70. Have your graphics calculator or computer plot the following functions for $-2 \leq x \leq 2$. Do they have inverses?

a. $h_1(x) = x + 2x$

b. $h_2(x) = x^2 + 2x$

c. $h_3(x) = x^3 + 2x$

d. $h_4(x) = x^4 + 2x$

e. $h_5(x) = x^5 + 2x$

Have your computer try to find the formula for the inverses of these functions and plot the results. Does the machine always succeed in finding an inverse when there is one? Does it sometimes find an inverse when there is none?

1.3 The Units and Dimensions of Measurements and Functions

Unlike the numbers and functions studied in many mathematics courses, the measurements and relations used by scientists and applied mathematicians have **units** and **dimensions.** Measurements of number, mass, height, and volume are fundamentally different from each other and are said to have different **dimensions.** Measurements of height in inches or in centimeters describe the same quantity but are presented in different **units.** In this section, we learn how to work with the units and dimensions of both measurements and functions. When we wish to express a measurement or relation in different units, we must use appropriate **conversion factors.** Changing the units of a function corresponds to **scaling** or **shifting** the graph of the function. When we wish to express a function in different dimensions, we must **translate** with a **fundamental relation.**

Converting Between Units

The equation

$$2 + 2 = 2$$

looks hopelessly wrong. But

$$2\ \mathrm{Na}^+ + 2\ \mathrm{Cl}^- = 2\ \mathrm{NaCl}$$

is a standard formula from chemistry. The difference is that the terms in the second equation have explicit units: ions of sodium, ions of chlorine, and molecules of salt. Similarly, although it is absurd to write

$$1 = 2.54$$

it is true that

$$1\ \text{in.} = 2.54\ \text{cm}$$

(This is the official definition of 1 in.) Numbers with units behave very differently from pure numbers.

Often, data are presented with more than one unit. To compare measurements, we must be able to convert between different units.

Example 1.3.1 Converting Miles to Centimeters

Suppose we want to know how many centimeters make up a mile. We can do this in steps, first changing miles to feet, then feet to inches, and then inches to centimeters. To convert between units, we first **write down the basic identities**

$$5280\ \text{ft} = 1\ \text{mile}$$
$$12\ \text{in.} = 1\ \text{ft}$$
$$2.54\ \text{cm} = 1\ \text{in.}$$

These define how many centimeters are in an inch, how many inches are in a foot, and so on. We next **divide** to find three **conversion factors**

$$1 = 5280 \, \frac{\text{ft}}{\text{mile}}$$

$$1 = 12 \, \frac{\text{in.}}{\text{ft}}$$

$$1 = 2.54 \, \frac{\text{cm}}{\text{in.}}$$

Units are manipulated exactly like the numerators and denominators of fractions. We next **multiply** the original measurement by the conversion factors (which are just fancy ways to write the number 1), finding

$$\begin{aligned} 1 \text{ mile} &= 1 \text{ mile} \times 1 \times 1 \times 1 \\ &= 1 \text{ mile} \times 5280 \, \frac{\text{ft}}{\text{mile}} \times 12 \, \frac{\text{in.}}{\text{ft}} \times 2.54 \, \frac{\text{cm}}{\text{in.}} \\ &\approx 160{,}934.4 \text{ cm} \end{aligned}$$

The units cancel just like the numerators and denominators of fractions. This method is often called the "factor-label" method in chemistry.

Algorithm 1.2 The Procedure for Converting Between Units

1. Write down the basic identities that relate the original units to the new units.
2. Divide the basic identities to create conversion factors equal to 1, placing unwanted units where they will cancel.
3. Multiply the original measurement by the appropriate conversion factors.

Example 1.3.2 Using the Algorithm to Change Units of Area

Suppose a house has an area of 2030 square feet. What is this in square meters? The basic identity relating the new unit to the original unit is

$$0.3048 \text{ m} = 1 \text{ ft}$$

We want to place feet in the denominator, so we divide to find the conversion factor

$$1 = 0.3048 \, \frac{\text{m}}{\text{ft}}$$

Square feet are feet times feet, so we can find

$$\begin{aligned} 2030 \text{ ft}^2 &= 1 \times 1 \times 2030 \text{ ft}^2 \\ &= 0.3048 \, \frac{\text{m}}{\text{ft}} \times 0.3048 \, \frac{\text{m}}{\text{ft}} \times 2030 \text{ ft}^2 \\ &\approx 188.6 \text{ m}^2 \end{aligned}$$

Alternatively, we can create a single conversion factor for changing square feet to square meters

$$1 \approx 0.3048^2 \, \frac{\text{m}^2}{\text{ft}^2} \approx 0.0929 \, \frac{\text{m}^2}{\text{ft}^2}$$

Then

$$\begin{aligned} 2030 \text{ ft}^2 &= 1 \times 2030 \text{ ft}^2 \\ &= 0.0929 \, \frac{\text{m}^2}{\text{ft}^2} \times 2030 \text{ ft}^2 \\ &\approx 188.6 \text{ m}^2 \end{aligned}$$

Example 1.3.3 Results of Mixing up Numerator and Denominator

In Example 1.3.1, it would be equally true that

$$1 = \frac{1}{5280}\frac{\text{mile}}{\text{ft}}$$

$$1 = \frac{1}{12}\frac{\text{ft}}{\text{in.}}$$

$$1 = \frac{1}{2.54}\frac{\text{in.}}{\text{cm}}$$

Multiplying by these conversion factors yields

$$\begin{aligned} 1\text{ mile} &= 1\text{ mile} \times 1 \times 1 \times 1 \\ &= 1\text{ mile} \times \frac{1}{5280}\frac{\text{mile}}{\text{ft}} \times \frac{1}{12}\frac{\text{ft}}{\text{in.}} \times \frac{1}{2.54}\frac{\text{in.}}{\text{cm}} \\ &\approx \frac{1}{160934}\frac{\text{mile}^2}{\text{cm}} \end{aligned}$$

The units did not cancel. Even though the result is true, miles and centimeters are left over in a rather inconvenient way. The trick to getting unit conversions to work is making sure that unwanted units in the numerator are canceled by using conversion factors with those same units in the denominator, and vice versa.

Translating Between Dimensions

Miles and centimeters measure the same quantity—length—with different rulers. Miles and grams measure completely different quantities—length and mass. **Dimensions** describe the underlying quantities. **Units** are a particular standard for measurement. Measurements in miles and centimeters share the same dimension and can be **converted** into one another. Measurements with different dimensions cannot. The dimensions and the units commonly used for some common biological measurements are listed in Table 1.3.1.

Suppose we want to measure a bacterial population in grams (mass) rather than numbers or to measure the size of a water droplet in cubic centimeters (volume) rather than centimeters (length of radius). We cannot apply a series of identities like 1 in. = 2.54 cm because we are translating between dimensions rather than converting between units. Instead of **identities,** we use **fundamental relations** among measurements with different dimensions (Table 1.3.2).

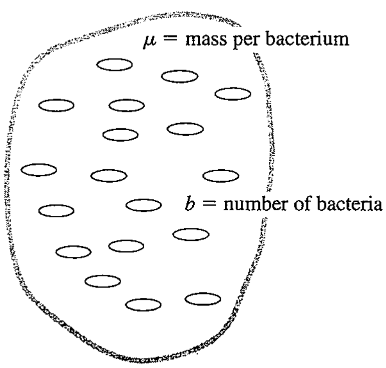
Table 1.3.1 Some quantities, their dimensions, and sample units

Quantity	Dimensions	Sample Units
length	length	meter, micron, inch
duration	time	second, minute, day
mass	mass	gram, kilogram
area	length^2	square meter, acre
volume	length^3	liter, cubic meter, gallon
speed	length/time	meters/second, mph
acceleration	length/time^2	meters/second^2
force	$\text{mass} \times \text{length/time}^2$	dynes, pounds
density	mass/length^3	grams/liter

Table 1.3.2 Important fundamental relations in biology

Relation	Variables	Formula
Geometric Relations		
Volume of a sphere	$V=$ volume $r=$ radius	$V=\frac{4\pi}{3}r^3$
Surface area of a sphere	$S=$ surface area $r=$ radius	$S=4\pi r^2$
Area of a circle	$A=$ area $r=$ radius	$A=\pi r^2$
Perimeter of a circle	$P=$ perimeter $r=$ radius	$P=2\pi r$
Volume, area, and thickness	$V=$ volume $A=$ area $T=$ thickness	$V=AT$
Relations Involving Mass		
Total number and mass	$m=$ total mass $\mu=$ mass per individual $b=$ number of individuals	$m=\mu b$
Mass, density, and volume	$M=$ mass $\rho=$ density $V=$ volume	$M=\rho V$

Example 1.3.4 Translating Between Number and Total Mass



total mass = mass per bacterium × number of bacteria $m=\mu b$

FIGURE 1.3.33
Fundamental relation between mass and number

The fundamental relation between number and total mass is

$$\text{total mass} = \text{mass per bacterium} \times \text{number of bacteria}$$

Let m represent the total mass, μ the mass per bacterium, and b the number of bacteria (Figure 1.3.33). The fundamental relation can be rewritten in mathematical symbols as

$$m=\mu b$$

Like numbers, variables representing measurements have both dimensions and units. The variable m has units of grams, μ has units of grams, and b has units of number of bacteria. If

$$b=2.0\times 10^5 \text{ and } \mu=3.1\times 10^{-9}\text{ g}$$

then

$$m=\left(3.1\times 10^{-9}\text{ g}\right)\cdot\left(2.0\times 10^5\right)=6.2\times 10^{-4}\text{ g}$$

Example 1.3.5 Computing the Volume of a Spherical Droplet from Its Radius

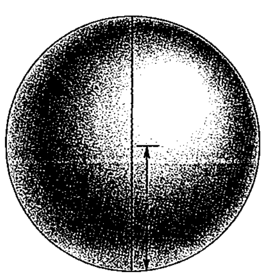


FIGURE 1.3.34
Volume and radius of a spherical droplet

Computing the volume of a droplet requires a fundamental relation between radius and volume, which depends on the **shape** of the droplet. Suppose that droplets are perfect spheres. The fundamental relation between radius and volume comes from geometry. The volume V of a sphere with radius r is

$$V=\frac{4\pi}{3}r^3$$

where r has units of centimeters and V has units of cubic centimeters, or cm^3 (Figure 1.3.34). If a droplet has radius 0.23 cm, the volume is

$$V=\frac{4\pi}{3}0.23^3\approx 0.051\text{ cm}^3$$

Example 1.3.6 Computing the Mass from the Volume

To compute the mass of the droplet in Example 1.3.5 from its volume, we use the fundamental relation

$$\text{mass} = \text{density} \times \text{volume}$$

If we denote the density by ρ and the mass by M, the fundamental relation can be rewritten in mathematical symbols as

$$M = \rho V$$

Suppose that the droplet is made of mercury, which has density of 13.58 g/cm^3. The mass M of a droplet with radius 0.23 cm is

$$M \approx 13.58 \frac{\text{g}}{\text{cm}^3} \cdot 0.051 \text{ cm}^3 \approx 0.693 \text{ g}$$

It is crucial to check the dimensions and units in any unfamiliar equation you encounter. In the equation

$$M = \rho V$$

the dimensions of M are mass, the dimensions of V are length^3, and the dimensions of ρ are mass/length^3. Rewriting this equation in dimensions yields

$$\text{mass} = \frac{\text{mass}}{\text{length}^3} \times \text{length}^3$$

The length^3 terms cancel and the dimensions of the two sides match, as they must. This procedure is called **dimensional analysis.** Many errors can be nipped in the bud by checking the dimensions. An equation with inconsistent dimensions is not merely incorrect, it is nonsensical.

Functions and Units: Composition, Scaling, and Shifting

Because functions describe relations between measurements, both their inputs and their outputs have units and dimensions. Care must be taken to ensure that functions are composed only when their units and dimensions match.

