Group work #1 - Models
MATH 306/506 - Jan 26, 2012

A *geometry* is a collection of points and lines that obeys certain axioms. A *geometric model* is a collection of points and lines properly described that will satisfy the axioms.

The undefined objects are a set $\mathcal{E}$ (the "plane") of *points*, and a collection $\mathcal{L}$ of special subsets of points, called the *lines*. Below we describe the points and lines for each of the models to be explored.

Each group will be given a different "geometric model". The goal of this project is to verify which axioms are valid for each model. To get us started we will concentrate on the *Incidence Axioms*:

I-1. The plane $\mathcal{E}$ contains at least three *non-collinear* points, that is three points that are not all contained on the same line.

I-2. Given two distinct points A and B, there is *exactly one* line ℓ containing both.

The models you will explore are:

- A Toy Model.
- $\mathbb{R}^2$: The Real Cartesian Plane
- $\mathbb{MP}^2$: The Moulton Plane
- $\mathbb{R}^3$: The Real Cartesian Space
- $\mathbb{P}^2$: The Poincaré Disk
- $\mathbb{RP}^2$: The Real Projective Plane

1 Toy model

The *points* in this model are three non-collinear points on $\mathbb{R}^2$.

A *line* in this model is is any two-point subset of the given three points.

Verify the incidence axioms for this model.

2 $\mathbb{R}^2$: The Real Cartesian Plane

The *points* in this model are all ordered pairs (x, y) where x and y are any two real numbers. The plane in this model is the set

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

A *line* in this model is the set of all points (x, y) that satisfy an equation of the form $ax + by + c = 0$ for some real numbers a, b, c, where at least one of a or b is not zero.

Verify the incidence axioms for this model.

3 $\mathbb{MP}^2$: The Moulton Plane

The *points* on the Moulton Plane are the same as for the real Cartesian plane $\mathbb{R}^2$, all ordered pairs (x, y) where x and y are any two real numbers. Therefore,

$$\mathbb{MP}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

There are three types of lines in this model.

1. Any vertical line in $\mathbb{R}^2$.
2. All lines in $\mathbb{R}^2$ with non-positive slopes (that lines with negative slope, and lines with zero slope -horizontal lines-).
3. All 'bent" lines in $\mathbb{R}^2$ with a positive slope $m > 0$ when $x < 0$, and a positive half-slope $m/2 > 0$ when $x > 0$.

Find analytic descriptions for the three types of lines (draw pictures of each type), then verify the incidence axioms.

4 $\mathbb{R}^3$: The Real Cartesian Space

The *points* in this model are all ordered triples (x, y, z) where x, y, and z are any three real numbers. The set of points is then,

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

A *line* in this model is the set of all points (x, y, z) that satisfy an equation of the line in $\mathbb{R}^3$ most conveniently described using parametric equations. A line ℓ in this model is the collection of points (x, y, z) that can be expressed as:

$$x = a_1 t + b_1, \quad y = a_2 t + b_2, \quad z = a_3 t + b_3,$$

for some real number t (the *parameter*, where $a_1, a_2, a_3, b_1, b_2, b_3$ are six constant real numbers. Could use vector notation to represent the line: $\vec{v} = t\vec{a} + \vec{b}$, where $\vec{v} = (x, y, z)$, $\vec{a} = (a_1, a_2, a_3)$, and $\vec{b} = (b_1, b_2, b_3)$

Verify the incidence axioms for this model.

5 $\mathbb{P}^2$: The Poincaré Disk

The *points* in this model are the ordered pairs $(x, y) \in \mathbb{R}^2$ that lie inside the unit disk, that is $x^2 + y^2 < 1$. The plane in this model is the open disk

$$\mathbb{P}^2 = \{(x, y) : x, y \in \mathbb{R} \text{ and } x^2 + y^2 < 1\}.$$

There are two types of lines in this model:

1. Any ordinary straight line containing the origin $(0, 0)$ and restricted to the open disk $\mathbb{P}^2$.

2. The other type of "line" will be any ordinary circle in $\mathbb{R}^2$ (restricted to $\mathbb{P}^2$) which intersects the unit circle $x^2 + y^2 = 1$ in a perpendicular fashion (that is the tangent lines to both circles at a point of intersection are perpendicular).

Find analytic descriptions for both types of lines (draw pictures of each type), then verify the incidence axioms. Here intuition may not be enough.

The Poincaré disk is a standard model for **hyperbolic geometry**.

6 $\mathbb{RP}^2$: The Real Projective Plane

The *points* in this model are pairs of *antipodal* points on the unit sphere $\mathcal{S}^2$ in $\mathbb{R}^3$. Recall that the unit sphere is

$$\mathcal{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

The antipodal point of $(x, y, z) \in \mathcal{S}^2$ is defined to be the point $(-x, -y, -z)$, that is the point which is directly opposite to it. The palne in this model is

$$\mathbb{RP}^2 = \big\{\big((x, y, z), (-x, -y, -z)\big) : x^2 + y^2 + z^2 = 1\big\}.$$

A *line* in this model is the set of all pairs of antipodal points on $\mathcal{S}^2$ that lie on a great circle (that is a circle whose center is $(0, 0, 0)$). In class we discussed how to find the great circles, as the intersection of the sphere and planes containing the origin. It will be helpfull to remember that a plane in $\mathbb{R}^3$ is described with an equation of the form $ax + by + cz = d$, for fixed reals a, b, c, d of which at least one of a, b, c is non zero, and if the plane is to contain the origin, that is the point $(0, 0, 0)$ then $d = 0$.

Verify the incidence axioms for this model.

Notice that this model fixes the spherical geometry we mentioned in class, so that it now satisfies the incidence axioms. If the points are just the points on the sphere, then through two points there infinitely many great circles (think North and South poles and all the meridians).