k \longrightarrow W$ will be denoted by $L(V_1 \ldots V_k; W)$, and like Hom (V,W) it can be turned into a vector space in a natural way. The elements in $L(V_1 \ldots V_k; \mathbb{R})$ are called <u>K-linear forms.</u>

TENSOR PRODUCT:

<u>DEFINITION</u>: Let $V_1, \ldots, V_k, W$ be vector spaces, and $\phi \in L(V_1 \ldots, V_k : W)$. The pair (W, ϕ) is called a <u>tensor product</u> of $V_1, \ldots, V_k$, if

(i) $\phi(V_1 \times \ldots \times V_k)$ spans W.

(ii) For every vector space U and every k-linear mapping $\psi : V_1 \times \ldots \times V_k \longrightarrow U$ there exists a linear mapping $f : W \longrightarrow U$ such that the following diagram commutes

$$\begin{tikzcd}
V_1 \times \ldots \times V_k \arrow[r] \arrow[d, "\psi"'] & W \arrow[dl, "f"] \\
U &
\end{tikzcd}$$

is property of (W,ϕ) is called the universal factorization

operty.

ERCISE: Show that condition (i) above, may be replaced by requiring

iqueness for the mapping f in (ii).

the pair (W, ϕ) is a tensor product of $V_1, \dots, V_k$, we denote

by $V_1 \otimes \dots \otimes V_k$ or $\overset{k}{\otimes} V_i$ and $\phi(v_1, \dots, v_k)$ by $v_1 \otimes \dots \otimes v_k$ or

v_i
=1

efore looking at the elementary properties we first consider the

estion of existence:

FINITION:

Let X be a set, and C(X) be the set of all mappings

X $\longrightarrow$ $\mathbb{R}$ such that $f(x) = 0$ for all but a finite number of

X. C(X) is readily made into a vector space by defining

$$f_1 + \beta f_2)\ (x) = \alpha f_1(x) + \beta f_2\ (x),\ \alpha,\ \beta \in \mathbb{R};\ f_1,\ f_2 \in C(X)$$

w, with each $x \in X$ we associate an $f_x \in C(X)$ defined by

$$f_x(y) = \begin{matrix} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{matrix}$$

e set of mappings of the form f_x constitutes a basis for C(X)

erify this), and the association $x \leftrightarrow f_x$ is bijective. So we can say that $\{x\}$ is a basis for $C(X)$ and consider $C(X)$ as the space of all the formal linear combinations

$$f = \sum_{x \varepsilon X} \alpha_x x \quad \text{with} \quad \alpha_x = f(x) \ \varepsilon\ R.$$

$C(X)$ is called the <u>free vector spaces</u> over the set X.

Now, let $V_1, \ldots, V_k$ be vector spaces and in the space $C(V_1 \times \ldots \times V_k)$ consider the sub space N generated by the elements of the form

$$(v_1, \ldots, \alpha v_i + \beta w_i, \ldots, v_k) - \alpha(v_1, \ldots, v_i, \ldots, v_k) - \beta(v_1, \ldots, w_i, \ldots, v_k)$$

with $\alpha, \beta \ \varepsilon\ R$; $v_i, w_i \ \varepsilon\ V_i$ for all $i = i, \ldots, k$.

We call $W = C(V_1 \times \ldots \times V_k)/N$, and π the natural projection onto the quotient. If we define $\phi\colon V_1 \times \ldots \times V_k \longrightarrow W$ as $\phi = \pi|_{V_1 \times \ldots \times V_k}$, we claim that the pair (W, ϕ) is a tensor product of $V_1 \times \ldots \times V_k$.

It is clear that ϕ is k-linear, and since $V_1 \times \ldots \times V_k$ is a basis for $C(V_1 \times \ldots \times V_k)$, we have that $\phi(V_1 \times \ldots \times V_k)$ spans W. Now, if U is any vector space and $\psi = V_1 \times \ldots \times V_k \longrightarrow U$ any k-linear mapping, there exist a well defined linear mapping $g = C(V_1 \times \ldots \times V_k) \longrightarrow U$, such that $g = \psi$ on $V_1 \times \ldots \times V_k$.

Since ψ is k-linear, we have $N \subset \ker g$, hence g induces a linear map $f\colon W \longrightarrow U$ such that $g = f \circ \pi$, and it follows that $f \circ \phi = f \circ \pi = g = \psi$ on $V_1 \times \ldots \times V_k$, so (W, ϕ) is a tensor product.

EXERCISE: Let V, V^* and U, U^* be two pairs of dual spaces of finite dimension. Consider the mapping

$$\beta\colon\; U \times V \longrightarrow L(U^*, V^*; R)\colon (u,v) \longrightarrow \beta_{u,v}$$

given by $\beta_{u,v}(u^*, v^*) = \langle u^*, u\rangle \langle u^*, u\rangle \langle u^*, v\rangle$.

Prove that the pair $(L(U^*, V^*; R), \beta)$ is a tensor product of U, V.

We see from this exercise that a tensor product can be constructed in several ways, but we can still speak of uniqueness of the tensor product in the sense that if (W, ϕ) and (W', ϕ') are tensor products of $V_1, \ldots, V_k$ then, by the universal factorization property we have unique linear maps $f\colon W \longrightarrow W'$ and $g\colon W' \longrightarrow W$ such that $\phi' = f \circ \phi$ and $\phi = g \circ \phi'$, that is, $g = f^{-1}$ so W and W' are naturally isomorphic.

Similarly we can prove:

Lemma 5.1: The following properties hold:

i) there is a unique isomorphism onto $g\colon U \otimes V \longrightarrow V \otimes U$ defined by $g(u \otimes v) = v \otimes u$.

ii) there is a unique isomorphism onto $\psi\colon (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$ defined by $\psi(u \otimes v) \otimes w = u \otimes (v \otimes w)$

iii) $(U_1 + U_2) \otimes V = U_1 \otimes V + U_2 \otimes V$

$$V \otimes (U_1 + U_2) = V \otimes U_1 + V \otimes U_2$$

where $U_1 + U_2$ means direct sum.

Proof: The proof of all of these uses the universal property just proved.

i) In (5.3) put $k=2$ and $W = V \otimes W$. Define f by $f(u, v) = v \otimes u$, then $v \otimes u = f(u,v) = g\circ\phi(u,v) = g(u \otimes v)$. Similarly interchange U and V, then $g' : V \otimes U \longrightarrow U \otimes V$ with $g'(v \otimes u) = u \otimes v$.