Example 1.3.7 Composing Functions with Dimensions

Suppose that F takes the radius r of a sphere as input and returns the volume of the sphere as output (as with the spherical droplet in Example 1.3.5). Then F has the formula

$$F(r) = \frac{4\pi}{3} r^3$$

Suppose G takes a volume V as input and returns the mass of an object with that volume as output according to mass = density × volume, or

$$G(V) = 13.58V$$

using the density $\rho = 13.58$ g/cm^3. The composition $G \circ F$ takes radius as input and returns mass as output in a single step (Figure 1.3.35). The composition is

$$\begin{aligned}(G \circ F)(r) &= G(F(r)) \\ &= G\left(\frac{4\pi}{3} r^3\right) \\ &= 13.58 \frac{4\pi}{3} r^3 \approx 56.88 r^3\end{aligned}$$

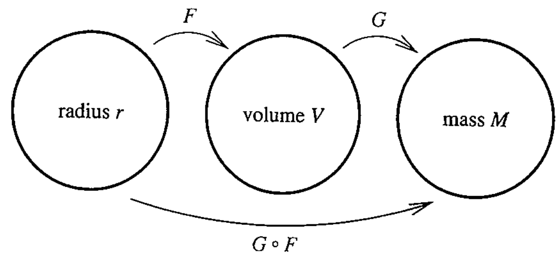


FIGURE 1.3.35
The composition of two functions with units

We could find the mass of a droplet with radius 0.23 cm in two steps by finding the volume and then the mass with the steps

$$V = F(0.23 \text{ cm}) = \frac{4\pi}{3} 0.23 \text{ cm}^3 \approx 0.051 \text{ cm}^3$$

$$M = G(0.051 \text{ cm}^3) = 13.58 \text{ g/cm}^3 \cdot 0.051 \text{ cm}^3 \approx 0.693 \text{ g}$$

Alternatively, we could find the mass in a single step by composing the functions G and F,

$$M = (G \circ F)(0.23 \text{ cm}) = 13.58 \text{ g/cm}^3 \frac{4\pi}{3} 0.23 \text{ cm}^3 \approx 0.692 \text{ g}$$

with the slight discrepancy caused by round-off error.

The function F accepts inputs with dimensions of length and returns outputs with dimensions of volume. G accepts inputs with dimensions of volume and returns outputs with dimensions of mass. Because G takes as input precisely what F provides as output, the composition makes sense.

What if we tried to compute $F \circ G$? F cannot accept an input with dimensions of mass, and such outputs are the only outputs that G can return. It is impossible to compute the volume of a sphere with a radius of 4.3 grams. This composition is nonsense.

Changing the units of a measurement that acts as the input or output of a function corresponds to **scaling** or **shifting** the graph of the function. In most cases, the measurement corresponding to a value of zero is the same in different units, and graphs of the function are scaled by changes in units. When units differ in the value corresponding to zero (as with temperature), then the graph of the function is shifted.

Example 1.3.8 Scaling Functions on the Vertical Axis by Changing Units of the Output

Consider the following function describing the growth of a bacterial population:

$$b(t) = 2.0t$$

where t is measured in hours and $b(t)$ is in millions of bacteria.

If bacteria are measured instead in thousands of bacteria, we choose a different variable to represent the new measurement, such as B. The relation becomes

$$B(t) = 2.0t \frac{\text{million bacteria}}{\text{hr}} \times \frac{1000 \text{ thousand bacteria}}{\text{one million bacteria}}$$
$$= 2000.0t \frac{\text{thousand bacteria}}{\text{hr}}$$

(Figure 1.3.36). The graph has been changed by **scaling** the vertical axis. It looks the same, except that the numbers that appear on the axis are 1000 times larger.

Example 1.3.9 Scaling Functions on the Horizontal Axis by Changing Units of the Input

Suppose again that $b(t) = 2.0t$ where b is measured in millions and t is measured in hours. If time is measured in minutes instead of hours, we must define a new variable

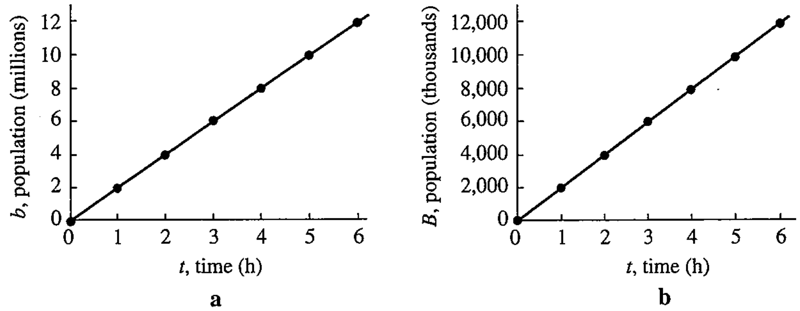


FIGURE 1.3.36
A growing bacterial population: new units on vertical axis

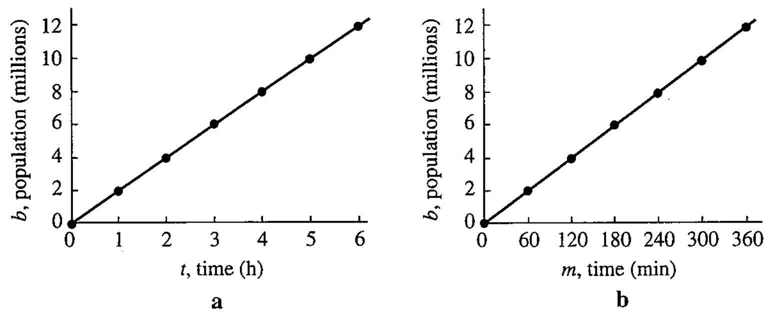


FIGURE 1.3.37
A growing bacterial population: new units on horizontal axis

for time, perhaps m. The relation becomes

$$\begin{aligned} b(m) &= 2.0m \frac{\text{million bacteria}}{\text{hr}} \times \frac{1\text{ hr}}{60\text{ min}} \\ &= \frac{1.0}{30.0}m \frac{\text{million bacteria}}{\text{min}} \\ &\approx 0.0333m \frac{\text{million bacteria}}{\text{min}} \end{aligned}$$

(Figure 1.3.37).

The graph has been changed by **scaling** the horizontal axis. Again, it looks the same, except that the numbers labeling the horizontal axis are 60 times larger. The data point $(1, 2)$ that indicated that there were 2.0 million bacteria after 1 hour becomes the point $(60, 2)$, indicating 2.0 million bacteria after 60 minutes.

t (h)	m (min)	b (millions)	B (thousands)
0	0	0	0
1	60	2	2,000
2	120	4	4,000
3	180	6	6,000
4	240	8	8,000
5	300	10	10,000
6	360	12	12,000

Example 1.3.10 Shifting Functions Vertically by Changing Units

Temperatures can be measured on scales with different values of zero (Figure 1.3.38a). For example, 0°C corresponds to 273.15 K. (The units of temperature on the Kelvin scale are referred to as kelvins, rather than as degrees, and no degree symbol is used.) Suppose that the temperature of a snake after digesting a mouse with mass m obeys the equation

$$T(m) = 10 + 0.06m$$

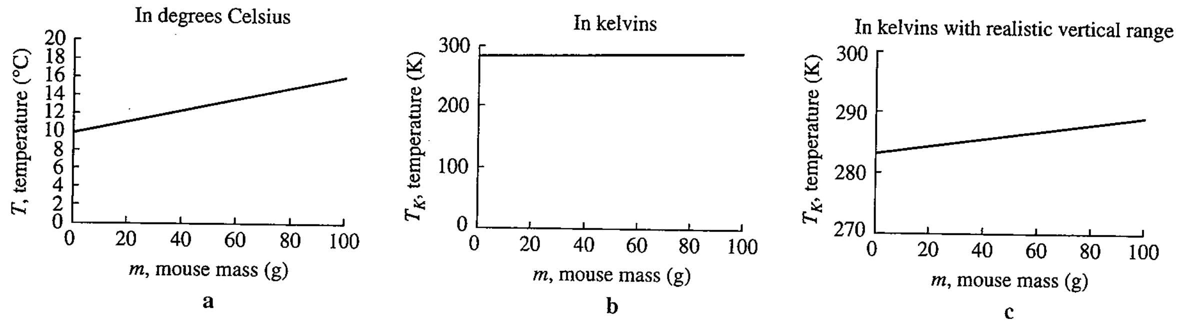


FIGURE 1.3.38

Snake temperature in different units

where temperature is measured in °C and mouse mass is measured in grams. The temperature in kelvins (K), which we denote by T_K, can be found by adding 273.15 to the temperature in °C, or

$$T_K(m) = 273.15 + T(m)$$

In the new units,

$$T_K(m) = 283.15 + 0.06m$$

(Figure 1.3.38b). The graph has been **shifted** vertically. It looks different because 0 K is so far from the temperatures measured. A graph with the horizontal axis set at 270 K is more informative (Figure 1.3.38c).

Example 1.3.11 Shifting Functions Horizontally by Changing Units

Suppose that the sprint speed of a snake is a function of temperature according to the equation

$$s(T) = 4 + 0.1T$$

where temperature is measured in °C and speed is measured in m/sec (Figure 1.3.39a). Temperature in K can be found by adding 273.15 to temperature in °C, or $T_K = T + 273.15$. To write the equation in the new units, we solve for $T = T_K - 273.15$, giving

$$s(T_K) = 4 + 0.1(T_K - 273.15)$$

(Figure 1.3.39b). The graph has been **shifted** horizontally. However, by including extremely cold temperatures, the function predicts impossible negative speeds. A graph showing only the vertical range from 275 K to 325 K is more informative (Figure 1.3.39c).

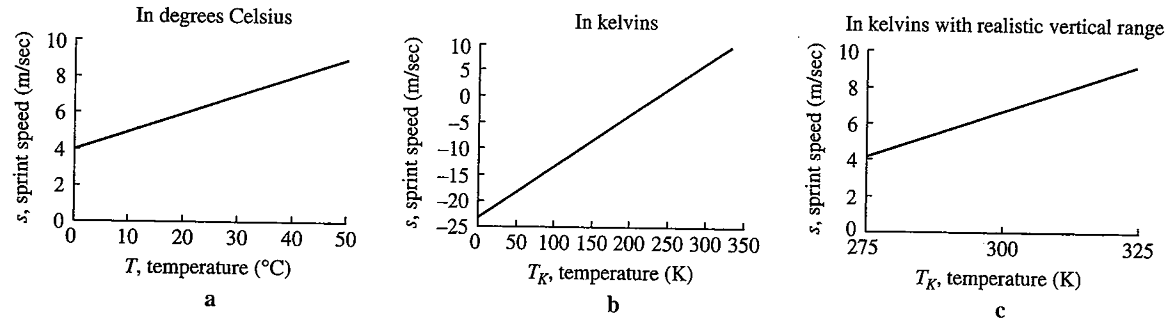


FIGURE 1.3.39

Snake speed in different units

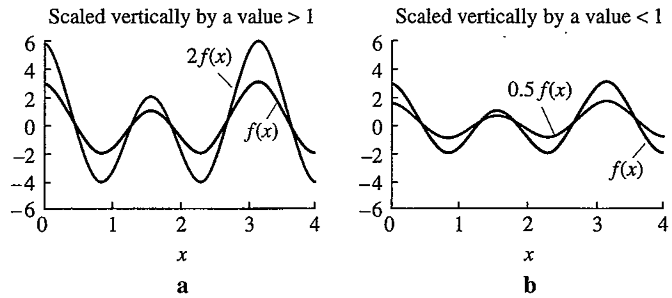


FIGURE 1.3.40
Vertically scaling a function

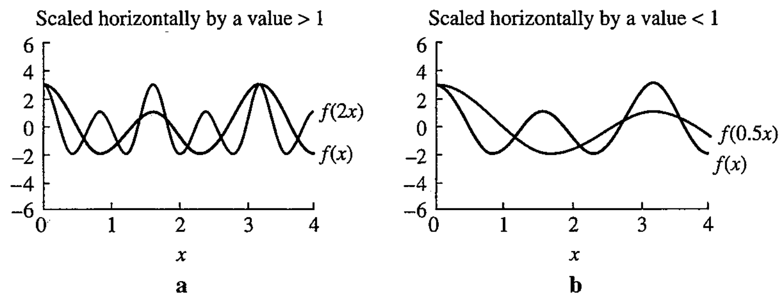


FIGURE 1.3.41
Horizontally scaling a function

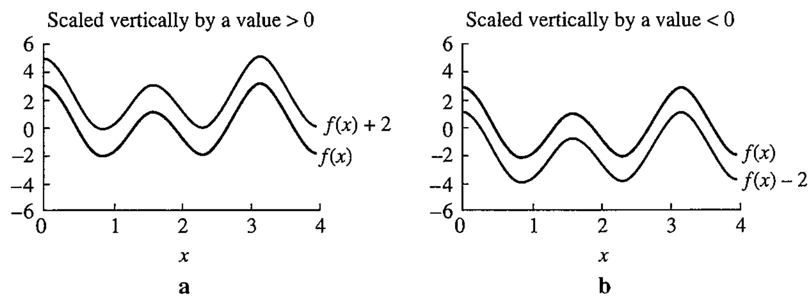


FIGURE 1.3.42
Vertically shifting a function

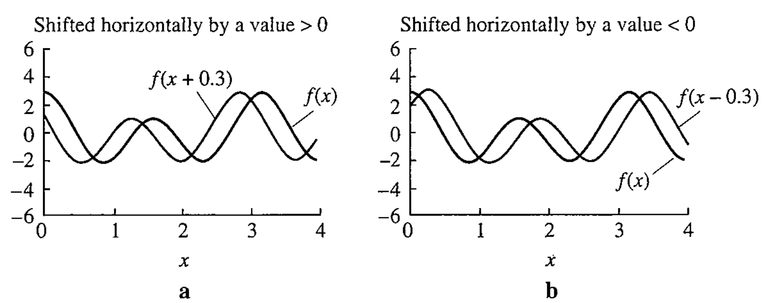


FIGURE 1.3.43
Horizontally shifting a function

Mathematically, scaling corresponds to multiplying the value or the argument of a function by a constant, whereas shifting corresponds to adding a constant to the value or argument. In particular, the function $f(x)$ can be scaled or shifted as follows:

- **Vertical Scaling** Multiply the value by the constant a to make the new function $af(x)$ (Figure 1.3.40).
- **Horizontal Scaling** Multiply the argument by the constant a to make the new function $f(ax)$ (Figure 1.3.41).
- **Vertical Shifting** Add the constant a to the value to make the new function $f(x) + a$. (Figure 1.3.42).
- **Horizontal Shifting** Add the constant a to the argument to make the new function $f(x + a)$ (Figure 1.3.43).

