NAME: ______________________________

Midterm 2 - math 313: Complex Analysis - Fall 2013

Instructor: Cristina Pereyra

There are a total of 110 points, 10 of them are bonus points. No books, notes or calculators are allowed. Good luck!

EXER. 1	EXER. 2	EXER. 3	EXER. 4	EXER. 5	EXER. 6	TOTAL

Formulas promised

- **Cauchy Integral Formulas:** if f is *analytic* on a *simply connected domain* containing a *simple closed positively oriented contour* Γ, and z_0 is a point *interior* to the contour, then
$$f(z_0) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-z_0}dz.$$
Furthermore, similar formulas hold for the nth-derivative of f at z_0,
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

- **Cauchy Product Formula:** If $\sum_{n=0}^\infty a_n(z-z_0)^n$ and $\sum_{n=0}^\infty b_n(z-z_0)^n$ converge for $|z-z_0|<r$, then their product is another power series
$$\sum_{n=0}^\infty c_n(z-z_0)^n \qquad \text{where} \qquad c_n = \sum_{k=0}^n a_k b_{n-k}.$$

Q1-Q4/120 %	HW1-8/760 %	MT1/100	MT2/100	MT3/100	FINAL/100

Third midterm is optional (Thursday Dec 5th, 2013). I will drop the worst among the six possible grades: quizes, homework, three midterms and a final (Tuesday Dec 10, 2013 from 10am-12noon). I will weight differently the five grades, giving more weight to the highest grade, and less to the lowest of the five grades. Weights to be decided.

1. (50 points) Please decide whether the following statements are true or false. If TRUE give a very short explanation as to why. If FALSE present an explanation, a counterexample, or a corrected formula/statement. All closed contours are positively oriented.

   (a) The function $f(z) = 1/z$ has an antiderivative in the disc $|z - 3i| < 2$

   ☐ TRUE ☐ FALSE

   (b) $$\oint_{|z|=1} \frac{1}{z}\, dz = \oint_{|z+2i|=1} \frac{1}{z}\, dz.$$

   ☐ TRUE ☐ FALSE

   (c) If f is analytic within and on the simple closed contour $|z| = 2$ then
   $$\oint_{|z|=2} \frac{f'(z)}{z+i}\, dz = \oint_{|z|=2} \frac{f(z)}{(z+i)^2}\, dz.$$

   ☐ TRUE ☐ FALSE

   (d) For every simple closed contour Γ that encloses the points z_0 and z_1 we have
   $$\oint_{\Gamma} \frac{1}{(z-z_0)(z-z_1)}\, dz = 0$$

   ☐ TRUE ☐ FALSE

(e) If the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 5 then the power series $\sum_{n=0}^{\infty} a_n z^{2n}$ has radius of convergence $\sqrt{5}$. ☐ TRUE ☐ FALSE

(f) A power series centered at z_0 and with radius of convergence R, converges for all $|z - z_0| \leq R$. ☐ TRUE ☐ FALSE

(g) A function f is analytic at a point z_0 if and only if f can be expanded in a convergent power series centered at z_0. ☐ TRUE ☐ FALSE

(h) Suppose a function f has a Taylor series representation centered at $z_0 = i$ with radius of convergence R. Then f is analytic everywhere inside the circle of convergence ($|z-i| < R$), and is not analytic everywhere outside the circle of convergence ($|z - i| > R$). ☐ TRUE ☐ FALSE

(i) $f(z) = \cos z$ is an entire function and is not constant hence it must be unbounded. ☐ TRUE ☐ FALSE

(i) If $|f(z)| \leq 7$ for all z on the contour $|z-3i| = 4$ then $\left| \oint_\Gamma \frac{f(z)}{z-3i}\, dz \right| \leq 14\pi$.

☐ TRUE ☐ FALSE

2. (12 points) Calculate the integral $\int_\Gamma \frac{1}{z+1}\, dz$ over the polygonal line that connects first $z = 0$ to $z = 1+2i$ and then it connects $z = 1+2i$ to $z = i$.

3. (12 points) The simple closed contour Γ is a figure eight crossing at $z = 0$ and containing $z = \pm i\pi$ (the top loop is traversed counterclockwise, the bottom loop clockwise). Calculate the integral

$$\oint_{\Gamma} \frac{\cos z}{z^2 + \pi^2}\, dz.$$

4. (12 points) Find the MacLaurin expansion for $f(z) = \dfrac{1}{(1-2z)^2}$ and determine its radius of convergence.

5. (12 points) Show that $f(z)$ defined below is an entire function and find $f^{(5)}(0)$, where

$$f(z) = \begin{cases} (e^z - 1)/z & z \neq 0, \\ 1 & z = 0. \end{cases}$$

Hint: find a MacLaurin expansion that coincides with the function.

6. (12 points) Suppose f is an entire function. Use the Cauchy Integral Formula to show that for all complex numbers z_0 and positive radius r,

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

In words: the value of f at a point z_0 is the average of the values of f along the circle centered at z_0 and of radius r.