Moreover,

$g\, g'(v \otimes u) = v\otimes u$ and $g'g(u \otimes v) = u\otimes v$, so $g' = g^{-1}$, and g is an isomorphism onto.

ii) Draw two diagrams and put them together taking $W = (U \otimes V) \otimes W$ in one case and $V\otimes(V \otimes W)$ in the other. Thus

$$\begin{tikzcd}
U \times V \times W \arrow[r, "f"] \arrow[d, "f'"'] \arrow[dr, "\phi"] & (V \otimes V)\otimes W \arrow[dl, "\psi'"] \\
U \otimes (V \otimes W) \arrow[ur, "\psi"] & U \otimes V \otimes W \arrow[l, "g'"] \arrow[u, "g"']
\end{tikzcd}$$

Define $f(u,v,w) = (u\otimes v)\otimes w$ and $f'(u,v,w) = u\otimes(v\otimes w)$ since both f and f' are multilinear, we can define ψ and ψ'

(we must show ψ' is the inverse of ψ), by $f = \psi \circ f'$ and $f' = \psi' \circ f$. We have $f(u,v,w) = (u \otimes v)\otimes w = \psi \circ f'(u,v,w)$ $= \psi(u\otimes(v\otimes w)$ and $f'(u,v,w) = u\otimes(v\otimes w) = \psi' \circ f(u,v,w) = \psi'((u\otimes v)\otimes w)$ so $\psi' = \psi^{-1}$ and ψ is an isomorphism onto. The uniqueness of ψ' follows from the commutativity of the lower triangle, since g and g' are unique and isomorphisms onto defined by $g(u\otimes v\otimes w)=(u\otimes v)\otimes w$ and $g'(u\otimes v\otimes w) = u\otimes(v\otimes w)$.

Before proving property (iii), we briefly discuss tensor products of linear mappings.

Let U_i, V_i be vector spaces and $f_i \in \text{Hom}(U_i, V_i)$, $i = 1,2$. (the discussion can readily be extended to $i = 1,\dots,k$). We can define a bilinear map

$\psi: U_1 \times U_2 \to V_1 \otimes V_2$ by $\psi(u_1,u_2) = f_1(u_1) \otimes f_2(u_2)$

then, in view of the factorization property there exists a unique linear map

$f_1 \otimes f_2: U_1 \otimes U_2 \to V_1 \otimes V_2$ such that $f_1 \otimes f_2(u_1 \otimes u_2)$
$= f_1(u_1) \otimes f_2(u_2)$ Now, the mapping

$$\beta: \text{Hom}(U_1, V_1) \times \text{Hom}(U_2, V_2) \to \text{Hom}(U_1\otimes U_2, V_1\otimes V_2)$$

defined by $\beta(f_1, f_2) = f_1 \otimes f_2$ is clearly bilinear. Moreover, it can be shown (see GREUB), that if U_i, V_i are finite dimensional, then the pair $(\text{Hom}(U_1 \otimes V_2, V_1 \otimes V_2), \beta)$ is a tensor product of $\text{Hom}(U_1, V_1)$, $\text{Hom}(U_2, V_2)$.

EXERCISE: Show that

(a) $(f_1 \otimes f_2) \circ (g_1 \otimes g_2) = (f_1 \otimes g_i) \circ (f_2 \otimes g_2)$

where $V_i \xrightarrow{g_i} V_i \xrightarrow{f_i} W_i$ are linear maps.

(b) $f_1 \otimes f_2$ is injective iff f_1 and f_2 are

(c) $\mathrm{Im}(f_1 \otimes f_2) = \mathrm{Im}(f_1) \otimes \mathrm{Im}(f_2)$.

Proof of property (iii): Define injections $i_k: U_k \longrightarrow U_1 + U_2$ $k = 1, 2$, and projections $\pi_k: U_1 + U_2 \longrightarrow U_k$. Then $\pi_k \circ i_k = 1_k$ the identiti on U_r. Moreover $\pi_1 \circ i_2 = \pi_2 \circ i_1 = 0$. Then it follows easily from the universality property that $i_r \otimes 1$: $V_r \otimes V \to (U_1 + U_2)\otimes V$ is an injection where 1 is the identity map on V. Also we have projections $\pi_r \otimes 1: (U_1 + U_2)\otimes V \to U_r \otimes V$. Moreover, $(\pi_r \otimes 1) \circ i_r \otimes 1: U_r \otimes V \to U_r \otimes V$ is the identity. Also $(\pi_r \otimes 1) \circ (i_e \otimes 1): U_\ell \otimes 1 \longrightarrow U_r \otimes 1$ $r \neq \ell$ is the zero map. It follows that

$$i_1 \otimes 1 + i_2 \otimes 1: U_1 \otimes V + U_2 \otimes V \longrightarrow (U_1 + U_2)\otimes V$$

is an isomorphism onto with inverse

$$\pi_1 \otimes 1 + \pi_2 \otimes 1: (U_1 + U_2)\otimes V \longrightarrow U_1 \otimes V + U_2 \otimes V.$$

The second identity is proved analogously.

By induction we can extend the above result to the multi sums, $(U_1 + \ldots + U_r)\otimes V$; $U\otimes(V_1 + \ldots + V_r)$. etc.

eorem 5.1; Let $V_1,\ldots,V_k$ be finite dimensional vector spaces
th dimensions n_i $i = 1,\ldots,k$, and let $e_{\mu_1},\ldots, e_{\mu_r}$, $\mu_i=1,\ldots,n_i$,
note a basis for $V_1,\ldots,V_k$, respectively, then $e_{\mu_1}\otimes\ldots\otimes e_{\mu_k}$
a basis for $V_1\otimes\ldots\otimes V_k$, Moreover, $\dim V_1\otimes\ldots\otimes V_k = \prod_{i=1}^{k} n_i$.

oof: We prove that $\{e_{\mu_1}\otimes e_{\mu_2}\}$ $\mu_i = 1,\ldots, n_i$ $i = 1,2$ is
asis for $V_1\otimes V_2$. Let V_i^{μ} be the 1 dimensional vector
aces spanned by e_{μ_i}. Then as a vector space direct sun we have
$= V_i^1 + \ldots + V_i^{n_i}$. Then by iii) of lemma 5.1 (extended to
lti-sums) we have that

$$V_1\otimes V_2 = \Sigma_{\mu_1,\mu_2} V_1^{\mu_1}\otimes V_2^{\mu_2}$$

we note that $u\otimes v = 0$ implies $u = 0$ or $v = 0$,
ause if $u \neq 0$ and $v \neq 0$, then there exists $f,g \in \mathrm{Hom}(V, R)$
h that $f(u) \neq 0$ and $g(v) \neq 0$, hence, given the bilinear
m $\psi(x, y) = f(x)\, g(y)$, there exists a unique linear form h
h that $h(x\otimes y) = f(x)\, g(y)$, and it follows that $h(u\otimes v)$
u) $g(v) \neq 0$, whence $u\otimes v \neq 0$. On the other hand condition
2) (i), applied to $V_1^{\mu_1}$, $V_2^{\mu_2}$ and $V_1^{\mu_1}\otimes V_2^{\mu_2}$ gives that
$\otimes V_2^{\mu_2}$ is spanned by the single element $e_{\mu_1}\otimes e_{\mu_2}$. Thus
$\otimes V_2$ is the direct sum of the one-dimensional subspaces
erated by the elements $e_{\mu_1}\otimes e_{\mu_2}$, and hence these elements

form a basis of $V_1 \otimes V_2$. By induction we easily extend this to $V_1 \otimes \ldots \otimes V_k$ making use of ii) of lemma 5,1. (This means we can write $V_1 \otimes \ldots \otimes V_k = (V_1 \otimes \ldots \otimes V_{k-1}) \otimes V_k$ and proceed with induction).