Vertical scaling and shifting work as one might expect: multiplying by a value greater than 1 stretches the function (Figure 1.3.40a), and adding a value greater than 0 raises the function (Figure 1.3.42a). Horizontal shifting and scaling, however, might seem to work backwards. Multiplying the argument by a value greater than 1 compresses the function (Figure 1.3.41a), and adding a positive constant to the argument moves the function to the left (Figure 1.3.43a).

Example 1.3.12 Vertically and Horizontally Scaling a Function

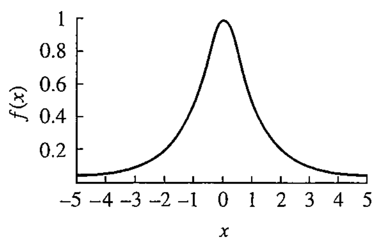


FIGURE 1.3.44
The original function

Consider the function

$$f(x) = \frac{1}{1+x^2}$$

shown plotted for $-5 \le x \le 5$ (Figure 1.3.44). We will scale the value and the argument of this function by values both greater than and less than 1.

Consider the input $x = -1.0$. Then

$$2f(-1.0) = 2 \cdot \frac{1}{1+(-1.0)^2} = 1.0$$

$$0.5f(-1.0) = 0.5 \cdot \frac{1}{1+(-1.0)^2} = 0.25$$

$$f(2 \cdot -1.0) = f(-2.0) = \frac{1}{1+(-2.0)^2} = 0.2$$

$$f(0.5 \cdot -1.0) = f(-0.5) = \frac{1}{1+(-0.5)^2} = 0.8$$

The following table gives several values of the scaled functions.

Argument	Original Function	Vertically Scaled by a Value > 1	Vertically Scaled by a Value < 1	Horizontally Scaled by a Value > 1	Horizontally Scaled by a Value < 1
x	$f(x)$	$2f(x)$	$0.5f(x)$	$f(2x)$	$f(0.5x)$
−5.0	0.038	0.077	0.019	0.010	0.14
−4.0	0.059	0.120	0.029	0.015	0.20
−3.0	0.10	0.20	0.050	0.027	0.31
−2.0	0.20	0.40	0.10	0.059	0.50
−1.0	0.50	1.0	0.250	0.20	0.80
0.0	1.0	2.0	0.50	1.0	1.0
1.0	0.50	1.0	0.250	0.20	0.80
2.0	0.20	0.40	0.10	0.059	0.50
3.0	0.10	0.20	0.050	0.027	0.31
4.0	0.059	0.120	0.029	0.015	0.20
5.0	0.038	0.077	0.019	0.010	0.14

Scaling vertically makes the graph of the function taller if it is scaled by a value greater than 1 or shorter if it is scaled by a value less than 1. Scaling horizontally makes the graph of the function thinner if it is scaled by a value greater than 1 or wider if it is scaled by a value less than 1.

Example 1.3.13 Vertically and Horizontally Shifting a Function

Consider again the function in Example 1.3.12 (Figure 1.3.44). We will shift the value and the argument of this function by values both greater than and less than 0.

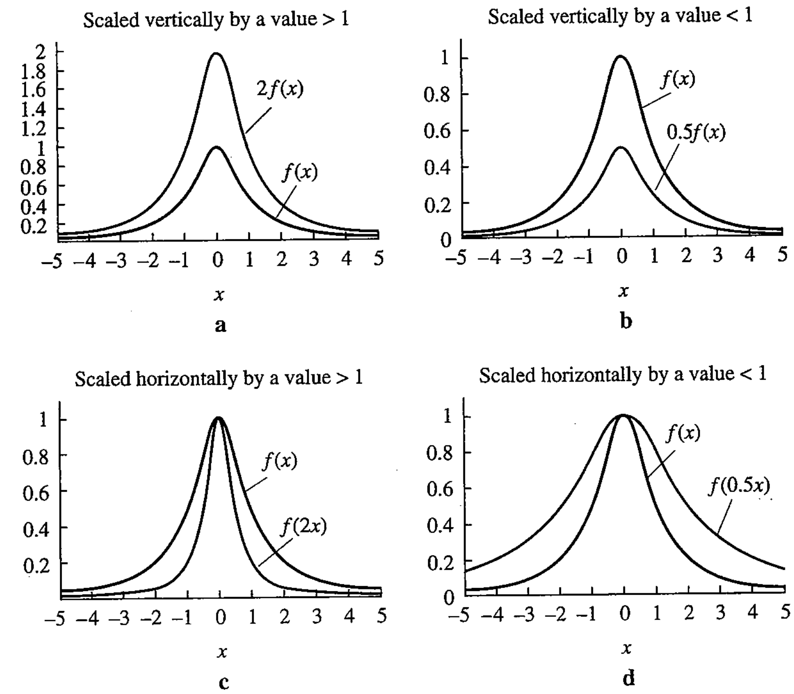


FIGURE 1.3.45
Vertically and horizontally scaling a function

Shifting the function vertically corresponds to adding a constant to f.

Argument	Original Function	Vertically Shifted by a Value > 0	Vertically Shifted by a Value < 0	Horizontally Shifted by a Value > 0	Horizontally Shifted by a Value < 0
x	$f(x)$	$f(x)+2$	$f(x)-2$	$f(x+2)$	$f(x-2)$
−5	0.038	2.04	−1.96	0.10	0.020
−4	0.059	2.06	−1.94	0.20	0.027
−3	0.10	2.1	−1.9	0.50	0.038
−2	0.20	2.2	−1.8	1.0	0.059
−1	0.50	2.5	−1.5	0.50	0.10
0	1.0	3.0	−1.0	0.20	0.20
1	0.50	2.5	−1.5	0.10	0.50
2	0.20	2.2	−1.8	0.059	1.0
3	0.10	2.1	−1.9	0.038	0.50
4	0.059	2.06	−1.94	0.027	0.20
5	0.038	2.04	−1.96	0.020	0.10

Shifting vertically moves the function up if it is shifted by a value greater than 0 or down if it is shifted by a value less than 0. Shifting horizontally moves the function to the right if it is shifted by a value greater than 0 or to the left if it is shifted by a value less than 0.

Checking: Dimensions and Estimation

Just as it is essential to check the dimensions and units of equations, it is essential to check the plausibility of the numerical results of calculations. Suppose you wanted to figure out how many tons all the people in the United States weigh. Each person (counting children) weighs on average about 100 pounds, or $1/20$ of a ton per person. If there are about 300 million people, they should weigh a net amount of around 15 million

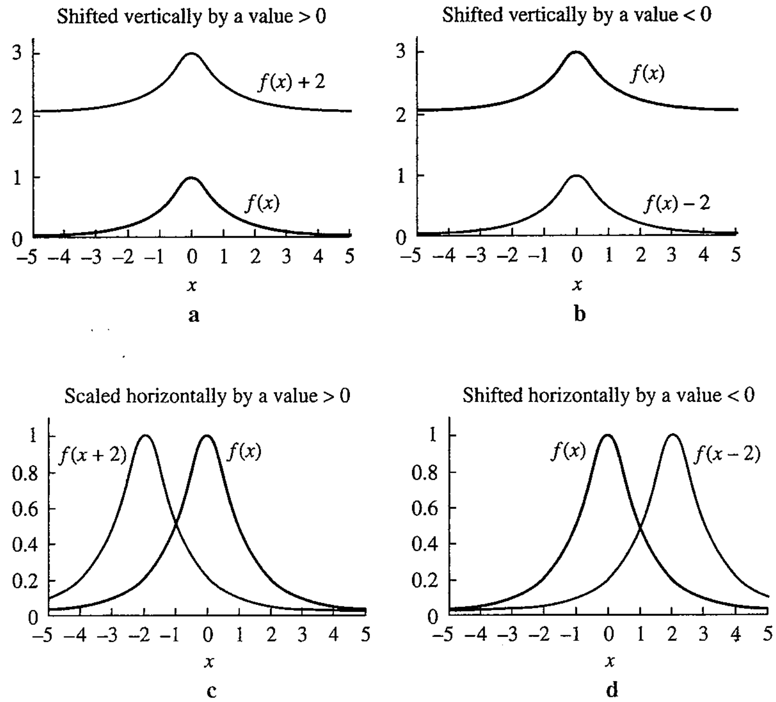


FIGURE 1.3.46
Vertically and horizontally shifting a function

tons (using the fundamental relation that total mass is equal to mass per individual times the number of individuals). If you had worked this out with a more complicated set of measurements and found a more precise answer of 14.7 million tons, everything is probably all right. If the complicated method gave an answer of 1.47 million tons, it needs to be checked.

Example 1.3.14 Estimating the Area of a Bacterial Colony

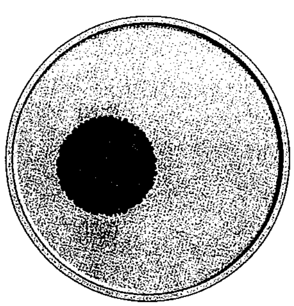


FIGURE 1.3.47
A bacterial colony on a Petri dish

How much area does a colony of 2.0×10^5 bacteria take up on a Petri dish (Figure 1.3.47)? One method is to use our computation of the mass (6.2×10^{-4} g in Example 1.3.4), convert to volume, and then to find the area by dividing by the thickness. If we assume that bacteria have approximately the density of water (which is 1 g/cm^3), the volume is

$$V = \frac{M}{\rho} = \frac{6.2 \times 10^{-4}\ \text{g}}{1.0 \times 10^{-12}\ \text{g}/\mu\text{m}^3}$$
$$= 6.2 \times 10^{8}\ \mu\text{m}^3$$

Here we used the fact that $1.0\,\mu\text{m}^3 = 10^{-12}\,\text{cm}^3$ to find density in g/μm^3. The next **fundamental relation** translates between volume and area and is

$$\text{volume} = \text{area} \times \text{thickness}$$

so that

$$\text{area} = \frac{\text{volume}}{\text{thickness}}$$

If we estimate the thickness of the colony to be about 20 μm (roughly the thickness of a cell),

$$\text{area} \approx \frac{6.2 \times 10^{8}\ \mu\text{m}^3}{20\ \mu\text{m}}$$
$$\approx 3 \times 10^{7}\ \mu\text{m}^2$$

This sounds rather large. To convert to square centimeters, we use the basic identity

$$1\ \mu\text{m} = 10^{-4}\ \text{cm}$$

so that the conversion factor is

$$1 = 10^{-4}\ \text{cm}/\mu\text{m}$$

Multiplying yields

$$\begin{aligned} 3 \times 10^7\ \mu\text{m}^2 &= 3 \times 10^7\ \mu\text{m}^2 \times 10^{-4}\ \text{cm}/\mu\text{m} \times 10^{-4}\ \text{cm}/\mu\text{m} \\ &= 0.3\ \text{cm}^2 \end{aligned}$$

To find the radius, we use the **fundamental geometric relation** between the area A and radius r for a circle,

$$A = \pi r^2$$

The radius r of this colony satisfies

$$\pi r^2 \approx 0.3\ \text{cm}^2$$

Solving for r yields

$$r \approx \sqrt{\frac{0.3\ \text{cm}^2}{\pi}} \approx 0.3\ \text{cm}$$

This colony is actually quite small, but large enough to be seen.

