p 233-234 Supplemental problems to Chapter 2

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#3 $\lim_{x\to 1} \frac{1}{x^2-1} = \frac{1}{0}$ $\boxed{\text{DNE}}$ (Does Not exist)

x	$1/x^2-1$
1.1	
1.01	
1.001	
-0.9	
-0.99	
-0.999	

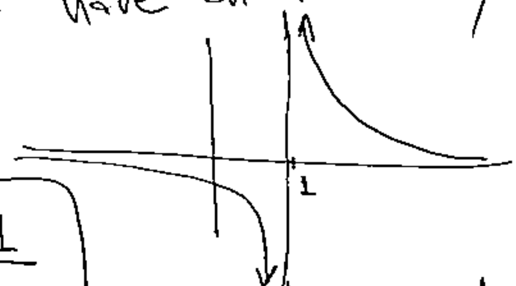
Notice large + numbers

Notice large − numbers

A more informed answer

$$\begin{cases} \lim_{x\to 1^+} \frac{1}{x^2-1} = +\infty \\ \lim_{x\to 1^-} \frac{1}{x^2-1} = -\infty \end{cases}$$

Limit on the right and limit on the left one different !! (as different as they can be, you have an infinitely large jump).



vertical asymptote

#4 $\lim_{x\to 0} \frac{e^x}{1+e^{2x}} = \frac{e^0}{1+e^{2\times 0}} = \boxed{\frac{1}{2}}$

substituting $x=0$ into the function,

This is legal because the function $f(x) = \frac{e^x}{1+e^{2x}}$ is a quotient of continuous functions and the denominator is different from zero when $x=0$, so f itself is a continuous function.

#8 $a(x) = 4x^7 + 7x^4 - 28$

$a'(x) = 28x^6 + 28x^3 = 28x^3(x^3+1)$

The derivative is defined for all real numbers x.

#12 For $z \geq 0$, $C(z) = \dfrac{z}{(1+z)(2+z)}$

(danger if $z=-1$ or $z=-2$, but since $z \geq 0$ danger is avoided)

$$C'(z) = \frac{1\,(1+z)(2+z) - z\,(3+2z)}{(1+z)^2\,(2+z)^2}$$

using quotient rule

$$\left(\frac{p}{q}\right)' = \frac{p'q - pq'}{q^2}$$

with $p(z) = z$, $p'(z) = 1$
$q(z) = (1+z)(2+z)$, $q'(z) = 3+2z$
$q(z) = 2+3z+z^2$

$$= \frac{2 + \cancel{3z} + z^2 - \cancel{3z} - 2z^2}{(1+z)^2(2+z)^2}$$

$$= \boxed{\frac{2-z^2}{(1+z)^2(2+z)^2}}$$

$z \geq 0$ well defined!

$$= \frac{(\sqrt{2}-z)(\sqrt{2}+z)}{(1+z)^2(2+z)^2}$$

#14 $C(x) = \left(1 + \dfrac{2}{x}\right)^5$, $x \neq 0$

Chain rule: $c(x) = f(g(x))$ where $f(g) = g^5$, $g(x) = 1 + \dfrac{2}{x}$

$f'(g) = 5g^4$

$g'(x) = \dfrac{-2}{x^2}$

$$c'(x) = f'(g(x))\,g'(x)$$

$$= f'\left(1+\frac{2}{x}\right)\cdot\left(-\frac{2}{x^2}\right)$$

$$= 5\left(1+\frac{2}{x}\right)^4\left(-\frac{2}{x^2}\right)$$

$$= \boxed{-\frac{10}{x^2}\left(1+\frac{2}{x}\right)^4}, \quad x \neq 0$$

#15 $S(t) = \ln(2t^3)$

<u>chain rule</u> $f(g) = \ln g$, $g(t) = 2t^3$ or
$f'(g) = \dfrac{1}{g}$, $g'(t) = 6t^2$

$$S'(t) = f'(g(t))\,g'(t) = \frac{6t^2}{2t^3} = \boxed{\frac{3}{t}}$$

using properties of ln
$S(t) = \ln(2t^3) = \ln 2 + 3\ln t$

$$\boxed{S'(t) = \frac{3}{t}}$$

same answer as it should be!