Theorem 5.1 says we can write any member $T \in V_1 \otimes \ldots \otimes V_k$ as

(5,4)
$$T = \Sigma \; T^{\mu_1 \ldots \mu_k} \; e_{\mu_1} \otimes \ldots \otimes e_{\mu_k}$$

The componentes $T^{\mu_1 \ldots \mu_k}$ are familiar from classical tensor analysis.

It is also straightforward to include the dual space V^* into our analysis. We know that V^* is the space of linear maps $V \longrightarrow \mathbb{R}$, e.e $V^* = \mathrm{Hom}(V, \mathbb{R})$ and for $v \in V$ and $f \in V^*$, $f(v) \in R$. Now consider $V \otimes V^*$ and define for $u, v \in V$, $f \in V^*$ $(u \otimes f)(v) = f(v) \,.\, u$. This can be extended linearly to all $V \otimes V^*$ and defines a map $V \longrightarrow V$. Thus we identify $V \otimes V^* = \mathrm{Hom}(V, V)$. Let's see what this looks like in components. Let e_i be a basis for V and e^{*i} be the dual basis for V^* that is

(5.5)
$$e^{*i}(e_j) = \delta^i_j.$$

Often this pairing between V and V^* is written as $\langle e_j, e^{*i} \rangle = e^{*i}(e_j) = \delta^i_j$. We will use both notations. Now from theorem 5.1

any member $T \varepsilon V \otimes V^*$ can be written as

$$T = \sum_{i,j} T^i_j \, e_i \otimes e^{*i}$$

and thus if $v = \sum_i v^i e_i \varepsilon V$ we can write

$$T(v) = \sum_{1,j} T^i_j \, e_i \otimes e^*_j (v)$$

$$= \sum_{i,j,k} T^i_j \, e_i \otimes v^k e^{*j}(e_k) = \sum_{i,j} e_i \, T^i_j \, v^j$$

Thus $T: V \longrightarrow V$ is a linear mapping. If u^i are the components of u, then the components of $T(u)$ are $\sum_j T^i_j u^j$. This clearly identifies $V \otimes V^*$ with $\mathrm{Hom}(V, V)$.

Now let us apply theorem 5.1 to the case when both V's and V*'s appear, i,e V_i in theorem 5.1 is either V or V*. Now suppose we have a tensor product of r copies of V and s copies of V*, A member of such a space is called a tensor of type (r,s). The problem is that the order ocurring in the tensor product is important, i.e $V \otimes V^*$ is different than $V^* \otimes V$, etc. So the tensor of type (r, s) are not uniquely determined. In any case r is the contraviariant degree and s is the coavariant degree. If s = 0, the tensor is called contravariant of degree r, whereas if r = 0, it is called covariant of degree s. Now suppose e_i is a basis for V and e^{*i} is the dual basis for V* then a tensor of type (r, s) belonging to $V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \times V^*$ is written as

$$T = \sum_{\substack{i_1,\ldots,i_r \\ j_1,\ldots j_s}} T^{i,\ldots i_r}{}_{j_1\ldots j_s} \; e_{i_r} \otimes \ldots \otimes e_{i_r} \otimes e^{*j_1} \ldots \otimes e^{*j_s}$$

whereas if the order is $\overset{r-1}{V \otimes \ldots, \otimes V} \otimes \overset{s}{V^* \otimes \ldots \otimes V^*} \otimes V$

We have

$$T = \sum_{\substack{i_1,\ldots i_r \\ j_1 j \ldots j_s}} T^{i,\ldots i_{r-1}}{}_{j,\ldots j_s}{}^{i_r} \; e_{i_1} \otimes \otimes e_{i_{r-1}} \otimes e^{j_1} \otimes \ldots \otimes e^{*j_s} \otimes e_{i_r}$$

and so on for the various possible orderings. Thus for a tensor of type (1,2) we have the possibilities

$$T^i{}_{jk} \; e_i \otimes e^{*j} \otimes e^{*k}, \qquad T_j{}^i{}_k \; e^{*j} \otimes e_i \otimes e^{*k},$$

$$T_{jk}{}^i \; e^{*j} \otimes e^{*k} \otimes e_i$$

It should be clear how this works in general.

Now that we understand the different possible orderings, we show how to get rid of them. This follows from part i) of lemma 5.1 which says that $U \otimes V$ and $V \otimes U$ are isomorphic, thus any tensor space of type (r, s) with any ordering is isomorphic to $V \otimes \ldots \otimes V \otimes V^* \otimes \ldots \otimes V^*$, and so, to any tensor of type (r, s) with any ordering, corresponds a unique tensor of the form

(5.6)

$$T = \sum_{\substack{i_1,\dots,i_r \\ j_1,\dots,j_s}} T^{i_1\dots i_r}_{j_1\dots j_s}\, e_{i_1}\otimes\dots\otimes e_{i_r}\otimes e^{*j_1}\otimes\dots\otimes e^{*j_s}$$

Thus in a certain sense (up to isomorphism) we may speak of <u>the</u> tensor space of type (r,s).

Any tensor of this space uniquely takes the form (5.6). However, in practice the ordering <u>does</u> matter. Many pure math books do not make any distinction concerning the order and this can lead to confusion when one tries to apply this analysis. We will, nevertheless, give the space $V\otimes\dots\otimes V\otimes V^*\otimes\dots\otimes V^*$ a name, $T^r_s(V)$, the space of tensors of type (r,s) over V. Then the space $T^r(V) = T^r{}_0(V)$ is the space of contravariant tensors of degree r; likewise, the space $T_s(V) \equiv T^0_s(V)$ is the space of covariant tensors of degree s. In this there is no ordering problem. It is clear that we can take tensor products of tensor. Thus if $t \in T^r_s(V)$ and $t' \in T^{r'}_{s'}(V)$, $t\otimes t' \in T^{r+r'}_{s+s'}(V)$. To see this just use i) and ii) of lemma 5.1.