Example 1.3.15 Fermi's Piano Tuner Problem

The great physicist Enrico Fermi emphasized our ability to combine educated guesses of ordinary quantities to estimate more complicated quantities. For example, we can estimate the number of piano tuners in Salt Lake City and vicinity knowing only that the population is about 1,000,000 people. First, we estimate the number of pianos. If the average family contains four members, the number of families is about 250,000. As a rough guess, suppose that one in five families owns a piano. There will then be 50,000 pianos. If the average piano tuner tunes 4 pianos every day and works for 250 days per year (50 weeks of 5 days), she will tune 1000 pianos per year. If each piano is tuned once in 2 years (another very rough guess), there will be 25,000 pianos tuned per year, requiring 25 piano tuners. A quick check of the phone book indicates that there are in fact about 30 piano tuners. Like all mathematical models, this method requires us to **analyze** the problem by breaking it into component parts. If our estimate proved to be extremely inaccurate, we could check each of our assumptions to find the source of the error.

Summary Understanding scientific equations and formulas requires understanding the **units** and **dimensions** of the measurements and variables. **Dimensions** describe the underlying quantities and tell what sort of thing is being measured. **Units** express numerical values based on a particular scale. Converting between units can be done by starting with **basic identities,** deriving **conversion factors,** and multiplying. Unit conversions correspond to **scaling** or **shifting** the graphs of functions describing the measurements. Translating between measurements with different dimensions requires using **fundamental relations,** such as those between mass and volume or between volume and radius. All such relations, and every scientific formula, should be checked for consistency with **dimensional analysis.** Using basic identities and fundamental relations, we can compute useful estimates of quantities, often without using a calculator. Checking results for plausibility can help locate mistakes.

1.3 Exercises

Mathematical Techniques

1–6 ▪ Convert the following into the new units.

1. Find 3.4 pounds in grams (1 oz is 28.35 g and 1 lb is 16 oz).

2. Find one yard in millimeters (1 in. is 25.4 mm and 1 yd is 36 in.).

3. Find 60 years in hours (1.0 year $\approx$ 365.25 days).

4. Find 65 miles per hour in centimeters per second (using information in Example 1.3.1).

5. Find 2.3 grams per cubic centimeter in pounds per cubic foot (using conversion factors in Exercises 1 and 2).

6. Find 9.807 $\mathrm{m/sec^2}$ (the acceleration due to gravity) in miles per hour per second.

7–10 ▪ Compute the answers by adding the given quantities.

7. A boy who is 1.34 m tall grows 2.3 cm. How tall is he then?

8. After waiting for 1.2 hr for a plane flight, you are told you will have to wait another 17 min. What is the total wait?

9. You purchase 6 apples that weigh 145 gm each, and 7 oranges that weigh 123 gm each. What is the total weight if you add the apples to the oranges?

10. The density of the apples in the previous problem is 0.8 $\mathrm{g/cm^3}$ and the density of the oranges is 0.95 $\mathrm{g/cm^3}$. What is the total volume if you add the apples to the oranges?

11–14 ▪ Figure out which is larger.

11. The area of a square with side length 1.7 cm or of a disk with radius 1.0 cm.

12. The perimeter of a square with side length 1.7 cm or of a circle with radius 1.0 cm.

13. The volume of a sphere with radius 100 m or of a lake 50 cm deep with an area of 3.0 $\mathrm{km^2}$.

14. The surface area of a sphere with radius 100 m or the surface area of a lake with area 3.0 $\mathrm{km^2}$.

15–18 ▪ Find the dimensions of the following quantities.

15. Pressure (force per unit area).

16. Energy (force times distance).

17. The rate of change of the area of a colony of bacteria growing on a plate.

18. The force of gravity between two objects is equal to Gm_1m_2/r^2 where m_1 and m_2 are the masses of the two objects, and r is the distance between them. What are the dimensions of the gravitational constant G?

19–22 ▪ Check whether the following formulas are dimensionally consistent.

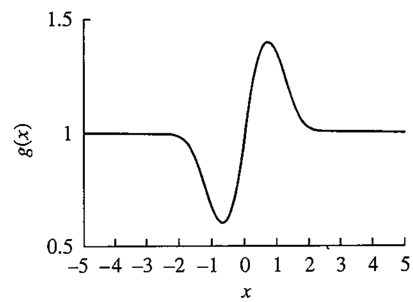
19. Distance = rate times time

20. Velocity = acceleration times time

21. Force = mass times acceleration

22. Energy = 1/2 mass times the square of velocity (see Exercise 16 for the units of energy)

23–26 ▪ Using the graph of the function $g(x)$, sketch a graph of the shifted or scaled function, say which kind of shift or scale it is, and compare with the original function.



23. $4g(x)$

24. $g(x) - 1$

25. $g(x/3)$

26. $g(x+1)$

Applications

27–30 ▪ Find the volumes of the following cartoon trees (drawing a sketch can help), assuming first that the height is 23.1 m and then that the height is 24.1 m. What is the ratio of the volume of the larger tree to that of the smaller tree?

27. A tree is a perfect cylinder with radius 0.5 m no matter what the height (the volume of a cylinder with height h and radius r is πhr^2).

28. A tree is a perfect cylinder with radius equal to 0.1 times the height.

29. A tree looks like the tree in Exercise 27, but with half the height in the cylindrical trunk and the other half in a spherical blob on top.

30. A tree looks like the tree in Exercise 27, but with 90% of the height in the cylindrical trunk and the remaining 10% in a spherical blob on top.

31–34 ▪ Find the masses in kilograms of the following objects (the density of water is 1.0 $\mathrm{g/cm^3}$).

31. A water bed that is 2 meters long, 20 cm thick, and 1.5 m wide.

32. A spherical cow with diameter 1.3 m and density 1.3 times that of water.

33. A coral colony consisting of 3200 individuals each weighing 0.45 g.

34. A circular colony of mold with diameter of 4.8 cm and density of 0.0023 $\mathrm{g/cm^2}$.

35–38 ▪ Change the units in the following functions, and compare a graph in the new units with that in the original units.

35. (Based on Section 1.2, Exercise 45) The number of bees b on a plant is given by $b = 2f + 1$ where f is the number of flowers. Suppose each flower has 4 petals. Graph the number of bees as a function of the number of petals.

36. The number of cancerous cells c as a function of radiation dose r is

$$c = r - 4$$

for r (measured in rads) greater than or equal to 5, and is zero for r less than 5 (as in Section 1.2, Exercise 46). Suppose that radiation is instead measured in millirads (1 rad = 1000 millirads).

37. Insect development time A (in days) obeys $A = 40 - \frac{T}{2}$ where T represents temperature in °C for C between 10 and 40 (as in Section 1.2, Exercise 47). Suppose that development time is measured in hours.

38. Tree height h (in meters) follows the formula

$$h = \frac{100a}{100 + a}$$

where a represents the age of the tree in years (as in Section 1.2, Exercise 48). Suppose that tree age is measured instead in decades.

39–44 ▪ Estimate the following.

39. The speed of light in centimeters per nanosecond (1 ns = 10^{-9} sec). (The speed of light is about 186,000 miles/s). A fast computer takes about 0.3 ns per operation. How far does light travel in the time required by one computer operation?

40. The speed that your hair grows in miles per hour (this problem was borrowed from the book *Innumeracy*).

41. The weight of the earth in kilograms. The earth is approximately a sphere with radius 6500 km and density five times that of water.

42. Suppose a person eats 2000 kCal per day. Using the facts that 1 kCal is approximately 4.2 kJ (a kilojoule is a unit of energy equal to 1000 joules) and 1 watt is one joule per second (a unit of power), about how many watts does a person use?

43. If a movie is about 2 hours long, how many movies could you watch if you spent half your time watching movies for 60 years?

44. The volume of all the people on earth in cubic kilometers. If a large mine is about 3 km across and 1 km deep, would they all fit?

45–48 ▪ The following problems give several ways to estimate the size or number of cells in your body. A cell is roughly a sphere 10 μm in radius, where 1 μm is 10^{-6} m.

45. Using the fact that the density of a cell is approximately the density of water, and that water weighs 1 g/cm^3, estimate the number of cells in your body.

46. Estimate your volume in cubic meters by pretending you are shaped like a board. Pretending that cells are cubes 20 μm on a side, what do you estimate the number of cells to be by this method?

47. The brain weighs about 1.3 kg, and it is estimated to have about 100 billion neurons and 10 to 50 times as many other cells (glial cells). Is this consistent with our previous estimates in Exercises 45 and 46?

48. The nematode *C. elegans* is a cylinder about 1 mm long and 0.1 mm in diameter, consisting of about 1000 cells. Are these cells about the same size as the ones in your body?

49–50 ▪ The following problems involve tying string around or gift-wrapping our planet, which can be thought of as a sphere with radius 6500 km.

49. How long would a piece of string have to be to go around the equator? If the string were made 1.0 m longer and stretched out all the way around, how high would it be above the surface? Does the result surprise you?

50. How large a piece of wrapping paper would be required to cover the entire planet? If the wrap were increased in area by 1.0 m^2 and stretched out all around, how high would it be above the surface? Why do you think the result is so different from that in the previous problem? (Working this out takes a lot of decimal places.)

1.4 Linear Functions and Their Graphs

Complicated models are built from simple pieces. Throughout the sciences, the simplest building blocks for mathematical models are **linear** functions, functions that have lines as their graphs. In this section, we derive formulas for linear functions, including the **point-slope formula** and the **slope-intercept formula.** Because linear functions are simple to work with algebraically, we use them to review methods for solving linear equations to answer scientific questions. In particular, we **interpolate** between known values to make predictions about the results of additional experiments.

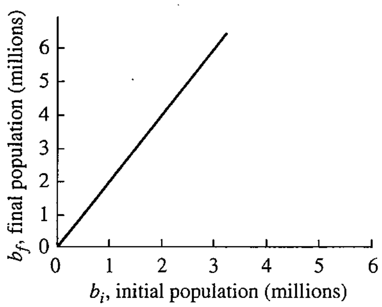


FIGURE 1.4.48

A proportional relation: bacterial populations

Proportional Relations

The simplest relations are **proportional** relations, meaning that the output is **proportional** to the input. Mathematically, this means that the ratio of the output to the input is a constant. The general formula for a proportional relation is

$$f(x) = ax$$

where a is some constant value. The ratio of the output ax to the input x is

$$\frac{\text{output}}{\text{input}} = \frac{ax}{x} = a$$

as long as $x \neq 0$. When the ratio is constant, the value a is called the **constant of proportionality.** Constants of proportionality, like all measurements, have units and dimensions.

Example 1.4.1 A Proportional Relation Between Population Sizes

The function describing the relation

$$b_f = 2.0 b_i$$

(Example 1.2.3) multiplies its input by 2.0 to produce the output. The ratio of the output population to the input population is

$$\frac{b_f}{b_i} = 2.0$$

a constant value. The graph of this proportional relation, as Figure 1.4.48 shows, is a line.

Example 1.4.2 A Proportional Relation Between Mass and Volume

In the fundamental relation between mass and volume $M = \rho V$ (Table 1.3.2), mass is found by multiplying volume by the constant value ρ. The ratio of mass to volume is

$$\frac{\text{mass}}{\text{volume}} = \text{density} = \rho$$

again a constant. As it must, the constant of proportionality, ρ, has the dimensions of density (mass per unit volume).

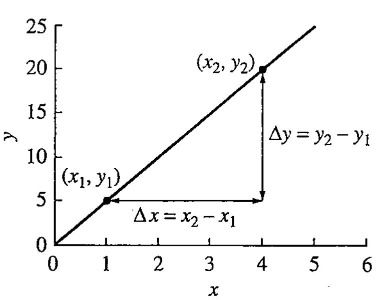


FIGURE 1.4.49

Slope and Δ notation

The proportion in a proportional relation is the **slope** of the graph. Slope is often defined as "rise over run," but we replace these archaic terms with more scientifically meaningful synonyms.