#16. $p(t) = t^2 e^{2t}$

product rule
$p(t) = f(t)\,g(t)$
$f(t) = t^2$, $f'(t) = 2t$
$g(t) = e^{2t}$, $g'(t) = 2e^{2t}$ (chain rule)

$p'(t) = f'(t)g(t) + f(t)g'(t)$
$= 2te^{2t} + t^2(2e^{2t})$
$= \boxed{2te^{2t}(1+t)}$

#19. Find derivative and <u>one</u> critical point of
$f(t) = e^t \cos t$

derivative

$f'(t) = e^t\cos t - e^t \sin t = \boxed{e^t(\cos t - \sin t)}$

product rule
$h(t) = e^t$, $h'(t) = e^t$
$g(t) = \cos t$, $g'(t) = -\sin t$

<u>Critical pt</u>: Want t such that $f'(t) = 0$

$e^t(\cos t - \sin t) = 0$, since $e^t > 0$

we need $\cos t - \sin t = 0$ or ~~cos t~~
$\cos t = \sin t$
$1 = \frac{\sin t}{\cos t} = \tan t$

either you know, or you use your calculator to find that $\boxed{t = \frac{\pi}{4}}$ ← critical pt

does the job

<u>Alternatives</u>: Ask calculator to plot $f'(t) = e^t(\cos t - \sin t)$
Ask calculator to locate a point where you cross the t-axis $(f'(t) = 0)$
(but you have to let me know what you are doing)

21 $h(y) = \frac{1-y}{(1+y)^3}$, $y \neq -1$ Find the derivative and all values where $h(y)$ is increasing (That is points where $h'(y) > 0$).

$$h'(y) = \frac{-(1+y)^3 - (1-y)\,3(1+y)^2}{(1+y)^6}$$

quotient rule
$h = \frac{p}{q}$ $h' = \frac{p'q - pq'}{q^2}$
$p(y) = 1-y$, $p'(y) = -1$
$q(y) = (1+y)^3$, $q'(y) = 3(1+y)^2$

$$= \frac{\cancel{(1+y)^2}\left[-(1+y) - 3(1-y)\right]}{(1+y)^{\cancel{6}\,4}}$$

$$h'(y) = \boxed{\frac{-4+2y}{(1+y)^4}}, \quad y \neq -1$$

derivative.

We now want to find those $y \neq -1$ such that $h'(y) > 0$. Notice that the denominator of $h'(y)$ is $(1+y)^4 > 0$ for all $y \neq -1$, so the denominator does not contribute to the sign, only the numerator: $-4 + 2y = 0$

$y = 2$

Note that the numerator is positive if $\boxed{y > 2}$ and negative if $y < 2$

$\boxed{\text{So } h(y) \text{ is increasing for all } y > 0}$

#24. Find all critical points and points of inflexion for $g(x) = e^{-x^4}$. Sketch graph.

<u>Critical pts</u>: points where $g'(x) = 0$ or it DNE.

$$\boxed{g'(x) = -4x^3 e^{-x^4}} \quad \text{(by chain rule)}$$

$$g'(x) = 0 \Rightarrow x^3 = 0 \quad \text{or} \quad \cancel{e^{-x^4} = 0}$$

Never happens since $e^y > 0$.

$$\Downarrow$$

Only one critical point $\boxed{x = 0}$

(g' is well defined for all x).

<u>Inflexion pts</u>: points where there is a change of concavity. First find points where $g''(x) = 0$. Then check whether $g''(x)$ changes sign at such x's (from + to − or from − to +). Those will be the inflexion points.

$$g''(x) = -12x^2 e^{-x^4} - 4x^3\left(-4x^3 e^{-x^4}\right) \quad \text{[product rule]}$$

$$= -12x^2 e^{-x^4} + 16x^6 e^{-x^4}$$

$$= 4x^2 e^{-x^4}\left(-3 + 4x^4\right) \leftarrow$$

[always good to factorize when trying to solve an eqn = 0]