Now we consider the direct sum (as vector spaces)

(5.7)

$$T(V) = \sum_{r,s=0}^{\infty} T^r_s(V)$$

where $t^{0}_{0}(V) \equiv \mathbb{R}$ (or $\mathbb{C}$ in the complex case). Recall that the infinite direct sum means that if $t \in T(V)$ then t has components in only finitely many of the $T^{r}_{s}(V)$'s.

The tensor product $\otimes$ of tensors discussed above turns $T(V)$ into an algebra. It is called the tensor algebra over V.

Now there is in $T(V)$ a useful operation known as contraction. Indeed consider for each $1 \leq p \leq s$ the linear map $C^{j_q}_{i_p}: T^{r}_{s}(V) \longrightarrow T^{r-1}_{s-1}(V)$ defined on a basis by

$$(5.8) \qquad C^{j_q}_{i_p}(e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{*j_1} \otimes \ldots \otimes e^{*jr})$$

$$= \delta^{j_q}_{i_p} \; e_{i_1} \otimes \ldots \otimes \hat{e}_{u_p} \otimes \ldots \otimes e_{i_r} \otimes e^{*j_1} \otimes \ldots \otimes \hat{e}^{*j_q} \otimes \ldots \otimes e^{*j_s}$$

where the notation $\hat{e}_{i_p}$ means that e_{i_p} is missing, i.e. is equivalent to $e_{i_{p-1}} \otimes e_{i_{p+1}}$.

It can be show that 5.8) is actually independent of basis.

Exercise: Show that if t has the form (5.6), then the components of $C^{i_p}_{j_q} t \in T^{r-1}_{s-1}(V)$ are

$$\sum_{k} t^{i_1 \ldots i_{p-1} k \, i_{p+1} \ldots i_r}_{j_1 \ldots j_{q-1} k j_{q+1} \ldots j_s}$$

Exercise: Show that the space $T_s(V)$ can be identified with the space of all s-linear maps of $V\times\ldots\times V \longrightarrow R$, i.e. $\mathrm{Hom}(V\times\ldots\times V, R)$

Exercise: Let $f_i = \sum_j \alpha_i^k e_k$ be a change of basis for V. For $t \in T_s^r(V)$ given in terms of the basis $e_{i_1}\otimes\ldots\otimes e_{i_r}\otimes e^{*j_1}\otimes\ldots\otimes e^{*j_s}$ by (5.6), give the components of t in terms of the new basis $f_{i_1}\otimes\ldots f_{i_r}\otimes f^{*j_1}\otimes\ldots\otimes f^{*j_s}$.

Exercise: If we define

$$T_0(V) = \sum_{r=0}^{\infty} T^r(V)$$

$$T^0(V) = \sum_{s=0}^{\infty} T_s(V)$$

Show that under $\otimes$ $T_0(V)$ and $T^0(V)$ are actually algebras —— the contravariant and covariant tensor algebras, respectively. Each, of course, is a subalgebra of $T(V)$.

In the following we will consider mostly the contravariant tensor algebra $T_0(V)$, but everything goes equally as well for $T^0(V)$ since if we identify $V^{**} = V$, then $T^0(V) = T_0(V^*)$. So the only differences are the differences between V and V*. Here is one with an "historical accident". Any homomorphism $f: V \longrightarrow W$ induces a homomorphism $f\otimes\ldots\otimes f: T^r(V) \longrightarrow T^r(V) \longrightarrow T^r(W)$. We will see that the rule which assigns to V the contravariant tensor algebra $T_0(V)$ is an example of a <u>covariant</u> functor. Likewise, the rule assigning to V the <u>covariant</u> tensor algebra is a <u>contravariant</u> functor.

We can also identify $T^r(V)^* = T^r(V^*) = T_r(V)$. To see this we introduce the pairing $\langle x,y\rangle$ $x \in T^r(V)$ $y \in T^r(V^*)$ and define it to be the total contraction. In coordinates this is

(5.9)

$$\langle x,y\rangle = \sum_{i_1,\dots i_r} x^{i_1,\dots i_r} y_{i_1\dots i_r}$$

In this way y can be viewed as a linear map

$$T^r(V) \longrightarrow \mathbb{R}, \quad \text{that is} \quad y \in T^r(V)^*.$$

Conversely any member of $T^r(V)^*$ can be considered as the total contraction of a $y \in T^r(V^*)$ with an $x \in T^r(V)$

<u>Exercise:</u> Show that

$$\langle x_1\otimes\dots\otimes x_r,\ y^1\otimes\dots\otimes y^r\rangle = \langle x_1, y^1\rangle\dots\langle x_r, y^r\rangle$$

for $x_i \in V$ $y^i \in V^*$.

Let us now formalize briefly the rule which assigns to a set of vector spaces its tensor product. Let $\mathcal{V}$ denote the set of all vector spaces together with all homomorphisms (or just morphisms for short) between them (Loosely speaking this is what is called a category). Let $\mathcal{F}: \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ be a map which assigns to each pair of vector spaces $V, W \varepsilon \mathcal{V}$ a new vector space $\mathcal{F}(V,W) \varepsilon \mathcal{V}$ and to each pair of morphisms $f: V \longrightarrow V'$, $g: W \longrightarrow W'$, $v, w, v', w' \varepsilon \mathcal{V}$ a new morphism

$$1) \quad \mathcal{F}(f,g): \mathcal{F}(V,W) \longrightarrow \mathcal{F}(V', W')$$

such that

$$2) \quad \mathcal{F}(1_{V}, 1_{W}) = 1_{\mathcal{F}(V,W)}$$

where 1_V is the identity morphism on V, and

$$3) \quad \mathcal{F}(f_1 \circ f_2) = \mathcal{F}(f_1, g_1) \circ \mathcal{F}(f_2, g_2)$$

for morphisms f_1, f_2, g_1, g_2. $\mathcal{F}$ is called a <u>covariant functor.</u>
If we replace 1), and 3) above by

$$1)' \quad \mathcal{F}(f,g): \mathcal{F}(V', W') \longrightarrow \mathcal{F}(V, W)$$

$$2) \quad \mathcal{F}(f_1 \circ f_2, g_1 \circ g_2) = \mathcal{F}(f_2, g_2) \circ \mathcal{F}(f_1, g_1)$$

then $\mathcal{F}$ is a <u>contravariant functor.</u> If $\mathcal{F}(f,g)$ depends continuously (or C^∞ differentiably) on f and g then $\mathcal{F}$ is a continuous (or C^∞) functor. We can also define <u>mixed functors</u> which are covariant in one argument and contravariant in the other. It should also be clear how to define functors in more than two arguments.