Definition 1.7 **Slope of a Line**

$$\text{slope} = \frac{\text{change in output}}{\text{change in input}}$$

The "change" is the change between two data points. Suppose we denote two points on the graph by (x_1, y_1) and (x_2, y_2) (Figure 1.4.49). Then

$$\text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{y_2 - y_1}{x_2 - x_1} \quad \textbf{(1.4.1)}$$

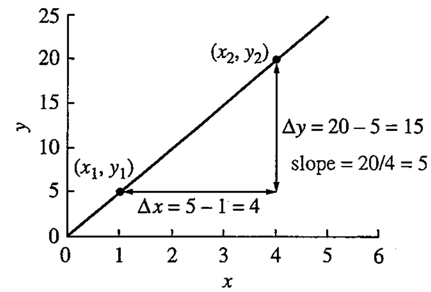


FIGURE 1.4.50
Finding the slope

Example 1.4.3 Finding the Slope Between Two Data Points

In Figure 1.4.49, the data points are $(x_1, y_1) = (1, 5)$ and $(x_2, y_2) = (4, 20)$. The slope (see Figure 1.4.50) is

$$\text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{20 - 5}{4 - 1} = \frac{15}{3} = 5$$

The changes in the input x and the output y are often written with the shorthand

$$\Delta x = x_2 - x_1$$
$$\Delta y = y_2 - y_1$$

where Δ (the Greek letter Delta) means "change in." The slope is Δy divided by Δx, or

$$\text{slope} = \frac{\Delta y}{\Delta x} \tag{1.4.2}$$

This notation will prove very useful when we study derivatives later in this book.

Example 1.4.4 The Slope of a Proportional Relation Between Populations

Recall the data in Example 1.2.3, graphed in Figure 1.4.51.

Colony	Initial Population, b_i	Final Population, b_f
1	0.47	0.94
2	3.30	6.60
3	0.73	1.46
4	2.80	5.60
5	1.50	3.00
6	0.62	1.24

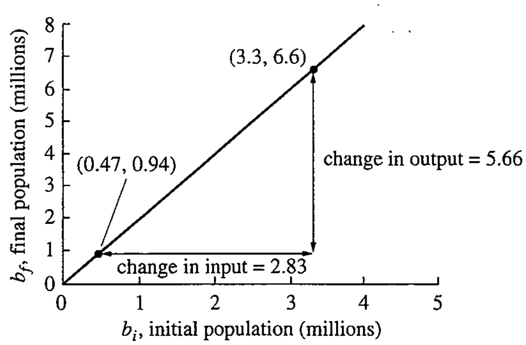


FIGURE 1.4.51
The slope of the proportional relation between bacterial populations

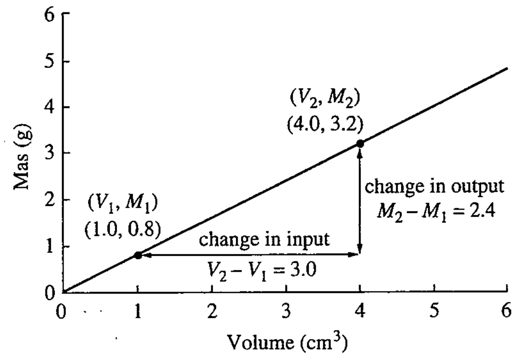


FIGURE 1.4.52
The slope of the proportional relation between mass and volume

The first two data points are (0.47, 0.94) and (3.30, 6.60). Then

$$\Delta b_i = 3.30 - 0.47 = 2.83$$
$$\Delta b_f = 6.60 - 0.94 = 5.66$$

The slope is

$$\text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta b_f}{\Delta b_i} = \frac{6.60 - 0.94}{3.30 - 0.47} = \frac{5.66}{2.83} = 2.0$$

The slope is equal to the constant of proportionality.

Example 1.4.5 The Slope of a Proportional Relation Between Mass and Volume

Suppose that $\rho = 0.8\ \frac{\text{g}}{\text{cm}^3}$. A first object with volume $V_1 = 1.0\ \text{cm}^3$ has mass $M_1 = 0.8$ g. A second object with volume $V_2 = 4.0\ \text{cm}^3$ has mass $M_2 = 3.2$ g (Figure 1.4.52). We then find

$$\text{change in output} = \Delta M = 3.2 - 0.8 = 2.4\ \text{g}$$
$$\text{change in input} = \Delta V = 4.0 - 1.0 = 3.0\ \text{cm}^3$$

The slope is then

$$\text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta M}{\Delta V} = \frac{2.4\ \text{g}}{3.0\ \text{cm}^3} = 0.8\ \frac{\text{g}}{\text{cm}^3}$$

Again, the slope is equal to the constant of proportionality, complete with the units of density.

Linear Functions and the Equation of a Line

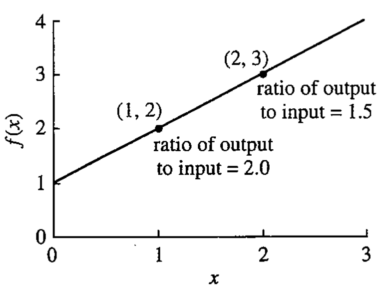


FIGURE 1.4.53
A linear function that is not a proportional relation

Proportional relations are described by functions that perform a single operation on their input: multiplication by a constant. The graphs of such functions are lines with slope equal to the constant of proportionality. Furthermore, these lines pass through the point (0, 0) because an input of 0 produces an output of 0.

Many functions other than those describing proportional relations have linear graphs.

Example 1.4.6 A Linear Function That Is not a Proportional Relation

The graph of the function

$$y = f(x) = x + 1$$

follows a line (Figure 1.4.53). But the relation between the input x and the output y is *not* a proportional relation. Two points on this line are (1, 2) and (2, 3). At the first, the ratio of output to input is $\frac{2}{1} = 2$. At the second, the ratio of output to input is $\frac{3}{2} = 1.5$. The ratio of output to input is not constant.

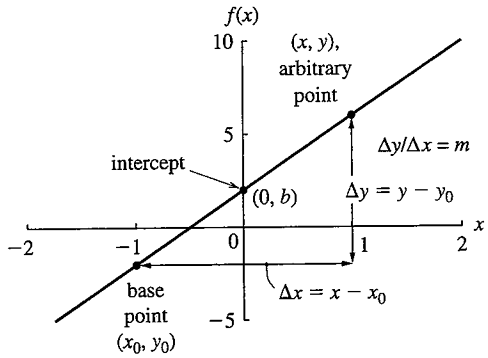


FIGURE 1.4.54
The elements of the general linear graph

For linear functions, it is the ratio of the **change in output** to the **change in input** that is constant. Suppose we start at the point (0, 1) on the graph. The ratio of change in output to change in input between this point and (1, 2) is

$$\frac{\text{change in output}}{\text{change in input}} = \frac{2-1}{1-0} = \frac{1}{1} = 1$$

The ratio of change in output to change in input between (0, 1) and (2, 3) is

$$\frac{\text{change in output}}{\text{change in input}} = \frac{3-1}{2-0} = \frac{2}{2} = 1$$

In general, a **line** is characterized by a **constant slope,** like a constant grade on a road. We use this fact to find a formula for a line. First, choose any point that lies on the graph of the function and call it the **base point** (Figure 1.4.54). If the base point has coordinates (x_0, y_0), the slope between it and an arbitrary point (x, y) on the line is

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y - y_0}{x - x_0}$$

Because the slope between any two points on the graph is constant,

$$\frac{y - y_0}{x - x_0} = m$$

for some fixed value of m. Multiplying both sides by $(x - x_0)$, we find

$$y - y_0 = m(x - x_0)$$

After solving for y by adding y_0 to both sides, we find the following form.

Definition 1.8 **The Point-Slope Form for a Line**

A line passing through the point (x_0, y_0) with slope m has formula

$$y = m(x - x_0) + y_0$$

Alternatively, we can multiply out the terms on the right-hand side of the point-slope form, finding

$$y = mx + (y_0 - mx_0)$$

We can combine the constants y_0, m, and x_0 into a single new parameter $b = y_0 - mx_0$. The letter b represents the point where the graph crosses the y-axis and is called the **y-intercept.**

Definition 1.9 **The Slope-Intercept Form for a Line**

A line with slope m and y-intercept b has formula

$$y = mx + b$$

In functional notation, if the function f has a linear graph passing through the base point (x_0, y_0) with slope m, then

$$f(x) = m(x - x_0) + y_0$$

in point-slope form. Similarly,

$$f(x) = mx + b$$

in slope-intercept form.

Example 1.4.7 Recognizing the Components of a Linear Function: Slope-Intercept Form

The function $f(x) = -4x + 5$ is a linear function in slope-intercept form. The slope is the factor multiplying the input x, or $m = -4$. The intercept is $b = 5$.

Example 1.4.8 Recognizing the Components of a Linear Function: Point-Slope Form

The function $f(x) = 3(x + 2) + 7$ is a linear function in point-slope form. To find x_0, we must write $x + 2$ as $x - (-2)$. This is in the form $x - x_0$ with $x_0 = -2$. The y-coordinate of the point is the added value, so $y_0 = 7$. Thus the base point is $(x_0, y_0) = (-2, 7)$. The slope is the factor multiplying the variable x, so $m = 3$.

Example 1.4.9 Recognizing the Components of a Biological Linear Function

The function describing the relation between initial and final bacterial populations in (Example 1.4.1),

$$b_f = f(b_i) = 2.0b_i$$

is a linear function with a slope of 2.0 and a y-intercept of 0 (and is therefore a proportional relation). In applications, inputs and outputs are rarely called x and y. Nonetheless, we recognize linear functions by the operations done to the input variable. If the formula involves only adding, subtracting, and multiplying by constants, the equation describes a linear function.

Example 1.4.10 Recognizing a Nonlinear Function

The function

$$b(t) = \frac{5.0}{1 + 2t}$$

is not a linear function because the input variable t appears in the the denominator. The function

$$b(t) = t^2 + 3t + 2$$

is not linear because the input variable t is squared.

Example 1.4.11 The Linear Relation Between Fahrenheit and Celsius

A once important linear function converts temperature in degrees Fahrenheit into temperature in degrees Celsius (Figure 1.4.55). Recall that

$$F = 1.8C + 32 \qquad \textbf{(1.4.3)}$$

where C represents temperature in degrees Celsius and F represents temperature in degrees Fahrenheit. Unlike almost all unit conversions, this formula does not express a proportional relation. The F-intercept of 32 indicates that 0°C corresponds to 32°F rather than 0°F. The slope, nonetheless, describes the number of degrees Fahrenheit per degree Celsius as in an ordinary conversion. To check the slope, we compute ΔF and ΔC between the points with $C = 0$ and $C = 20$. Because 20°C corresponds to 68°F, the

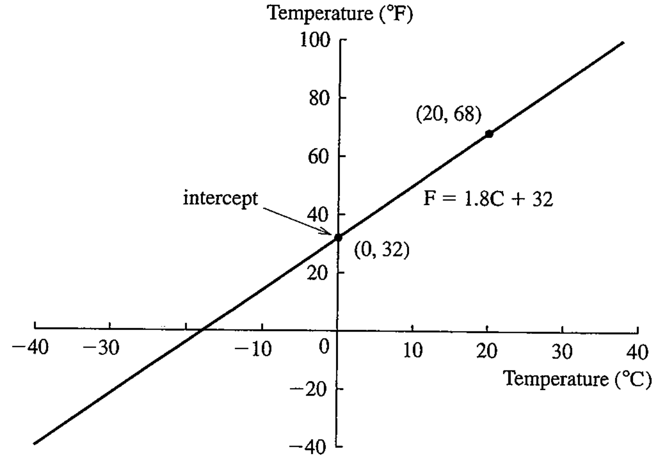


FIGURE 1.4.55
The relation between Fahrenheit and Celsius temperatures

change in °F (the output) is

$$\Delta F = 68°\text{F} - 32°\text{F} = 36°\text{F}$$

and the change in °C (the input) is

$$\Delta C = 20°\text{C} - 0°\text{C} = 20°\text{C}$$

Therefore

$$\text{slope} = \frac{36°\text{F}}{20°\text{C}} = 1.8\,\frac{°\text{F}}{°\text{C}}$$

It takes 1.8°F to make up 1.0°C.