$$= 0 \quad \text{if} \quad x^2 = 0 \Rightarrow x = 0$$

$$\text{or} \quad -3 + 4x^4 = 0 \Rightarrow x = \pm\sqrt[4]{\frac{3}{4}}$$

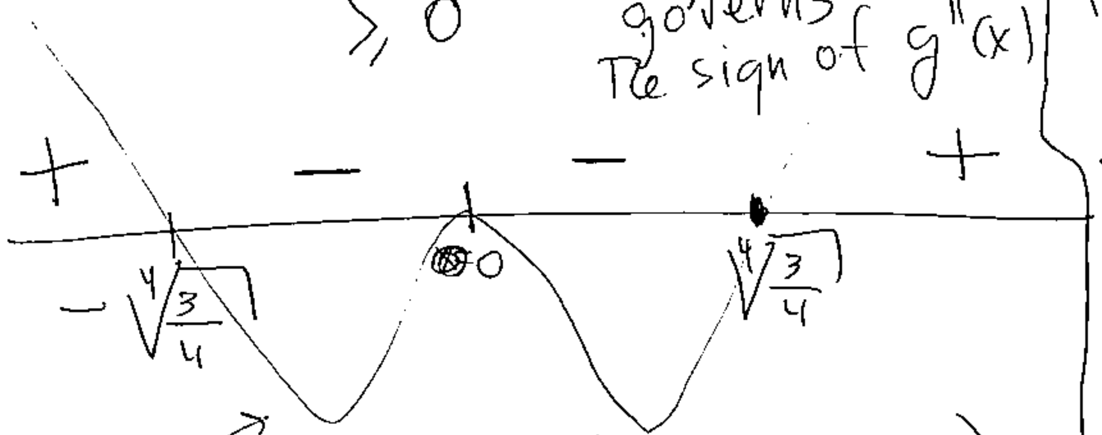
(e^{-x^4} is never zero).

#24 (continuation).

So $g''(x) = 0$ if $x = 0$ or $x = \pm\sqrt[4]{\frac{3}{4}}$

$$g''(x) = \underbrace{4x^2 e^{-x^4}}_{\geq 0} \underbrace{\left(-3 + 4x^4\right)}_{\text{governs the sign of } g''(x)}$$

clearly $-3+4x^4 > 0$ if $|x| > \sqrt[4]{\frac{3}{4}}$ and $-3+4x^4 < 0$ if $|x| < \sqrt[4]{\frac{3}{4}}$



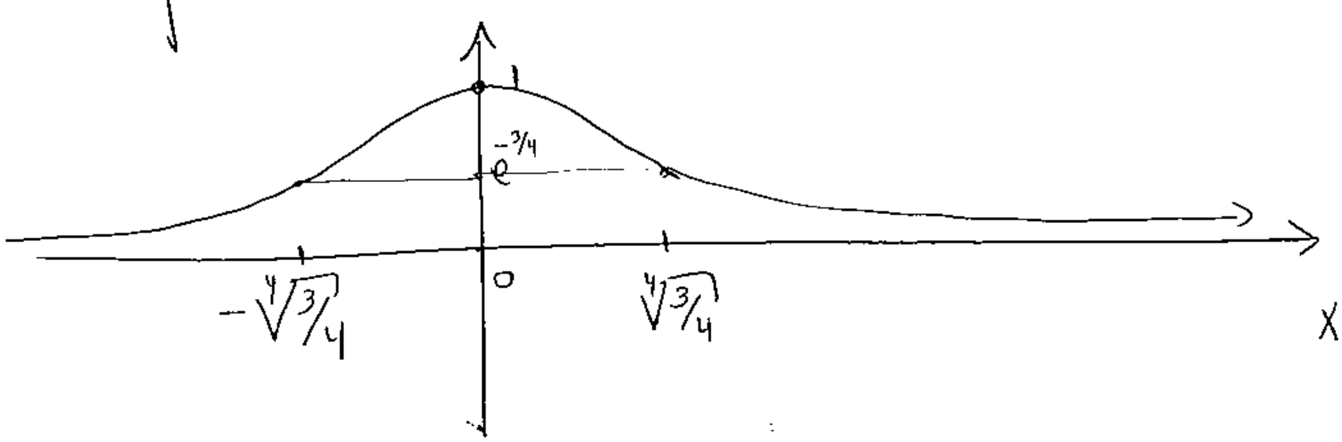
(could use your calculator to decide on the sign of $g''(x)$) (without calculator)