The tensor product defines a covariant functor $F(V,W) = V \otimes W$ and $F(f,g) = f \otimes g\colon V\otimes W \longrightarrow V' \otimes W'$ as defined previously $f \otimes g\,(v \otimes w) = f((v) \otimes f(w)$. Property 2) is easy to verify and property 3) is a previous exercise.

Exercise: Show that the dual functor $F(V) = V^*$, $F(f) = f^*$ is a contravariant functor, while the double dual $F(V) = V^{**}$, $F(f) = f^{**}$ is a covariant functor.

Exercise: Show that if F is any functor and $f \in \mathrm{Hom}(V,V')$, $g \in \mathrm{Hom}(W, W')$ are isomorphisms then $F(f,g)$ is an isomorphism

Exercise: Show that $F(V) = T^r(V)$, $T_s(v)$, and $T^r_s\,(v)$ define respectively covariant, contravariant, and mixed functors

Exercise: Show that the direct sum functor $F(V,W)=V+W$, $F(f,g)=f+g$ is a covariant functor.

Let us now introduce on $T^r(V)$ the permutation group P_r on r indices. For a tensor of the form $x_1\otimes\ldots\otimes x_r \in T^r(V)$ we define for $\sigma \in P_r$

(5.10a)

$$\sigma(x_1\otimes\ldots\otimes x_r) = x_{\sigma(1)}\otimes\ldots\otimes x_{\sigma(r)}$$

where $\sigma(1)\ldots\sigma(r)$ just gives us a rearrangement of the indices $1,\ldots,r$. We then extend this definition to all members of $T^r(V)$

by linearity. In terms of coordinates it is easy to see that we have

(5.10b)

$$\sigma(x^{i_1,\ldots i_r}) = x^{i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(r)}}$$

where σ^{-1} is the inverse permutation. Notice that we can pick $\sigma(1)$ in r different ways, then we can pick $\sigma(2)$ in $(r - 1)$ different ways, etc, so that P_r has $r!$ elements.

We are interested in two special cases of (5.10). First the case when for all $\sigma \varepsilon P_r$

(5.11)

$$\sigma(x^{i_1,\ldots,i_r}) = x^{i_1,\ldots,i_r} \quad \text{or} \quad \sigma(x_1 \otimes \ldots \otimes x_r) = x_1 \otimes \ldots \otimes x_r$$

this says that the tensor is independent of the arrangement of the indices that is the tensor is totally symmetric. Thus we define

(5.12)

$$S^r(V) = \{v \varepsilon T^r(V) : \sigma v = v\}$$

We can again form the direct sum

(5.13)

$$S(V) = \sum_{r=0}^{\infty} S^r(V)$$

where $S^0(V) \equiv R$. Clearly $S(V)$ is a subspace of $T_0(V)$ since $S^r(V) \subset T^r(V)$. Indeed we have a projection operator

$S: T^r(V) \longrightarrow S^r(V)$ called the symmetrizer and defined by

(5.14)

$$Sv = \sum_{\sigma \varepsilon P_r} \sigma v$$

This is a projection since $S^2 = S$ and it is onto.

<u>Exercise:</u> Show that $S: T^r(V) \longrightarrow S^r(V)$ is indeed a projection onto $S^r(V)$.

<u>Exercise:</u> We can define a product in $S(V)$ by $uv = S(u \otimes v)$. Show that with this product $S(V)$ is a commutative associative algebra. (Remember that $T_0(V)$ is associative).

<u>Exercise:</u> Show that $S(V^*)$ is just the ring of polynomials in the indeterminates $x_1,\ldots,x_n \varepsilon V^*$ over $\mathbb{R}$. Where $x_1,\ldots,x_n$ is a basis for V^*.

Now consider $S^2(V^*)$. This is just the space of symmetric bilinear forms on V. A member $g \varepsilon S^2(V^*)$ is called <u>nondegenerate</u> if $g(u,v) = 0$ for all $v \varepsilon V$ then $u = 0$. A nondegenerate $g \varepsilon S^2(V)$ is called an <u>indefinite</u> <u>metric</u> on V. It is a well known theorem of algebra that every indefinite metric can be brought to the form

$$g(x,x) = \sum_{i=1} (x^i)^2 - \sum_{i=k+1} (x^i)^2$$

where $n = \dim V$. The number $\operatorname{sig}(g) = |n-2k|$ is called the <u>signature</u> of g. If $k=n$, i.e $\operatorname{sig}(g)=n$, then the <u>metric</u> is positive definite and the adjective indefinite is dropped. For $n=4$, $k=1$ (or equivantly $r=3$) we have $\mathrm{sig}(g)=2$. This gives us the "Lorentz metric" (indefinite) of special relativity. It is sometimes convenient to abuse the word signature and say $\mathrm{sig}(g) = (+++-)$ or $(k,n-k)$, enumerating explicitly the number of $+$ and $-$ signs. In fact we will often use this terminology.

Exercise: Show that an indefinite metric on V establishes a canonical isomorphism $V \longrightarrow V^*$

Exercise: Show that $F(V) = S^r(V)$, $F(f)\ (u_1 \ldots u_r) = f(u_1)\ldots f(u_r)$ is a covariant functor.

5.2 Exterior Algebra.

The second case of (5.10) which interests us is the following. Consider the permutation known as an inversion which interchanges any two numbers which are next to each other. For example

$$\sigma(1,\ldots,i,j,\ldots n) = 1,\ldots,j,i,\ldots n$$

or for tensors

$$\sigma(x_1 \otimes x_2 \ldots \otimes x_r) = x_2 \otimes x_1 \otimes x_3 \otimes \ldots \otimes x_r$$

Any permutation can be obtained by an appropriate number of inversions. There are two types them of permutations, ones with an even number or odd number of inversions. Let $N(\sigma)$ denote the total number of inversions. The number $\varepsilon_\sigma = (-1)^{n(\sigma)}$ is the sign of the permutation. For any contravariant or covariant tensor t we are interested in the case of (5.10) for which

(5.15a)

$$\sigma(x_1 \otimes \cdots \otimes x_r) = \varepsilon_\sigma \, (x_1 \otimes \cdots \otimes x_r) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}$$

for $x_i \varepsilon V$. In terms of components we have

(5.15b)

$$\sigma(t^{i_1 \cdots t_r}) = \varepsilon_\sigma \; t^{i_1 \cdots i_r} = t^{i_{\sigma^{-1}(1)} \cdots i_{\sigma^{-1}(r)}}$$

Notice that $N(\sigma) = N(\sigma^{-1})$. The subspace

(5.16)

$$\Lambda^r(V) = \{t \; \varepsilon \; T^r(V) : \; \sigma(t) = (-1)^{n(\sigma)} t,$$

is called the space of antisymmetric tensors of degree r. Again we can construct the direct sum

(5.17)

$$\Lambda(V) = \sum_{r=0}^{\infty} \Lambda^r(V)$$

where $\Lambda^\circ(V) \equiv \mathbb{R}$. However, we will see that to the contrary of $S(V)$, $\Lambda(V)$ is actually finite dimensional. Similarly to (5.14) we introduce the alternator

$$A: t^r(V) \longrightarrow \Lambda^r(V)$$

defined by

(5.18)

$$T(t) = \sum_{\sigma \varepsilon P_r} \varepsilon_\sigma \, \sigma(t)$$

for $t \varepsilon T^r(V)$.