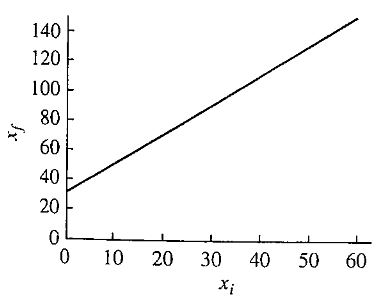


FIGURE 1.4.56
Graph describing a changing mite population

Finding Equations and Graphing Lines

Pet store owners can be plagued by parasites. Suppose the employees spend a week observing populations of mites on several lizards and collect the following data, plotted in Figure 1.4.56.

Initial Number, x_i	Final Number, x_f
20	70
30	90
40	110
50	130

Here x_i is the initial number of mites and x_f is the final number. Suppose we wish to estimate the number of mites we would find after a week on a lizard that has 45 mites today. To do so, we must first find an equation for the function relating x_f and x_i and then evaluate it at $x_i = 45$.

We can find the equation with the following steps.

Algorithm 1.3 Finding the Equation of a Line from Data

1. Graph the data and check that the points lie on a line.
2. Pick two data points.
3. Find the slope as the change in output divided by the change in input.

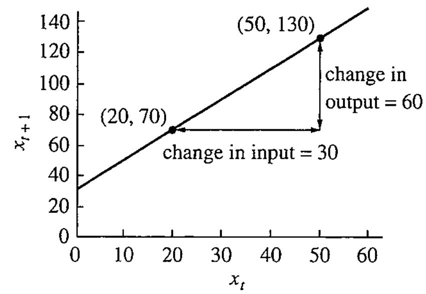


FIGURE 1.4.57
Finding the equation of the function describing mites

4. Find the equation by plugging one point and the slope into the point-slope form for a line (Definition 1.8).
5. If needed, convert this equation into the slope-intercept form.

Example 1.4.12 Finding the Equation of a Line from Data

We can follow this algorithm to find the equation of the function describing our data.

1. The graph in Figure 1.4.56 looks like a line.
2. Pick the first and last data points (any others could be chosen as long as the data lie on a line (Figure 1.4.57)).
3. The slope m is
$$m = \frac{\Delta x_f}{\Delta x_i} = \frac{130-70}{50-20} = \frac{60}{30} = 2.0$$
4. If we choose the point (20, 70) as (x_0, y_0) in the point-slope form for a line, the equation is
$$\begin{aligned} x_f &= m\left(x_i - (x_i)_0\right) + (x_f)_0 \\ &= 2.0(x_i - 20) + 70 \end{aligned}$$
5. We can expand this to find the slope-intercept form
$$x_f = 2.0x_i + 30$$
This defines a function $h(x_i)$ with formula
$$x_f = h(x_i) = 2.0x_i + 30$$

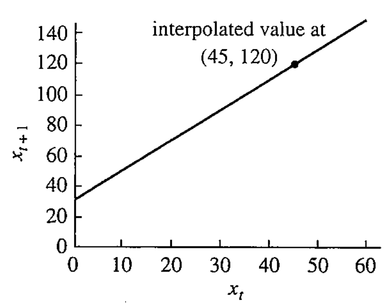


FIGURE 1.4.58
Interpolating a value

This function $h(x_i)$ can be interpreted in biological terms. The x_f-intercept of 30 is the number of mites we would find on a lizard that started out with no mites. These mites probably arrived from other lizards. The slope of 2.0 is the number of additional mites we would find after a week if we added one mite at the beginning. For example, $h(1) = 32$, two more than $h(0) = 30$. The one additional mite left two offspring.

Example 1.4.13 Using a Linear Function to Interpolate

To predict the number of mites we will find after a week on a lizard that has 45 mites today, we substitute $x_i = 45$ into the function $h(x_i) = 2.0x_i + 30$, finding
$$x_f = h(x_i) = 2.0 \cdot 45 + 30 = 120$$
(See Figure 1.4.58.) We have used the formula for the function to **interpolate** a prediction between known values.

In this example, we used a set of numerical data to plot a graph and derive an algebraic formula. In other cases, we are given a formula and need to produce a graph. The easiest way to graph a linear function from its formula is to plug two reasonable values of the input into the equation, graph the points, and connect them with a line.

Example 1.4.14 Plotting a Line from an Equation

Suppose we wish to plot the linear function $F(x)$ given by

$$F(x) = -2x + 30$$

Plugging in $x = 0$ gives $F(0) = 30$, and $x = 10$ gives $F(10) = 10$ (Figure 1.4.59). The graph of the line connects the points (0, 30) and (10, 10). This line goes down with a negative slope of -2.

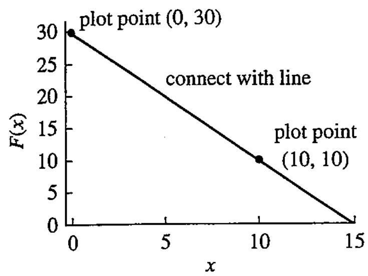


FIGURE 1.4.59
Graphing a line from its equation

When the graph goes down when read from left to right (as in Example 1.4.14), we say that the function is a **decreasing function.** A larger input produces a smaller output. Linear functions with negative slopes are decreasing functions. In contrast, a positive slope corresponds to an **increasing function.** Larger inputs produce larger outputs. A slope of exactly 0 corresponds to a function with equation

$$f(x) = 0 \cdot x + b = b$$

Such a function always takes on the constant value b, the y-intercept, and has as its graph a horizontal line.

Slope	Graph	Function
positive	goes up	increasing
negative	goes down	decreasing
zero	flat	constant

An example of each type is shown in Figure 1.4.60.

Solving Equations Involving Lines

Answering questions about linear relations requires solving **linear equations,** which are among the simplest equations to solve.

Example 1.4.15 Solving a Linear Equation

Suppose we wish to find where the line

$$y = 3x + 1$$

takes on the value $y = 7$ (Figure 1.4.61).

$7 = 3x + 1$	substitute the value of y
$6 = 3x$	subtract 1 from both sides
$2 = x$	divide both sides by 3

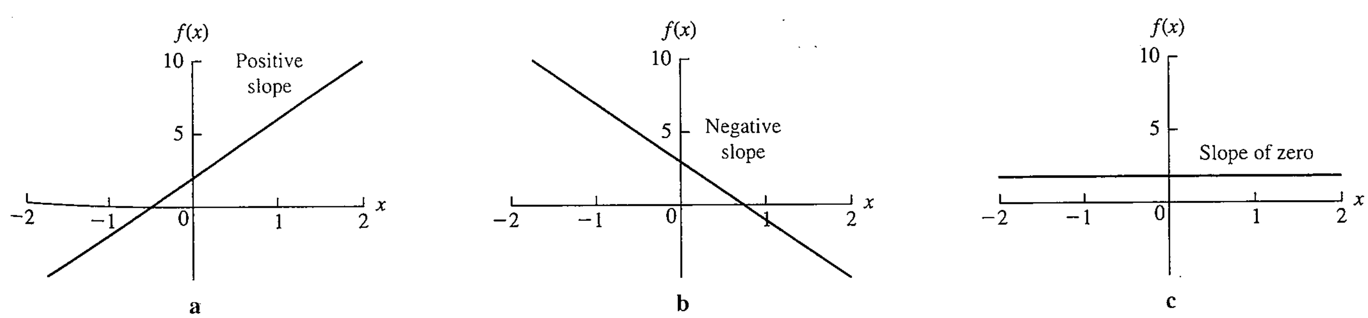


FIGURE 1.4.60
Linear functions with positive, negative, and zero slopes

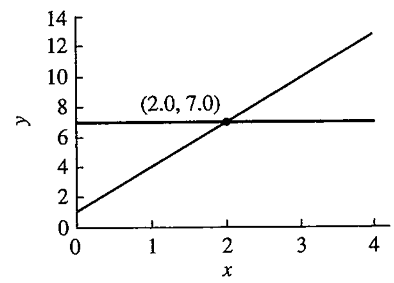


FIGURE 1.4.61
Solving an equation involving a linear function

Graphically, this equation corresponds to finding where the line $y = 3x + 1$ crosses the horizontal line that represents $y = 7$.

Example 1.4.16 Solving a Linear Equation Involving a Parameter

Suppose we wish to find where the line

$$y = mx + 1$$

takes on the value $y = 7$ for any value of the slope m.

$$7 = mx + 1 \quad \text{substitute the value of } y$$
$$6 = mx \quad \text{subtract 1 from both sides}$$
$$\frac{6}{m} = x \quad \text{divide both sides by } m$$

This solution makes sense for any value of $m \neq 0$.

Example 1.4.17 Finding the Intersection of Two Lines

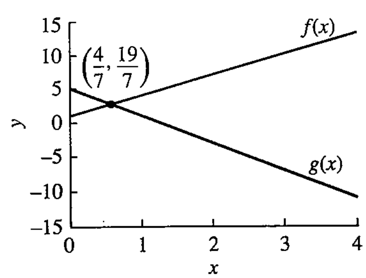


FIGURE 1.4.62
Finding where two lines intersect

Suppose we wish to find where the lines $f(x) = 3x + 1$ and $g(x) = -4x + 5$ intersect (Figure 1.4.62). We set the two values equal, and solve for x as follows:

$$3x + 1 = -4x + 5 \quad \text{set the two formulas equal to each other}$$
$$7x + 1 = 5 \quad \text{add } 4x \text{ to both sides}$$
$$7x = 4 \quad \text{subtract 1 from both sides}$$
$$x = \frac{4}{7} \quad \text{divide both sides by 7}$$

This gives the value of x where the two intersect. The value of y can be found by substituting $x = \frac{4}{7}$ into either function, or

$$f\left(\frac{4}{7}\right) = 3 \cdot \frac{4}{7} + 1 = \frac{19}{7}$$
$$g\left(\frac{4}{7}\right) = -4 \cdot \frac{4}{7} + 5 = \frac{19}{7}$$

Both functions give the same result, as they must.

Example 1.4.18 Solving a Classic Word Problem with Linear Equations

Little Billy's father is three times as old as Billy in 2001. Ten years later, Billy's father will be only twice as old as Billy. What year was Billy born, and how old was his dad

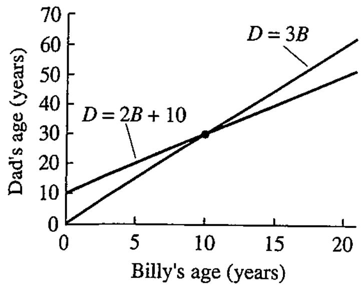


FIGURE 1.4.63
Graphical method to find ages

at that time? Let B represent Billy's age in 2001, and D his dad's age. Then

$$D = 3B$$

Ten years later, Billy is $B + 10$ and his dad is $D + 10$. Because his dad is then twice as old,

$$D + 10 = 2(B + 10)$$

We can rewrite this (in slope-intercept form) to find that

$$D = 2B + 10$$

This gives two equations for D (Figure 1.4.63). Setting the right-hand sides equal gives the single linear equation

$$3B = 2B + 10$$

Subtracting $2B$ from both sides gives $B = 10$, Billy's age in 2001. His dad was three times as old, or 30. Thus Billy was born in 1991, when his dad was 20. Ten years later, in 2011, Billy will be 20, exactly half as old as his 40-year-old father.

Example 1.4.19 A Linear Equation with No Solution

Suppose we are given the following variant of the classic word problem. Little Billy's father is three times as old as Billy in 2001. Ten years later, Billy's father will be 10 years less than three times Billy's age. What year was Billy born, and how old was his dad at that time? Let B represent Billy's age in 2001, and D his dad's age. Then

$$D = 3B$$

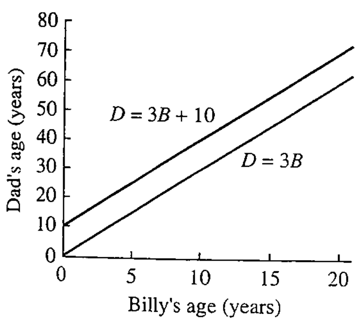


FIGURE 1.4.64
Failure of graphical method to find ages

Ten years later, Billy is $B + 10$ and his dad is $D + 10$. Because his dad is then 10 years less than three times as old,

$$D + 10 = 3(B + 10) - 10$$

Subtracting 10 from both sides and solving for D gives

$$D = 3B + 10$$

Setting these two equations for D equal gives

$$3B = 3B + 10$$

Subtracting $3B$ from both sides gives $0 = 10$, which is impossible. The original problem has no solution. Graphically, this corresponds to trying to find the intersection of two parallel lines (Figure 1.4.64).

