Review 3 - math 313: Complex Analysis - Fall 2013
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For the third optional midterm the topics to be evaluated are:

1. Laurent series (Section 5.5).
2. Classification of isolated singularities and zeros of analytic functions (Section 5.6). Including possibly at $z = \infty$ (Section 5.7).
3. Residue theorem (Section 6.1).
4. Calculating integrals on $\mathbb{R}$ or on an interval using complex analysis techniques:
   - trigonometric integrals on an interval of length 2π (Section 6.2),
   - improper integrals of rational functions with denominator at least two degrees more than numerator (Section 6.3),
   - improper integrals of trigonometric functions times rational functions with denominator at least one degree more than numerator (Section 6.4),
   - indented contours (Section 6.5),
   - integrals involving multiple-valued functions (Section 6.6).
5. Conformal mappings and specifically Moëbius transformations (Section 7.2, 7.3).
6. Applications to Harmonic functions: solve Dirichlet problem on a domain with given boundary conditions by first conformally mapping the domain onto another, where we know how to solve Dirichlet's problem and pulling back, since composition with analytic maps preserves harmonicity (Sections 7.1 and 7.6, you must remember also Sections 2.5 and 3.4 to effectively use this technique).

Here are a few useful facts (I will give you the formulas for the residues at poles, Jordan's lemma and small circle lemma in the exam):

1. **Behavior near an isolated singularity:** If f has an isolated singularity at z_0 (f is analytic on $0 < |z - z_0| < r$ for some $r > 0$) then
   (i) f is bounded iff z_0 is a removable singularity,
   (ii) $\lim_{z\to z_0} f(z) = \infty$ iff z_0 is a pole,
   (iii) $\lim_{z\to z_0} f(z)$ does not exist iff z_0 is an essential singularity.

   Furthermore you can pin down the order $m > 0$ of a zero or of a pole by checking the following limits:
   (i) $\lim_{z\to z_0} \frac{f(z)}{(z-z_0)^m}$ exists and is not zero then z_0 is a zero of order m.
   (ii) $\lim_{z\to z_0} (z - z_0)^m f(z)$ exists and is not zero then z_0 is a pole of order m.
2. **Behavior at $z = \infty$:** A function $f(z)$ analytic for all $|z| > R$ for some $R \geq 0$ is said to be analytic at $z = \infty$, to be conformal at $z = \infty$, to have a zero of order m at $z = \infty$, to have a pole of order m at $z = \infty$, to have a removable singularity at $z = \infty$, or to have an essential singularity at $z = \infty$ if the function $g(w) = f(1/w)$ is analytic at $w = 0$, is conformal at $w = 0$, has a zero of order m at $w = 0$, has a pole of order m at $w = 0$, has a removable singularity at $w = 0$, or has an essential singularity at $w = 0$ respectively.
3. **Residue at poles:** If a function f has a pole of order $m \geq 1$ at z_0 then

$$\mathrm{Res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z - z_0)^m f(z)].$$

4. **Jordan's Lemma:** If $\xi > 0$ and $P(z)/Q(z)$ is the quotient of two polynomials such that degree $Q \geq 1 +$ degree P then
$$\lim_{R\to+\infty} \int_{\gamma_R^+} e^{i\xi x} \frac{P(z)}{Q(z)} dz = 0,$$
where γ_R^+ is the upper semi-circle of radius R, that is $\gamma_R^+ = \{z \in \mathbb{C} : |z| = R \text{ and } \mathrm{Im} z \geq 0\}$.
If $\xi < 0$ same holds integrating over the lower semi-circle $\gamma_R^- = \{z \in \mathbb{C} : |z| = R \text{ and } \mathrm{Im} z \leq 0\}$.

5. **Small Circles' Lemma:** If a function f has a simple pole at z_0, and γ_r is the arc of the circle centered at z_0 of radius r parametrized by $z(t) = z_0 + re^{it}$ with $\alpha \leq t \leq \beta$ (counterclockwise), then
$$\lim_{r\to 0^+} \int_{\gamma_r} f(z)dz = i(\beta - \alpha)\mathrm{Res}(f; z_0).$$