<u>Exercise:</u> Show that $A: T^r(V) \longrightarrow \Lambda^r(V)$ is a projection, i.e. $A^2 = A$.

We can introduce a product $\wedge$ in $\Lambda(V)$, called exterior multiplication, defined by

(5.19)

$$s \wedge t = A(s \otimes t)$$

for $s \varepsilon \Lambda^p(V)$ $t \varepsilon \Lambda^q(V)$. It follows that $s \wedge t \varepsilon \Lambda^{p+q}(V)$.

<u>Exercise:</u> Show that $\Lambda(V)$ with the product (5.19) is a associative algebra over V and that

(5.20)

$$s \wedge t = (-1)^{pq} \, t \wedge s$$

This algebra is called the <u>exterior algebra</u> over V. It is most commonly used in conjunction with $V = T_p^*(M)$, and was invented by E. Cartan for this purpose. We will prove an important theorem about $\Lambda(V)$. But first notice from (5.20) that if $p = q$ is <u>odd</u> and $s = t$, we have $t \wedge t = 0$. In particular let $t \varepsilon \Lambda^1(V) = V$, then $t \wedge t = 0$.

<u>Theorem 5.2</u>: The space $\Lambda^r(V)$ of antisymmetric tensors over V is a vector space of dimension $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. If $e_1, \ldots, e_n$ is a basis for V, then

$$e_{i_1} \wedge \cdots \wedge e_{i_r}, \quad 1 \leq i_1 < i_2 < \cdots < i_r \leq n,$$

a basis for $\Lambda^r(V)$. Moreover, the exterior algebra $\Lambda(V)$ has dimension 2^n and a basis given by 1 and $e_{i_1} \wedge \cdots \wedge e_{i_r}$ for $1 \leq i_1 < \cdots i_r \leq n$ and all $r = 1, \ldots, n$.

<u>Proof:</u> The vector space property of $\Lambda^r(V)$ is immediate and is left as an exercise for the reader. By theorem 5.1 $e_{i_1} \otimes \cdots \otimes e_{i_r}$ is a basis for $T^r(V)$ and by an exercise $A(T^r(V)) = \Lambda^r(V)$ so the set of elements $\{A(e_{i_1} \otimes \cdots \otimes e_{i_r})\} = e_{i_r} \wedge \cdots \wedge e_{i_r}$ spans $\Lambda^r(V)$. Now, it is clear that $e_{i_r} \wedge \cdots \wedge e_{i_r}$ and $\sigma(e_{i_r} \wedge \cdots \wedge e_{i_r})$, generate the same one-dimensional subspace for all $t \in P_r$, hence we may choose our spanning set as the set $\{e_{i_r} \wedge \cdots \wedge e_{i_r} \mid i_1 < \cdots < i_r\}$ We must show that those are also linearly independent. Consider

$$\sum_{i_1 < \cdots < i_r} a^{i_1 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} = 0$$

For a fixed set

$j_1, \ldots, j_r$ of the $i_1, \ldots, i_r$ let $j_{r+1}, \ldots j_n$

be the remaining set such that $e_{j_1},\ldots,e_{j_n}$ is a basis for V. Multiply the above expression by $e_{j_{r+1}}\wedge\cdots\wedge e_{j_n}$, then all members of the sum vanish except those with $e_{j_1}\wedge\cdots\wedge e_{j_r}$. This is so since the other members will have a repeated factor and an we have seen for all such members say e_k, $e_k \wedge e_k = 0$. We, thus, have

$$\sigma^{j_1\ldots j_r}\, e_{j_1}\wedge\cdots\wedge e_{j_n} = 0$$

But $e_{j_1},\ldots,e_{j_n}$ is a basis for V, so we have, (see exercise below), that $e_{j_1}\wedge\cdots\wedge e_{j_n} = A(e_{j_n}\otimes\cdots\otimes e_{j_n}) \neq 0$. It follows that $a^{j_1\ldots j_r} = 0$. Now from what has been said about repeated factors, we have clearly that $\Lambda^r(V) = 0$ for $r > n$. Moreover, there are $\binom{n}{r}$ ways to choose r objects out of n. Thus dim $\Lambda^r(V)=\binom{n}{r}$ $r<n$. All of what was said holds for each $0 \leq r \leq n$, so we can take the vector space direct sum (5.17) and a basis for $\Lambda(V)$ is given by the basis $1 \in \Lambda^\circ(V) = \mathbb{R}$ plus the basis for each $\Lambda^r(V)$. Furthermore, $\dim \Lambda(V) = \Sigma \dim \Lambda^r(V) = \sum_{r=0}^{n} \binom{n}{r} (1)^{n-r} = (1+1)^n = 2^n$.Q.E.D,

<u>Exercise:</u> Show that for $A\colon T^r(V) \longrightarrow \Lambda^r(V)$ we have that ker A is the space spanned by all elements $v_1 \otimes\cdots\otimes v_r$ in $T^r(V)$ such that $v_i = v_j$ for some $i \neq j$, $1 \leqslant i, j \leqslant r$.

Thus for any $t \in \Lambda^r(V)$ we can write

(5.20)

$$t = \sum_{i_1<\dots<i_r} t^{i_1\dots i_r} e_{i_1} \wedge\dots\wedge e_{i_r}$$

The components $t^{i_1\dots i_r}$ are totally antisymmetric in the sense that it changes sign upon the inversion of any two indices. Thus it changes sign under an odd number of inversions and retaining its sign under an even number of inversions. Members of $\Lambda^r(V)$ are sometimes called r-vectors. An r-vector $\omega \varepsilon \Lambda^r(V)$ is said to be <u>decomposable</u> if there exists r - vectors $v_i \varepsilon V$ $i = 1,\dots,r$ such that

$$\omega = v_1 \wedge\dots\wedge v_r$$

We will now prove some lemmas on linear dependence.