Sketch graph of g

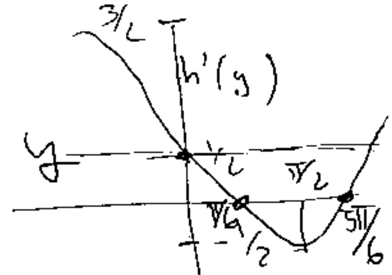
x	$-\infty$		$-\sqrt[4]{\frac{3}{4}}$		0		$\sqrt[4]{\frac{3}{4}}$		$+\infty$
$g(x)$	0	$\nearrow$	$e^{-3/4}$	$\nearrow$	1	$\searrow$	$e^{-3/4}$	$\searrow$	0
$g'(x)$		+ (increasing)			0	— (decreasing)			
$g''(x)$		+ (concave up)	0	− concave down	0	− concave down	0	+ (concave up)	

$g(0) = 1$

$g\left(\pm\sqrt[4]{\frac{3}{4}}\right) = e^{-3/4}$



#25 Find all critical points and points of inflections of $h(y) = \cos(y) + \frac{y}{2}$. Sketch graph.



critical pts : ($h'(y)=0$ or $h'(y)$ DNE)

$$\boxed{h'(y) = -\sin y + \frac{1}{2}}$$ defined for all y

$h'(y) = 0 = -\sin y + \frac{1}{2}$

$\sin y = \frac{1}{2} \Rightarrow \boxed{y = 30° = \pi/6 \quad \text{also } y = 150° = 5\pi/6}$ ⟵ also $\frac{\pi}{6} \pm 2\pi$ and $\frac{5\pi}{6} \pm 2(\pi)$ etc.

all solutions are $\boxed{y = \frac{\pi}{6} + 2K\pi \quad K \text{ integer}}$ ⟵ They repeat periodically with period 2π

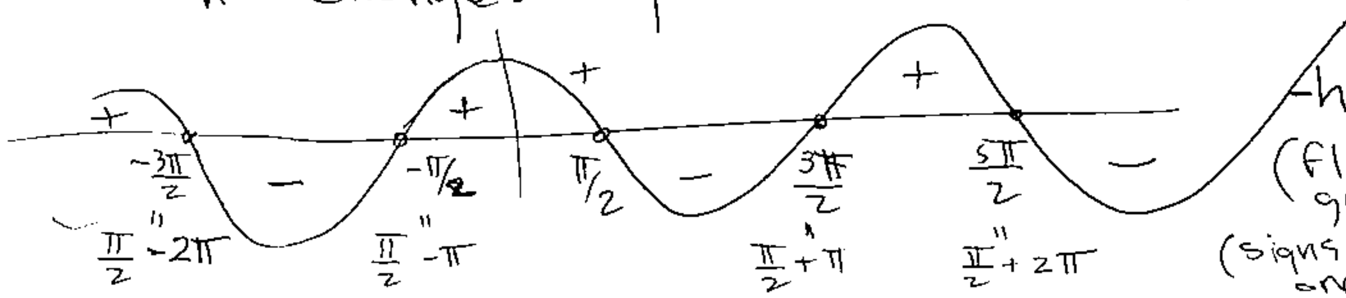
$\boxed{y = \frac{5\pi}{6} + 2K\pi}$ are critical pts

Inflexion pts First find points where $h''(y) = 0$, Then check whether h'' changes sign at those y's (hence There will be a change of concavity for h at that point).

$$\boxed{h''(y) = -\cos y}$$

$h''(y) = 0 = -\cos y \Rightarrow \boxed{y = \frac{\pi}{2} + K\pi, \quad K \text{ integer}}$