Example 1.4.20 A Linear Equation with Many Solutions

Suppose we are given yet another variant of the classic word problem. Little Billy's father is three times as old as Billy in 2001. Ten years later, Billy's father will be 20 years less than three times Billy's age. What year was Billy born, and how old was his dad at that time? Then

$$D = 3B$$

Ten years later, Billy is $B + 10$ and his dad is $D + 10$. Because his dad is then 20 years less than three times as old,

$$D + 10 = 3(B + 10) - 20$$

Solving for D, we find

$$D = 3B$$

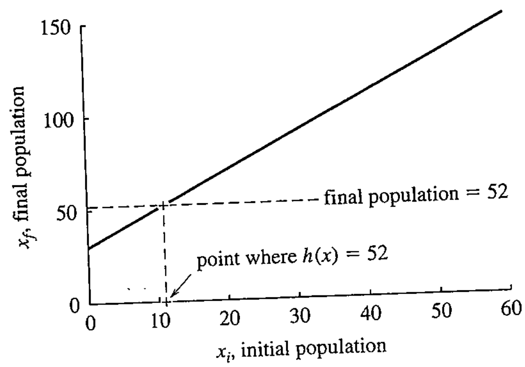


FIGURE 1.4.65
Going backwards with the mite population

This matches our original equation and works for any value of B. For example, if Billy was born in 1988, and consequently was 13 in 2001, his dad was three times as old in 2001, or 39. Ten years later, Billy would be 23, and his dad would be 49, exactly 20 years less than 3 times 23. But if Billy had been born in 1992, he would have been 9 in 2001, and his dad would have been 27. Ten years later, Billy would be 19, and dad would be 37, again exactly 20 years less than 3 times 19.

Example 1.4.21 Solving Another Word Problem with Linear Equations

Recall the lizards with mites obeying the equation

$$x_f = 2.0x_i + 30$$

(Example 1.4.12). Suppose a lizard ends up with $x_f = 52$ mites. How many did it have the week before? This is a kind of "inverse" problem; starting from where we ended up, we want to try to end up where we started. Fortunately, these problems can be expressed more clearly in equations than in words. In terms of the variables x_i and x_f, our question can be rephrased. What was x_i if x_f is 52? We want to solve the equation $h(x_i) = 52$ for x_i, or

$$2.0x_i + 30 = 52$$

The two sides of the equation say the same thing in two ways. The right-hand side gives our measured value 52. The left-hand side gives the measured value as a function of the unknown x_i. We can solve for x_i

$$2.0x_i = 52 - 30 = 22 \quad \text{subtract 30 from both sides}$$

$$x_i = \frac{22}{2.0} = 11 \quad \text{divide both sides by 2}$$

We can check this answer by plugging in, finding

$$h(11) = 2 \cdot 11 + 30 = 52$$

(Figure 1.4.65).

Summary The graphs of many important functions in biology are lines. We derived the link between lines and **linear functions.** A **proportional relation** is a special type of linear function in which the ratio of the output to the input is always the same. This constant ratio is the **slope** of the graph of the relation. Lines other than proportional relations can be expressed in **point-slope form** or **slope-intercept form.** The slope can be found as the change in output divided by the change in input. Equations of linear functions can be used to **interpolate**—that is, to estimate outputs from untested inputs.

1.4 Exercises

Mathematical Techniques

1–4 ▪ For the following lines, find the slope between the two given points by finding the change in output divided by the change in input. What is the ratio of the output to the input at each of the points? Which are proportional relations? Which are increasing and which are decreasing? Sketch a graph.

1. $y = 2x + 3$, using points with $x = 1$ and $x = 3$
2. $z = -5w$, using points with $w = 1$ and $w = 3$
3. $z = 5(w - 2) + 8$, using points with $w = 1$ and $w = 3$
4. $y - 5 = -3(x + 2) - 6$, using points with $x = 1$ and $x = 3$

5–6 ▪ Check that the point indicated lies on the line and find the equation of the line in point-slope form using the given point. Multiply out to check that the point-slope form matches the original equation.

5. The line $f(x) = 2x + 3$ and the point $(2, 7)$.
6. The line $g(y) = -2y + 7$ and the point $(3, 1)$.

7–12 ▪ Find equations in slope-intercept form for the following lines. Sketch a graph indicating the original point from the point-slope form.

7. The line $f(x) = 2(x - 1) + 3$.
8. The line $g(z) = -3(z + 1) - 3$.
9. A line passing through the point $(1, 6)$ with slope -2.
10. A line passing through the point $(-1, 6)$ with slope 4.
11. A line passing through the points $(1, 6)$ and $(4, 3)$.
12. A line passing through the points $(6, 1)$ and $(3, 4)$.

13–16 ▪ Check whether the following are linear functions.

13. $h(z) = \frac{1}{5z}$
14. $F(r) = r^2 + 5$
15. $P(q) = 8(3q + 2) - 6$
16. $Q(w) = 8(3w + 2) - 6(w + 4)$

17–18 ▪ Check that the following curves do not have constant slope by computing the slopes between the points indicated. Compare with the graphs in Section 1.2, Exercises 5 and 6.

17. $h(z) = \frac{1}{5z}$ at $z = 1$, $z = 2$, and $z = 4$, as in Section 1.2, Exercise 5. Find the slope between $z = 1$ and $z = 2$, and the slope between $z = 2$ and $z = 4$.
18. $F(r) = r^2 + 5$ at $r = 0$, $r = 1$, and $r = 4$, as in Section 1.2, Exercise 6. Find the slope between $r = 0$ and $r = 1$, and the slope between $r = 1$ and $r = 4$.

19–24 ▪ Solve the following equations. Check your answer by plugging in the value you found.

19. $2x + 3 = 7$
20. $\frac{1}{2}z - 3 = 7$
21. $2x + 3 = 3x + 7$
22. $-3y + 5 = 8 + 2y$
23. $2(5(x - 1) + 3) = 5(2(x - 2) + 5)$
24. $2(4(x - 1) + 3) = 5(2(x - 2) + 5)$

25–28 ▪ Solve the following equations for the given variable, treating the other letters as constant parameters.

25. Solve $2x + b = 7$ for x.
26. Solve $mx + 3 = 7$ for x.
27. Solve $2x + b = mx + 7$ for x. Are there any values of b or m for which this has no solution?
28. Solve $mx + b = 3x + 7$ for x. Are there any values of b or m for which this has no solution?

29–32 ▪ Most unit conversions are proportional relations. Find the slope and graph the relations between the following units.

29. Place inches on the horizontal axis and centimeters on the vertical axis. Use the fact that 1 in. = 2.54 cm. Mark the point corresponding to 1 in. on your graph.
30. Place centimeters on the horizontal axis and inches on the vertical axis. Use the fact that 1 in. = 2.54 cm. Mark the point corresponding to 1 in. on your graph.
31. Place grams on the horizontal axis and pounds on the vertical axis. Use the identity 1 lb $\approx$ 453.6 g. Mark the point corresponding to 1 lb on your graph.
32. Place pounds on the horizontal axis and grams on the vertical axis. Use the identity 1 lb $\approx$ 453.6 g. Mark the point corresponding to 1 lb on your graph.

33–34 ▪ Not very many functions commute with each other (Section 1.2). The following problems ask you to find all linear functions that commute with the given linear function.

33. Find all functions of the form $g(x) = mx + b$ that commute with the function $f(x) = x + 1$. Can you explain your answer in words?
34. Find all functions of the form $g(x) = mx + b$ that commute with the function $f(x) = 2x$. Can you explain your answer in words?

Applications

35–38 ▪ Many fundamental relations express a proportional relation between two measurements with different dimensions. Find the slopes and the equations of the relations between the following quantities.

35. Volume = area × thickness. Find the volume V as a function of the area A if the thickness is 1.0 cm.
36. Volume = area × thickness. Find the volume V as a function of thickness T if the area is 7.0 cm^2.

37. Total mass = mass per bacterium × number of bacteria. Find the total mass M as a function of the number of bacteria b if the mass per bacterium is 5.0×10^{-9} g.

38. Total mass = mass per bacterium × number of bacteria. Find the total mass M as a function of mass per bacterium m if the number of bacteria is 10^6.

39–42 ▪ A ski slope has a slope of −0.2. You start at an altitude of 10,000 feet.

39. Write the equation giving altitude a as a function of horizontal distance moved d.

40. Write the equation of the line in meters.

41. What will be your altitude when you have gone 2000 feet horizontally?

42. The ski run ends at an altitude of 8000 feet. How far will you have gone horizontally?

43–46 ▪ The following data give the elevation of the surface of the Great Salt Lake in Utah.

Year, y	Elevation, E (ft)
1965	4193
1970	4196
1975	4199
1980	4199
1985	4206
1990	4203
1995	4200

43. Graph these data.

44. During which periods is the surface elevation changing linearly?

45. What was the slope between 1965 and 1975? What would the surface elevation have been in 1990 if things had continued as they began? How different is this from the actual elevation?

46. What was the slope during the period between 1985 and 1995? What would the surface elevation have been in 1965 if things had always followed this trend? How different is this from the actual elevation?

47–50 ▪ Graph the following relations between measurements of a growing plant, checking that the points lie on a line. Find the equations in both point-slope and slope-intercept form. What do the y-intercepts mean?

Age, a (days)	Mass, M (g)	Volume, V (cm^3)	Glucose Production, G (mg)
0.5	2.5	5.1	0.0
1.0	4.0	6.2	3.4
1.5	5.5	7.3	6.8
2.0	7.0	8.4	10.2
2.5	8.5	9.5	13.6
3.0	10.0	10.6	17.0

47. Mass as a function of age. Find the mass on day 1.75.

48. Volume as a function of age. Find the volume on day 2.75.

49. Glucose production as a function of mass. Estimate glucose production when the mass reaches 20.0 g.

50. Volume as a function of mass. Estimate the volume when the mass reaches 30.0 g. How will the density at that time compare with the density when $a = 0.5$?

51–54 ▪ Consider the data in the following table (adapted from *Parasitoids* by H. C. F. Godfray), describing the number of wasps that can develop inside caterpillars of different weights.

Weight of Caterpillar (g)	Number of Wasps
0.5	80
1.0	115
1.5	150
2.0	175

51. Graph these data. Which point does not lie on the line?

52. Find the equation of the line connecting the first two points.

53. How many wasps does the function predict would develop in a caterpillar weighing 0.72 g?

54. How many wasps does the function predict would develop in a caterpillar weighing 0.0 g? Does this make sense? How many would you really expect?

55–58 ▪ The world record times for various races are decreasing at roughly linear rates (adapted from *Guinness Book of Records*, 1990).

55. The men's Olympic record for the 1500 meters was 3:36.8 (3 minutes and 36.8 seconds) in 1972 and was 3:35.9 in 1988. Find and graph the line connecting these. (Don't forget to convert everything into seconds.)

56. The women's Olympic record for the 1500 meters was 4:01.4 in 1972 and was 3:53.9 in 1988. Find and graph the line connecting these.

57. If things continue at this rate, when will women finish the race in exactly no time? (Set the time equal to 0 and solve for the date.) What might happen before that date?

58. If things continue at this rate, when will women be running this race faster than men? (Set the two speeds equal and solve for the date.)

Computer Exercises

59. Try Exercise 58 on the computer. Compute the year when the times will reach 0. Give your best guess of the times in the year 1900.

60. Graph the ratio of temperature measured in Fahrenheit to temperature measured in Celsius for $-273 \leq C < 200$. What happens near $C = 0$? What happens for large and small values of C? How would the results differ if the zero for Fahrenheit were changed to match that of Celsius?