<u>Lemma 5.1</u>: Let $x_i \varepsilon V$ $i = 1,\dots,r$. A necessary and sufficient condition that the vectors $x_1,\dots,x_r$ be linearly dependent is that

$$x_1\wedge\dots\wedge x_r = 0$$

Proof: Suppose $x_1,\dots,x_r$ are linearly independent, then there are vectors $x_{r+1},\dots,x_n$ such that $x_1,\dots,x_n$ is a basis for V. Thus $x_1\wedge\dots\wedge x_n \neq 0$ so $x_1\wedge\dots\wedge x_r \neq 0$. Conversely suppose that $x_1,\dots,x_r$ are linearly dependent, then $x_1 = \sum_{i=2}^{r} a^i x_i$ for some $a^i \varepsilon \mathbb{R}$. It follows that $x_1\wedge\dots\wedge x_r = \sum_{i=2}^{r} a^i x_i\wedge x_2\wedge\dots\wedge x_r = 0$

since each summand will have a repeated factor.

Lemma 5.2: (Cartan's lemma). Let $x_1,\ldots,x_r$ be linearly independent members of V and suppose that for $y_1,\ldots,y_r \in V$ we have

$$\sum_{i=1}^{r} x_i \wedge y_i = 0$$

Then there are real numbers $a_j^i = a_i^j$ such that

$$y_i = \sum_{j=1}^{r} a_i^j x_j$$

Proof: Choose $x_{r+1},\ldots,x_n$ so that $x_1,\ldots,x_n$ is a basis for V. Then we can write

$$y^i = \sum_{j=1}^{r} a_i^j x_j + \sum_{j=r+1}^{n} b_i^j x_j$$

and so

$$\sum_{i=1}^{r} x_i \wedge y_i = \sum_{i=1}^{r} \sum_{j=1}^{r} a_i^j x_i \wedge x_j + \sum_{i=1}^{r} \sum_{j=r+1}^{n} b_i^j x_i \wedge x_j = 0$$

the first term gives zero if $i = j$ so we get

$$0 = \sum_{i<j}^{r} (a_i^j - a_j^i) x_i \wedge x_j + \sum_{i=1}^{r} \sum_{j=r+1} b_i^j x_i \wedge x_j = 0$$

But all the $x_i \wedge x_j$ are linearly independent by theorem 5.2. Thus $a_i^j = a_j^i$ and $b_i^j = 0$ proving the lemma.

<u>Exercise:</u> Show that $F(V) = \Lambda^r(V)$ and $F(V) = \Lambda(V)$ define covariant functors. Hint: Define $F(f)(v_1 \wedge \cdots \wedge v_r) = f(v_1) \wedge \cdots \wedge f(v_r)$ on decomposable elements and extend by linearity.

Now the previous identification $T^r(V^*) = T^r(V)^*$ can easily be seen to carry over to $\Lambda^r(V)$ by the projection map A. Indeed we have $\Lambda^r(V^*) = \Lambda^r(V)^*$.

<u>Exercise:</u> Let $x \in \Lambda^r(V)$ and $y \in \Lambda^r(V^*)$ be decomposable. Show that

$$<x,y> = \det <x_i, y^j>$$

The pairing $<x, y>$ of $\Lambda^r(V)$ and $\Lambda^r(V^*)$ can be extended easily to all of $\Lambda(V)$ and $\Lambda(V^*)$. We just put $<x, y> = 0$ $x \in \Lambda^r(V)$, $y \in \Lambda^s(V^*)$ if $r \neq s$. This pairing allows us to define the adjoint operator to exterior multiplication. If $y \in \Lambda(V)$ and $z \in \Lambda(V^*)$ define $y \lrcorner$ by

(5.21)

$$<x, y \lrcorner z> = <y \wedge x, z>$$

for all $x \in \Lambda(V)$. We will make frequent use of both operations $\wedge$ and $\lrcorner$.

Exercise: Show that $u_1 \lrcorner (u_2 \lrcorner y) = (u_2 \wedge u_1) \lrcorner y = -u_2 \lrcorner (u_1 \lrcorner y)$, $u_i \in \Lambda(V)$ and $y \in \Lambda(V^*)$.

We can use $\lrcorner$ to define an isomorphism between $\Lambda^r(V)$ and $\Lambda^{n-r}(V^*)$. Let $e_1, \ldots, e_n$ be a basis for V and $e^{1^*}, \ldots, e^{n^*}$ the dual basis in V^*. Let $\omega^* = e^{1^*} \wedge \ldots \wedge e^{n^*}$, then the map $\lrcorner \omega^*: \Lambda^r(V) \longrightarrow \Lambda^{n-r}(V^*)$ exists since

$\langle x, u \lrcorner \omega^* \rangle = \langle u \wedge x, \omega^* \rangle$, for $u \in \Lambda^r(V)$, differs from zero only if $u \wedge x \in \Lambda^n(V)$ which implies $x \in \Lambda^{n-r}(V)$ and hence $u \lrcorner \omega^* \in \Lambda^{n-r}(V)$. In fact $\lrcorner \omega^*$ is an isomorphism onto, which will be seen explicitly. It is convenient here to write ω^* in a form which does not specify an order of the products. Indeed, it is easy to see that if we denote as $\varepsilon_{i_1 \ldots i_n}$ the sign of the permutation $(1, \ldots, n) \longrightarrow (i_1, \ldots, i_n)$ then we can write

(5.22)

$$\omega^* = \frac{1}{n!} \sum_{i_1, \ldots, i_k} \varepsilon_{i_1 \ldots i_n} e^{*i_1} \wedge \ldots \wedge e^{*i_n}$$

We then find using (5.21) that

$$u \lrcorner \omega^* = \frac{1}{(n-r)!r!} \sum_{i_1,\ldots i_n} u^{i_1\ldots i_r} \varepsilon_{i_1\ldots i_r i_{r+1}\ldots i_n} e^{*i_{k+1}} \wedge\ldots\wedge e^{*i_n}$$

(5.23)

$$= \sum_{i_1<\ldots<i_r} \sum_{i_{r+1}<\ldots<i_n} u^{i_1\ldots i_r} \varepsilon_{i,\ldots i_n} e^{*i_{r+1}} \wedge\ldots\wedge e^{*i_n}$$

This shows that $\lrcorner\,\omega^*$ is onto. It is also clear that $u \lrcorner \omega^*$ can only vanish if $u = 0$; hence $\lrcorner\,\omega^*$ is 1-1. you should check the counting in the above formulas.