we know h'' changes sign at all this points



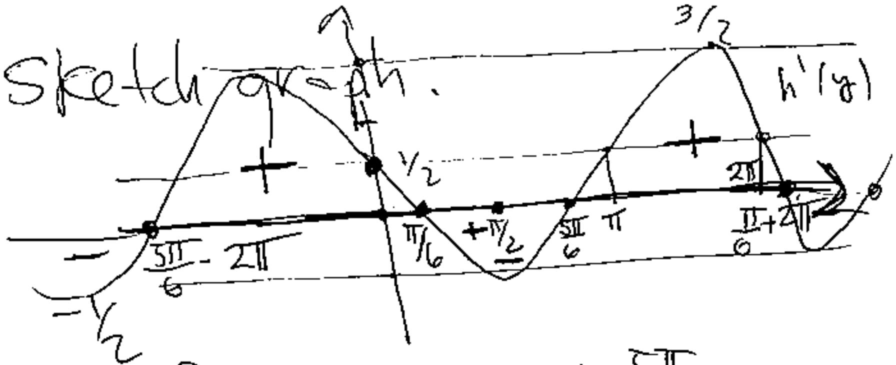
$-h''(x) = \cos y$ (flip to get h'') (signs of $h''(x)$ are opposite)

so h has an inflexion at each point

$$\boxed{y = \frac{\pi}{2} + K\pi, \quad K \text{ integer}}$$ ⟵ inflexion points

#25 (continuation) Sketch graph.

$$h'(y) = -\sin y + \frac{1}{2}$$



Notice that $h'(y) < 0$ if $\frac{\pi}{6} < y < \frac{5\pi}{6}$

in general if $\frac{\pi}{6} + 2k\pi < y < \frac{5\pi}{6} + 2k\pi$ k integer

and $h'(y) > 0$ if $\frac{5\pi}{6} - 2\pi < y < \frac{\pi}{6}$

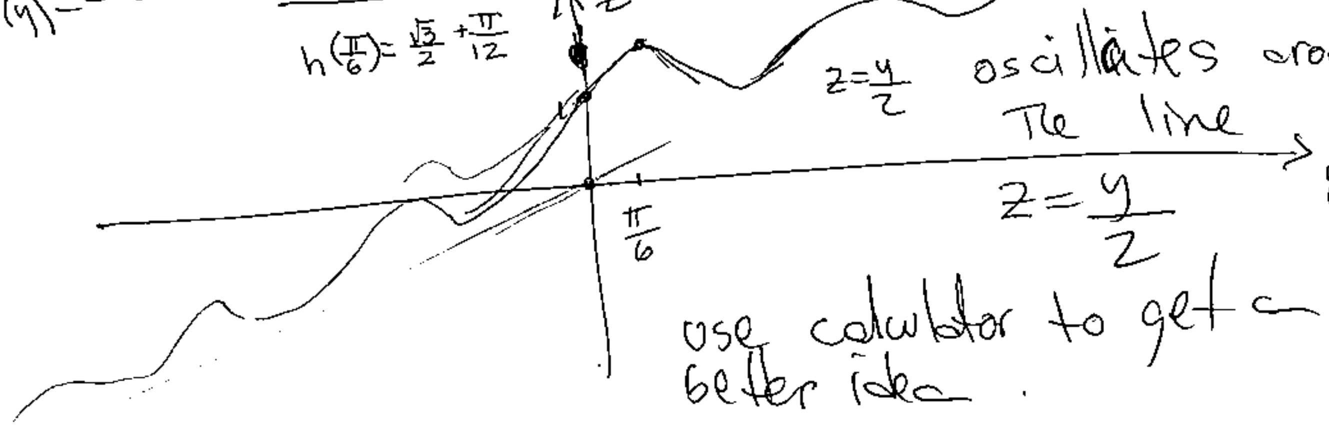
or in general if $\frac{5\pi}{6} - 2\pi + 2k\pi < y < \frac{\pi}{6} + 2k\pi$) k integer

y	$-\infty$	$\frac{\pi}{6}-2\pi$	$\frac{5\pi}{6}-2\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	$\frac{3\pi}{2}$	$\frac{\pi}{6}+2\pi$	$+\infty$
h(y)	$-\infty$	max	min		1	max		min			$+\infty$
h'(y)	+	0	- 0	+	1/2	0	-	0	+	0	
h''(y)		- 0	+ 0		-1		0	+	0	-	

$\lim_{y \to +\infty} h(y) = +\infty$

$\lim_{y \to -\infty} h(y) = -\infty$

$h\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{12}$



oscillates around the line $z = \frac{y}{2}$

use calculator to get a better idea.

