NAME: ______________________________

Midterm 1 - math 313: Complex Analysis - Fall 2013

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There are a total of 100 points, plus ten possible bonus points. No books, notes or calculators are allowed. Good luck!

EXER. 1	EXER. 2	EXER. 3	EXER. 4	EXER. 5	EXER. 6	TOTAL

Blackboard formulas and results promised

1. **Definitions of trigonometric, hyperbolic, logarithmic and power functions:**
   - $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}.$
   - $\cosh z = \dfrac{e^{z} + e^{-z}}{2}, \qquad \sinh z = \dfrac{e^{z} - e^{-z}}{2}.$
   - For $z \neq 0$, $\;\mathrm{Log} z = \ln|z| + i\mathrm{Arg} z$ (single-valued).
   - For $z \neq 0$, $\;\log z = \ln|z| + i \arg z$ (multiple-valued).
   - For $z \neq 0$, $a \in \mathbb{C}$, $\; z^a = e^{a \log z}$ (multiple-valued when a is not an integer).
2. **Lemma:** If $\phi : S \subset \mathbb{R}^2 \to \mathbb{R}$, S is a domain (an open and connected set), $\dfrac{\partial \phi}{\partial x} = \dfrac{\partial \phi}{\partial y} = 0$ on S then ϕ is a constant function.
3. **Cauchy-Riemann Equations in polar coordinates**: Given $f(z) = u(r,\theta) + iv(r,\theta)$ for $z \neq 0$, then u and v satisfy the CR equations if and only if
$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$
4. **Limits at infinity and equal to infinity**
   - We say $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \dfrac{1}{f(z)} = 0$.
   - We say $\lim_{z \to \infty} f(z) = L$ if and only if $\lim_{z \to 0} f(1/z) = L$.
   - We say $\lim_{z \to \infty} f(z) = \infty$ if and only if $\lim_{z \to 0} \dfrac{1}{f(1/z)} = 0$.

1. (50 points + 10 bonus points) Please decide whether the following statements are true or false. If TRUE give a very short explanation as to why. If FALSE present a a counterexample or a corrected formula/statement. (There are 12 questions: 10 correct give you full credit, everything additional is bonus).

(a) If $z^5 = 2e^{i\pi}$ then $z = 2^{1/5}e^{i\pi/5}$. ☐ TRUE ☐ FALSE

(b) $|-i^3| = i$ ☐ TRUE ☐ FALSE

(c) $f(z) = e^z$ is a one-to-one function on $\mathbb{C}$. ☐ TRUE ☐ FALSE

(d) $\lim\limits_{z\to(-3i)} z^2e^z = -9\cos 3 + i\,9\sin 3$. ☐ TRUE ☐ FALSE

(e) $\mathrm{Arg}(z)$ is a continuous function on $\mathbb{C}\setminus\{0\}$. ☐ TRUE ☐ FALSE

(f) $\mathrm{Log}(z^4) = 4\mathrm{Log}(z)$ for all $z \neq 0$. ☐ TRUE ☐ FALSE

(g) $|\cos z| \leq 1$ for all $z \in \mathbb{C}$. ☐ TRUE ☐ FALSE

(h) If f is an entire functions then $h(z) = e^z f(1/z^2)$ is analytic on $\mathbb{C} \setminus \{0\}$, and its derivative $h'(z) = h(z) + e^z f'(1/z^2)(-2/z^3)$. ☐ TRUE ☐ FALSE

(i) The function $u(x, y) = 2y - 3$ is harmonic in $\mathbb{C}$. ☐ TRUE ☐ FALSE

(j) If u and v are harmonic functions on $\mathbb{C}$ then $f(z) = u(x, y) + iv(x, y)$ is analytic on $\mathbb{C}$. ☐ TRUE ☐ FALSE

(k) If $f(z)$ is an entire function with $u(x, y)$ its real part and $v(x, y)$ its imaginary part. Then u is the harmonic conjugate of v. ☐ TRUE ☐ FALSE

(m) If f is real-valued on $\mathbb{C}$ and analytic then f must be a constant function.

☐ TRUE ☐ FALSE

2. (10 points) Let $z = 1 - i$. Write z in polar coordinates.

   Write (a) $z^{1/3}$, (b) $\log z$ and (c) $\dfrac{i}{z}$ in the form $a + ib$ with $a, b \in \mathbb{R}$ (note: some of these are multiple-valued).

3. (10 pts) Find the limit if it exists otherwise justify why it does not exist. Here $y \in \mathbb{R}$ and $z \in \mathbb{C}$.

   (a) $\displaystyle\lim_{z \to 0} \frac{\overline{z}}{z}$

   (b) $\displaystyle\lim_{z \to i\pi} \frac{e^z + 1}{z - i\pi}$

4. (10 pts) Given the function $f(z) = \left(x + \frac{y^3}{3}\right) + i\left(y - x + \frac{x^3}{3}\right)$ defined on the complex plane $\mathbb{C}$.

(a) Determine all points of continuity of f.

(b) Determine all points at which the function f is differentiable.

(c) Determine all points at which the function is analytic.

5. (10 pts) Can you find an entire function $f(z)$ whose real part is $u(x, y) = e^x \sin y$? If yes, find such function $f(z)$ and write it as a function of z only.

6. (10 pts) Find a function $\phi(r, \theta)$ that is harmonic in the domain $\{z \in \mathbb{C} : 1 < |z-3i| < 5\}$ and that has boundary values $\phi = 2$ when $|z - 3i| = 1$ and $\phi = 12$ when $|z - 3i| = 5$.