On page 5.20 we had seen that we can put an indefinite metric on V and the exercise on page 5.21 shows that this provides an isomorphism onto between V and V^*. Explicitly, we put

(5.24)

$$g(u,v) = \langle u, v^* \rangle$$

and this identifies V with V^*. Now we can extend the indefinite metric to $T^r_o(V)$ by defining

(5.25)

$$g(u_1 \otimes\ldots\otimes u_r,\ v_1 \otimes\ldots\otimes v_r) = g(u_1,v_1)\ldots g(u_r,v_r)$$

on decomposable elements and then extending by linearity. By restriction we can get an indefinite metric on $S^r(V)$ and $\Lambda^r(V)$. Using (5.24) and the exercise on the previous page we find

(5.26)

$$g(u_1\wedge\ldots\wedge u_r, v_1\wedge\ldots\wedge v_r) = \det \langle u_i, v^{*j} \rangle = \det g(u_i, v_j)$$

Now given a basis of V we can construct an orthonormal basis $(e_1,\dots e_n)$ of V by the Gram-Schmidt orthonormalization process. (You should convince yourselves that this works for indefinite metrics, too). Then $e_{i_1}\wedge\dots\wedge e_{i_r}$ is an orthonormal basis for $\Lambda^r(V)$. (check this). (check this)

As with (5.24), (5.26) defines an isomorphism onto $G\colon \Lambda^r(V^*) \longrightarrow \Lambda^r(V)$ defined by $G(v^*) = v$, where v is the (unique) element in $\Lambda^r(V)$ such that

$$g(v^*, u^*) = \langle v, u^*\rangle \quad \text{for all } u^* \in \Lambda^r(V^*).$$

From G and $\lrcorner\,\omega^*$ we construct a new isomorphism onto $*\colon \Lambda^r(V) \longrightarrow \Lambda^{n-r}(V)$, such that the following diagram commutes

$$\begin{tikzcd}
\Lambda^r(V) \arrow[r, "\lrcorner\,\omega^*"] \arrow[dr, "*"'] & \Lambda^{n-r}(V^*) \arrow[d, "G"] \\
& \Lambda^{n-r}(V)
\end{tikzcd} \tag{5.27}$$

This is called Hodge's star operator. Let us see what $*$ looks like in components. Choose orthonormal dual bases $(e_1,\dots,e_n)$ and $(e^{*1},\dots,e^{*n})$ for V and V^*, and suppose that the indefinite metric has signature $|n-2k|$, saw

(5.28)

$$g(e_i, e_j) = \begin{cases} \delta_{i_j} & i,j = 1,\dots,k \\ -\delta_{i_j} & i,j = k+1,\dots,n \end{cases}$$

and similarly for V^*. Then applying $\lrcorner\, \omega^*$ by (5.23) and then G, we find, for $u = \sum_{i_1<\dots<i_r} u^{i_1\dots i_r} e_{i_1}\wedge\dots\wedge e_{i_r} \in \Lambda^r(V)$

$$*u = \sum_{\substack{i_1<\dots<i_r \\ i_{r+1}<\dots<i_n \\ j_{r+1}<\dots<j_n}} g(e^{*i_{r+1}}\wedge\dots\wedge e^{*i_n}, e^{*j_{r+1}}\wedge\dots\wedge e^{*j_n})\, \varepsilon_{i_1\dots i_n} u^{i_1\dots i_r} e_{j_{r+1}}\wedge\dots\wedge e_{j_n}$$

It is convenient to let $\bar{I} = \{i_1<\dots<i_r>$, $I = \{i_{r+1}<\dots<i_n\}$, $J=\{j_{r+1}<\dots<j_n\}$ $e_I = e_{i_{r+1}}\wedge\dots e_{i_n}$, etc. Then the last equation becomes

(5.29)

$$*u = \sum_{\substack{\bar{I} \\ I,J}} g(e^{*I}, e^{*J})\, \varepsilon_{\bar{I}I}\, u^{\bar{I}} e_J$$

We can also note that $g(e^{*I}, e^{*J}) = \pm\, \delta^{I,J}$ where the $\pm$ depends on the $\pm$ according to (5.28).

Lemma 5.3: Let $u \in \Lambda^r(V)$ and k be the number of plus signs in the metric g, then

(5.30)

$$**u = (-1)^{r(n-r)+n-k}\, u$$

Proof: Taking $\bar{I} = \{1,\ldots r\}$, and putting $I = \{r+1, \ldots n\}$ in (5.29)

$$* e_{\bar{I}} = g(e^{*I}, e^{*I})\, e_I$$

On the other hand interchanging I and $\bar{I}$ in (5.29) gives

$$*e_I = (-1)^{n(n-r)} g(e^{*\bar{I}}, e^{*\bar{I}})\, e_{\bar{I}}$$

the factor $(-1)^{n(n-r)}$ comes from $\varepsilon_{r+1\ldots n,1,\ldots r} = (-1)^{r(n-r)}\, \varepsilon_{1\ldots n}$. thus

$$**e_{\bar{I}} = (-1)^{n(n-r)} g(e^{*I}, e^{*I})\, g(e^{*\bar{I}}, e^{*\bar{I}})\, e_{\bar{I}}$$

But $$g^*(e^{*\bar{I}}, e^{*\bar{I}})\; g(e^{*\bar{I}}, e^{*I}) = g(\omega, \omega) = (-1)^{n-k}$$

so (5.30) follows.

Before ending this section, I will discuss briefly a topic which we probably will not treat in its generality —— the theory of exterior equations. For complete treatments see the books of E. Cartan or Slob dzinski. This was E. Cartan's method of treating arbitrary systems of partial differential equations. Given a vector space V and a subspace W with the natural injection map $i: W \longrightarrow V$, we have an induced map (pullback) $i^*: \Lambda(V^*) \longrightarrow \Lambda(W^*)$. Now let $\{y^*_\alpha\} \varepsilon \Lambda(V^*)$ be a set of forms, the subspace W is said to annihilate y^*_α if $i^* y^*_\alpha = 0$. Clearly if $\lambda \varepsilon \Lambda(V^*)$ is arbitrary then $i^*(\lambda \wedge y^*_\alpha) = (i^*\lambda) \wedge (i^* y^*_\alpha) = 0$. So $\{y^*_\alpha\}$ generates an ideal in $\Lambda(V^*)$. The problem of exterior equations is for a given set $\{y^*_\alpha\}$ to find those subspaces W which annihilate this set. We will make use of these ideas later.

Exercise: Show that $*$ is an isometry on $\Lambda(V)$, i.e. $g(*u,*v) = g(u,v)$

Exercise: Show that $X \lrcorner$ is a graded derivation of $\Lambda(V^*)$, i.e for $y_1 \varepsilon \Lambda^r(V^*)$, $y_2 \varepsilon \Lambda^s(V^*)$, $X \varepsilon V$

$$X \lrcorner (y_1 \wedge y_2) = (X \lrcorner y_1) \wedge y_2 + (-1)^r y_1 \wedge (X \lrcorner y_2)$$

Exercise: In 3-dimensional Euclidean space $\mathbb{R}^3$ with positive definite metric, show that for $x, y \varepsilon \mathbb{R}^3$

$$*(x \wedge y) = x \times y$$

where × is the vector cross product.