#32 (p 234) $p(t) = 2\cos\left(\frac{2\pi t}{3.2}\right)$ position at time t

(a) $p'(t) = 2\left(-\sin\left(\frac{2\pi t}{3.2}\right) \times \left(\frac{2\pi}{3.2}\right)\right)$ chain rule.

$$\boxed{p'(t) = -\frac{4\pi}{3.2}\sin\left(\frac{2\pi t}{3.2}\right)}$$ ← velocity at time t

(b) $p''(t) = \frac{-4\pi}{3.2}\cos\left(\frac{2\pi t}{3.2}\right)\left(\frac{2\pi}{3.2}\right)$

$$\boxed{p''(t) = -\frac{8\pi}{(3.2)^2}\cos\left(\frac{2\pi t}{3.2}\right)}$$ ← acceleration at time t

(c) $p(0) = 2\cos(0) = 2$, $p'(0) = 0$

(d) $p(1.6) = 2\cos\left(\frac{2\pi \times (1.6)}{3.2}\right) = 2\cos\pi = -2$

$p'(1.6) = -\frac{4\pi}{3.2}\sin(\pi) = 0$

(e) $p(3.2) = 2\cos(2\pi) = 2$

$p'(3.2) = 2\sin(2\pi) = 0$.

(f) Notice that, so p obeys this differential equation.

$$\boxed{p''(t) = \frac{-4\pi}{(3.2)^2}\, p(t)}$$

#34. $P(t) = \dfrac{t}{1+2t}$ $t \neq -\frac{1}{2}$ $P(t)$ in grams (g), t time in hours (h)

(a) $\dfrac{P(2)-P(1)}{2-1} = \dfrac{\frac{2}{1+4} - \frac{1}{1+2}}{1} = \frac{2}{5} - \frac{1}{3} = \boxed{\frac{1}{15}}$

(Average rate of change between $t=1$ e $t=2$)

(b) Secant line $m = \frac{1}{15}$ (slope)

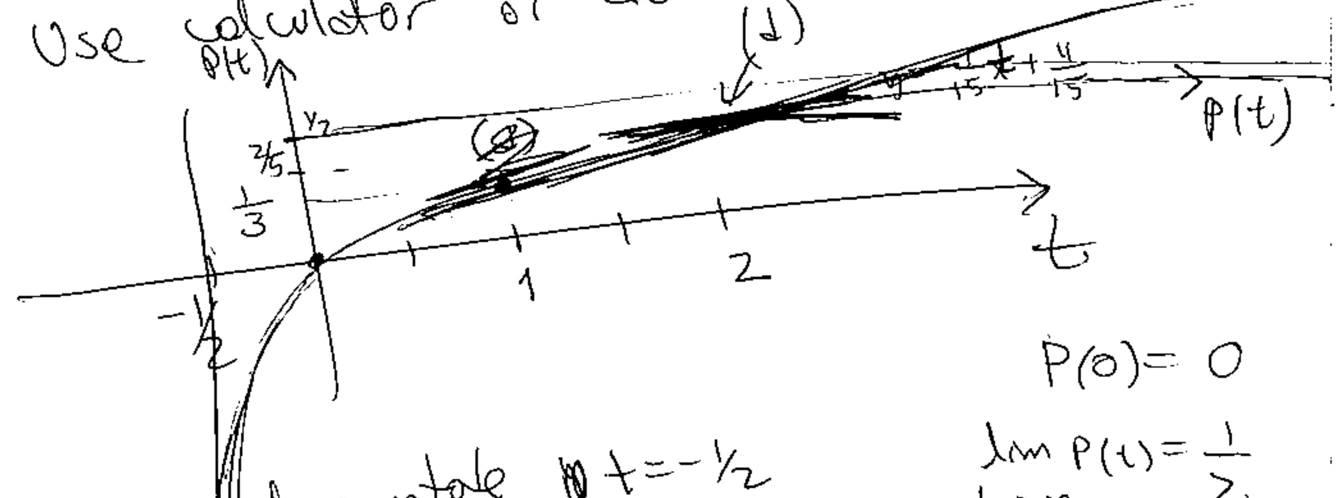
Through $(1, P(1)) = (1, \frac{1}{3})$

equation secant line: $y - \frac{1}{3} = \frac{1}{15}(x-1)$

$y = \frac{1}{15}x - \frac{1}{15} + \frac{1}{3} = \frac{1}{15}x + \frac{-1+5}{15}$