6. When using the theory of residues to calculate integrals on $\mathbb{R}$ you need to specify:
   (i) The contour(s) you will be using.
   (ii) The function $f(z)$ to be integrated which should be analytic except for finitely many isolated singularities inside all the contours.
   (iii) If you are sending any parameter defining the contours to infinity or to zero.
   (iv) That you are using the Cauchy residue Theorem to calculate the integral of $f(z)$ over each of the contours, giving you a number independent of the parameters defining the contours (more precisely you will get the complex number $\pm 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f; z_k)$ where $\{z_k\}_{k=1}^{N}$ are the N singularities enclosed by ALL the contours, and the sign is determined by the orientation of the contour).
   (v) That you are breaking each contour onto subcontours that in the limit, as the parameters defining the contour go to infinity and/or to zero, will yield the principal value integral you are trying to calculate, and integrals on, for example, big or small circles/half-circles that will either go to zero (Jordan's Lemma) or will give you a portion of the residue at the center of the small circles (Small Circle's Lemma).
   (vi) If integrating a multi-valued function such as z^α or $\log z$ know that the function has a discontinuity accross the slit so that if you approach from above you get a different value than when you approach from below. Specifically if $x > 0$, $\lim_{y\to 0^+} Log_0(x + iy) = \ln x$, and $\lim_{y\to 0^-} Log_0(x + iy) = \ln x + i2\pi$.

7. **Harmonicity preserved under conformal maps:** If a domain D_1 is mapped in a one-to-one fashion onto another domain D_2 by a conformal mapping $h : D_1 \to D_2$ and $\phi : D_2 \to \mathbb{R}$ is a harmonic function then the function $\psi(z) = \phi(h(z))$ is harmonic on D_1.

8. **Möebius Transformations:** These are analytic functions on the extended complex plane $\mathbb{C} \cup \{\infty\}$ of the following form,
$$h(z) = \frac{az + b}{cz + d}, \quad \text{with} \quad ad - bc \neq 0.$$
Properties:
   (a) h has exactly one single pole (at $z = \infty$ when $c = 0$, otherwise at $z = -d/c$).
   (b) h is conformal everywhere except at the pole. In particular this means that h preserves angles between intersecting smooth curves at points other than the pole.
   (c) h maps the extended complex plane one-to-one onto itself, hence h is invertible in the extended complex plane. Its inverse is another Moëbius transformation found solving for z in terms of w the equation $w = \frac{az+b}{cz+d}$.
   (d) h map circles and lines to circles and lines.
   (e) h is completely determined by three different points and their images.

Review Problems

1. Classify all the singularities of the function $f(z) = \frac{\sin z}{z^5(z-2)}$ in the extended complex plane $\mathbb{C} \cup \{\infty\}$ (identify all poles with their order, all removable singularities and all essential singularities).

2. Evaluate the following integrals by explicitely calculating appropriate Laurent series

   (a) $\displaystyle\oint_{|z|=1} z^5 \cos^{1/z}\, dz$

   (b) $\displaystyle\oint_{|z+1|=3} \frac{e^{1/(z+1)}}{z+2}\, dz$

3. Evaluate the following integrals (please write all details and justifications):

   (a) $\displaystyle\int_0^{2\pi} \frac{1}{1+a\cos\theta}\, d\theta$, where a is a real parameter $-1 < a < 1$.

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\, dx,$

   (c) p.v. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)(x+2)}\, dx,$

   (d) $\displaystyle\oint_{|z|=2} e^{-1/z} \sin 1/z\, dz.$

4. Compute $\displaystyle\int_0^{\infty} \frac{1}{\sqrt{x}(x+1)}\, dx.$

5. Find a function $\phi(z)$ harmonic in the intersection of the discs $|z| < 1$ and $|z-1| < 1$ and with boundary values $\phi = 2$ on the arc of the circle $|z| = 1$ and $\phi = 5$ on the arc of the circle $|z-1| = 1$. Hint: map the domain using an appropriate Moëbius transformation to a wedge, notice that the angle at the intersection points of the circles is $\pi/2$.

6. Find a function $\phi(z)$ harmonic in the domain $\{z \in \mathbb{C} : |z| < 2 \text{ and } |z-1| > 1\}$ and with boundary values $\phi = 4$ on the larger circle $|z| = 2$ and $\phi = 3$ on the smaller circle $|z-1| = 1$. Hint: map the domain using an appropriate Moëbius transformation to two parallel lines (this time the angle at the tangent point is zero.

7. Find a function ϕ harmonic in the annuli $\{z \in \mathbb{C} : |z+1| > 1 \text{ and } |z| < 3\}$ that has boundary values $\phi = 0$ on the smaller circle $|z+1| = 1$ and $\phi = 1$ on the larger circle $|z| = 3$. Hint: map the domain using an appropriate Moëbius transformation to two concentric circles, then use washers. Model on Example 2 in page 421.