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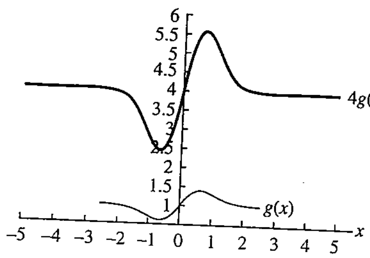
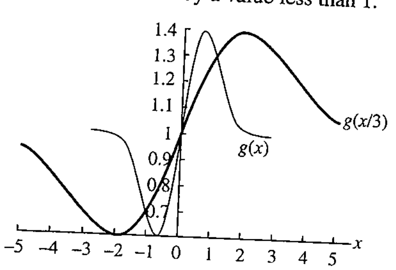
1. 3.4 lb $\times 16$ oz/lb $\times$ 28.35 g/oz $\approx$ 1542.24 g.
7. 2.3 cm is 0.023 m, so the final height is $1.34 + 0.023 = 1.363$ m.
11. The area of the square is $1.7^2 = 2.89$ cm^2, but the area of the disk is $\pi r^2 = \pi \cdot 1^2$ cm$^2 \approx 3.1415$ cm^2. The disk has a larger area.
15. Pressure is force per unit area, or

$$\frac{\text{force}}{\text{area}} = \frac{\frac{ML}{T^2}}{L^2} = \frac{M}{LT^2}$$

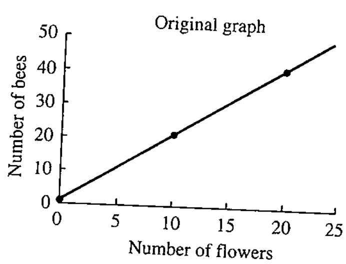
17. Rate of spread of bacteria on a plate has dimensions of L^2/T, or area per time.
19. This checks, because length $= \frac{\text{length}}{\text{time}} \times$ time.
21. This checks, because $\frac{\text{mass} \times \text{length}}{\text{time}^2} = \text{mass} \times \frac{\text{length}}{\text{time}^2}$.
23. The vertical axis is scaled by a value greater than 1.

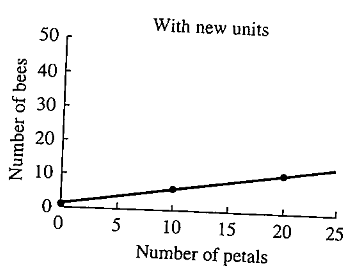


25. The horizontal axis is scaled by a value less than 1.



27. The tree with height 23.1 m will have volume of $\pi \cdot 23.1 \cdot 0.5^2 \approx 18.14$ m^3. When the height is 24.1, the volume is 18.93 m^3. The ratio is $18.93/18.14 \approx 1.043$.
31. The volume is $2.0 \cdot 0.2 \cdot 1.5 = 0.6\,\text{m}^3 = 6.0 \times 10^5\,\text{cm}^3$. This gives a mass of 6.0×10^5 g $= 600$ kg.
33. $3200 \cdot 0.45$ g/individual $= 1440$ g $= 1.44$ kg.
35. Let p be the number of petals. Then $f = p/4$, so $b = p/2 + 1$. When $p = 0$, $b = 1$; when $p = 10$, $b = 6$; when $p = 20$, $b = 11$. The number of bees goes up more slowly as a function of the number of petals.





39. 186,000 mile/s $\approx$ 200,000 mile/s $\approx$ 200,000 mile/s $\times$ 60,000 in./mile
$= 1.2 \times 10^{10}$ in./s $\approx 1.2 \times 10^{10}$ in./s $\times$ 2.5 cm/in.
$= 3.0 \times 10^{10}$ cm/s $= 30$ cm/ns

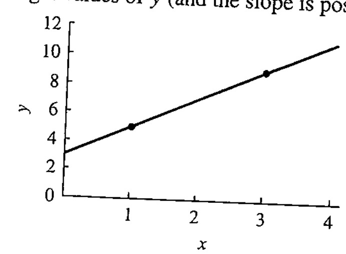
If a computer is supposed to do an operation in 0.3 ns, it had better not need to move information for more than the distance light can travel in that time, or about 9 cm.
43. You'd catch 6 movies per day, or about 2000 per year, for a total of 120,000 in your life.
45. The volume of a sphere of radius r is $4\pi r^3/3 \approx 4r^3$. The radius of the cell is 10^{-3} cm, so the volume is about 4×10^{-9} cm^3. The mass of a cell is therefore around 4×10^{-9} g. I weigh about 60 kg, which is 6×10^4 g. The number of cells is then

$$\frac{6 \times 10^4 \text{ g}}{4 \times 10^{-9} \text{ g/cell}} = 1.5 \times 10^{13} \text{ cells.}$$

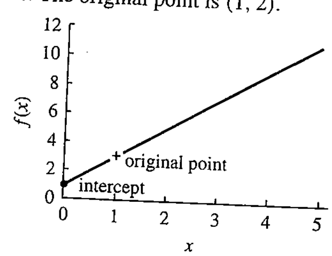
47. The brain is about 2% of my weight and should have about 2% of my cells, or 3×10^{11} cells. The number of neurons is 1×10^{11}, but the total number of cells in the brain is between 1×10^{12} and 5×10^{12}, a bit higher than the previous estimates.
49. The length of the string would be $2\pi r \approx 40840.704$ km. Adding 1 m would make it 40840.705 km. The radius corresponding to this is $r = 40840.705/2\pi \approx 6500.0002$ km. The string would be 0.0002 km, or 0.2 m, above the earth. It is amazing that such a relatively tiny change in the length of the string would produce such a big effect.

Section 1.4, page 51

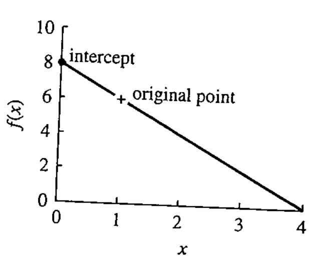
1. The points are (1, 5) and (3, 9). The change in input is 2, the change in output is 4, and the slope is 2. This is not a proportional relation because the ratio of output to input changes from 5 at the first point to 3 at the second point. This relation is increasing because larger values of x lead to larger values of y (and the slope is positive).



5. The point lies on the line because $f(2) = 2 \cdot 2 + 3 = 7$. The point-slope form is $f(x) = 2(x - 2) + 7$. Multiplying out gives $f(x) = 2x - 4 + 7 = 2x + 3$, as it should.
7. Multiplying out, we find that $f(x) = 2x + 1$. The slope is 2 and the y-intercept is 1. The original point is (1, 2).



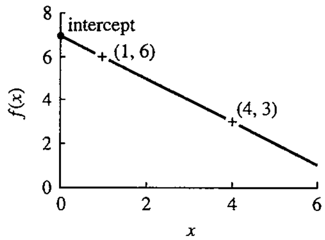
9. In point-slope form, this line has equation $f(x) = -2(x - 1) + 6$. Multiplying out, we find that $f(x) = -2x + 8$. The slope is -2 and the y-intercept is 8.



11. The slope between the two points is

$$\text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{3-6}{4-1} = -1$$

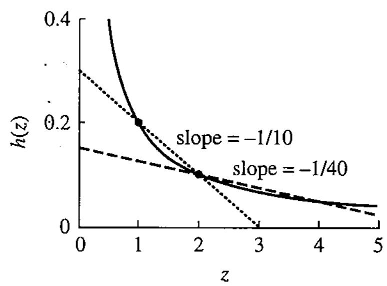
In point-slope form, the line has equation $f(x) = -1 \cdot (x-1) + 6$. In slope-intercept form, it is $f(x) = -x + 7$. This line has slope -1 and y-intercept 7.



13. This is not linear because the input z appears in the denominator.

15. This is linear because the input q is multiplied only by constants and has constants added to it.

17. $h(1) = 1/5$, $h(2) = 1/10$, $h(4) = 1/20$. The slope between $z = 1$ and $z = 2$ is $-1/10$, and that between $z = 2$ and $z = 4$ is $-1/40$.



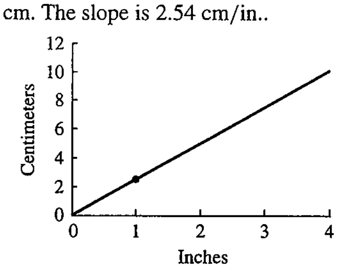
19. $2x = 7 - 3 = 4$, so $x = 4/2 = 2$. Plugging in, $2 \cdot 2 + 3 = 7$.

23. Multiplying out, we get $10x - 4 = 10x + 5$. This has no solution.

25. $2x = 7 - b$, so $x = \frac{7-b}{2}$.

27. $(2-m)x = 7 - b$, so $x = \frac{7-b}{2-m}$. There is no solution if $m = 2$. However, if $m = 2$ and $b = 7$, both sides are identical and any value of x works.

29. 1 in. = 2.54 cm. The slope is 2.54 cm/in..



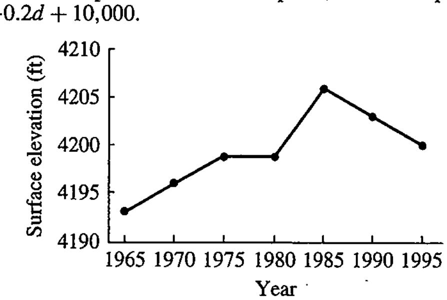
33. $(f \circ g)(x) = (mx + b) + 1$ and $(g \circ f)(x) = m(x+1) + b = mx + m + b$. These match only if the intercepts are equal, or $b + 1 = m + b$. This is true for any b as long as $m = 1$. In this case, both f and g have slope 1, meaning that each just adds a constant to its input. The order cannot matter because addition is commutative.

35. The slope is 1.0 cm, and the equation is $V = 1.0A$.

37. The slope is 5.0×10^{-9} g, and the equation is $M = 5.0 \times 10^{-9}b$.

39. The line has slope -0.2 and intercept 10,000. The equation is thus $a = -0.2d + 10{,}000$.

43.



45. The slope is 3 ft every 5 years. It would have been up to 4208 by 1990, 5 ft higher than the actual level.

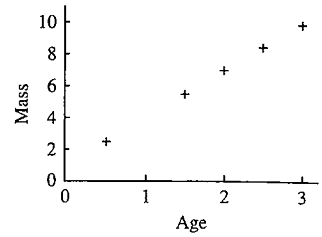
47. Using the first two rows for mass, we find a slope of

$$\text{slope} = \frac{\text{change in mass}}{\text{change in age}} = \frac{4.0 - 2.5}{1.0 - 0.5} = 3.0.$$

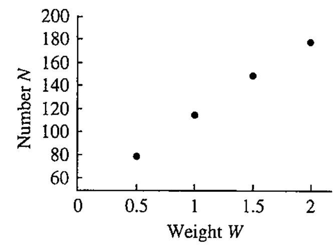
Using (1.0, 4.0) as the base point,

$$M = 3.0(a - 1.0) + 4.0 = 3.0a + 1.0.$$

The y-intercept is 1.0, meaning that the mass was 1.0 g at age 0. This might be the mass of a new seedling. Interpolating at $a = 1.75$, $M = 3.0 \cdot 1.75 + 1.0 = 6.25$.



51. The point (2.0, 175) lies below the line.



53. $N = 70(0.72 - 0.5) + 80 = 95.4$.

55. The time decreases from 216.8 to 215.9 seconds, giving a change of -0.9 second in 16 years or a slope of $-0.9/16 \approx -0.0563$ second per year. In point-slope form, the men's time m as a function of the year y is

$$m = -0.0563(y - 1972) + 216.8$$

Section 1.5, page 64

1. The updating function is $f(p_t) = p_t - 2$, and $f(5) = 3$, $f(10) = 8$, $f(15) = 13$. This is a linear function.

3. The updating function is $f(x_t) = x_t^2 + 2$, and $f(0) = 2$, $f(2) = 6$, $f(4) = 18$. This is not a linear function because the input x_t is squared.

5. Denote the updating function by $f(v) = 1.5v$. Then $(f \circ f)(v) = f(1.5v) = 1.5(1.5v) = 2.25v$, so $v_{t+2} = 2.25v_t$. Applying f to the initial condition twice gives $f(1220) = 1830$ and $f(1830) = 2745$, which is equal to $2.25 \cdot 1220$.

9. Solving for v_t gives $v_t = v_{t+1}/1.5$. Then $v_0 = 1220/1.5 = 813.3$.

13.

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{x}{1+x}\right) = \frac{\frac{x}{1+x}}{1 + \frac{x}{1+x}}$$

$$= \frac{\frac{x}{1+x}}{\frac{1+2x}{1+x}} = \frac{x}{1+2x}.$$

To find the inverse, set $y = f(x)$ and solve

$$\begin{aligned} y &= \frac{x}{1+x} \\ (1+x)y &= x \\ y + xy &= x \\ y &= x - xy \\ \frac{y}{1-y} &= x. \end{aligned}$$

Therefore, $f^{-1}(y) = \frac{y}{1-y}$.