$$\boxed{y = \frac{1}{15}x + \frac{4}{15}}$$

(c) Use calculator or do by hand: $t \neq -\frac{1}{2}$



vertical asymptote $t = -1/2$

$P(0) = 0$

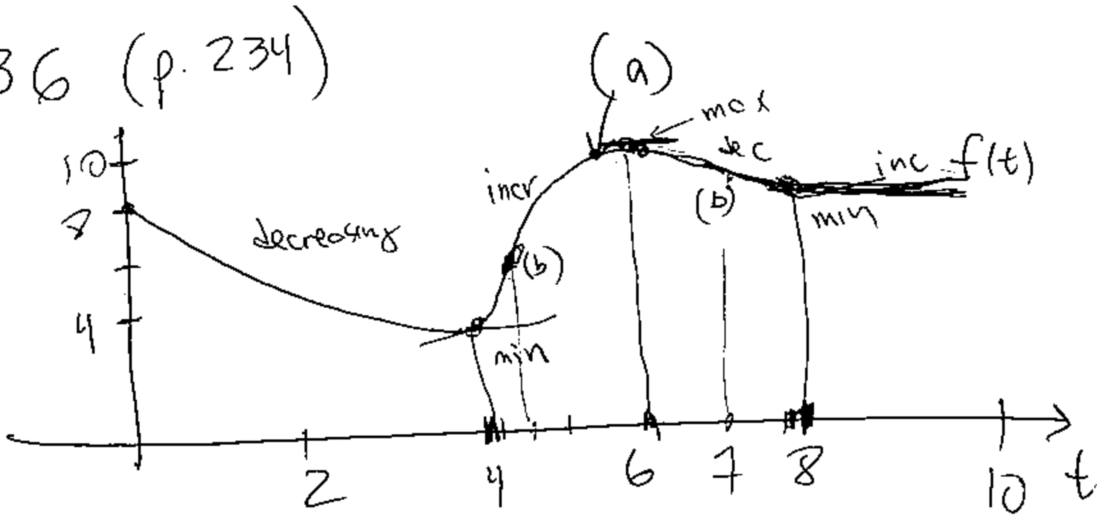
$\lim_{t\to\infty} P(t) = \frac{1}{2}$

(d) Slope of secant seems larger than slope of tangent at $t=2$

(e) $P'(2) = \lim_{h\to 0} \dfrac{P(2+h)-P(2)}{h}$

(can calculate $P'(t)$ and evaluate at $t=2$).

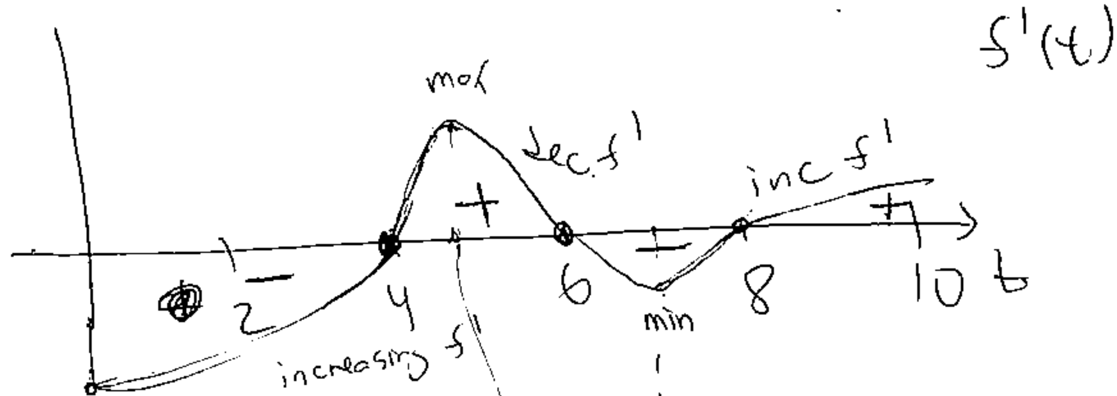
36 (p. 234)



(a) $f'(t) > 0$ and $f''(t) < 0$ (increasing & concave down)

(b) Change of concavity = inflexion pt

(c)



(d)