# Review and Practice Problems for Exam # 2 - MATH 401/501 - Fall 2017

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**Real numbers**

- Real numbers are closed under addition, multiplication, negation, subtraction and division by non-zero real numbers. You are free to use usual arithmetic properties (commutative and associative properties of addition and multiplication, distributive property, etc).
- Real numbers have an order, and obey a trichotomy if $x, y$ are real numbers then exactly one of the following holds: $x = y$, $x < y$ or $x > y$.
- Should know and be able to use
  - (i) the definition of absolute value of a real number,
  - (ii) the triangle inequality (and "reverse" triangle inequality).
- Understand Archimedean properties and their implications: interspersing of integers by $\mathbb{R}$, density of rationals and irrationals.
- Understand the meaning of upper and lower bounds for a set in $\mathbb{R}$, and the meaning of the supremum (least upper bound or l.u.b.) and infimum (greatest lower bound or g.l.b.) of a set of real numbers.
- Be able to show that a given number is the supremum (infimum) of a set by showing that
  - (i) it is an upper (lower) bound for the set,
  - (ii) it is the smallest upper (largest lower) bound.
- Appreciate the Least Upper Bound and Greatest Lower Bound properties of real numbers: every non-empty and bounded set of real numbers has a unique supremum and a unique infimum.

**Sequences of real numbers**

- Know the definition of bounded sequences, bounded above sequences and bounded below sequences. More precisely a sequence $\{x_n\}_{n\geq 0}$ is bounded (respectively bounded above or bounded below) iff there is $M > 0$ such that for all $n \geq 0$ we have $|x_n| \leq M$ (respectively $x_n \leq M$ or $M \leq x_n$).
- Know the $\epsilon, N$ definition of Cauchy sequences and of convergent sequences in $\mathbb{R}$ to a limit $L \in \mathbb{R}$. More precisely, a sequence $\{x_n\}_{n\geq 0}$ of real numbers
  - is Cauchy iff given $\epsilon > 0$ there is $N > 0$ such that for all $n, m \geq N$ then $|x_n - x_m| \leq \epsilon$,
  - converges to $L$ iff given $\epsilon > 0$ there is $N > 0$ such that for all $n \geq N$ then $|x_n - L| \leq \epsilon$.
- Be able to show that limits are unique (that is if a sequence converges it converges to a unique limit).
- Be able to prove or disprove that a given sequence converges or is Cauchy by using the "$\epsilon, N$ definition". E.g. $a_n = 1/n$, $b_n = 2^{-n}$.
- Be able to show that a convergent sequence is a Cauchy sequence.
- Be able to show and use that Cauchy sequences (and hence convergent sequences) are bounded sequences. However not all bounded sequences are convergent, e.g . $b_n = (-1)^n$ for all $n \geq 0$.
- Be able to show that the sum/product of two Cauchy sequences (or two convergent sequences) is a Cauchy sequence (or a convergent sequence and convergent to the sum/product of the limits of the given convergent sequences "limit laws").
- Understand that if a Cauchy (convergent) sequence is bounded away from zero then the sequence of reciprocals is Cauchy (hence convergent and to the reciprocal of the limit which is necessarily non-zero, another "limit law").
- Be able to prove or disprove that a given sequence converges by appealing to additive/multiplicative/reciprocal properties of limits (limit laws), and using known basic limits.

- Know and be able to use the Monotone Bounded Sequence Convergence Theorem:
  (i) an increasing and bounded above sequence is convergent and to the sequence's supremum,
  (ii) a decreasing and bounded below sequence is convergent and to the sequence's infimum.

- You should know and use some basic limits :
  - $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$, is 1 if $x = 1$, and does not exist if $x = -1$ or $|x| > 1$;
  - $\lim_{n\to\infty} x^{1/n} = 1$ if $x > 0$;
  - $\lim_{n\to\infty} 1/n^{1/k} = 0$ for all integers $k \geq 1$.
  - $\lim_{n\to\infty} n^{1/n} = 1$;
  - Let $a_n > 0$, $q \in \mathbb{R}$, if $\lim_{n\to\infty} a_n = 1 > 0$ then $\lim_{n\to\infty} a_n^q = 1$.

- Appreciate the deep fact that Cauchy sequences are convergent sequences in $\mathbb{R}$ (completeness of the real numbers) .

**Limit points, limsup, liminf**

- Appreciate the definition of "limit points" of a sequence as the collection of "subsequencial limits" (the limits of convergent subsequences of the sequence.

- Know that $c$ is a limit point for a sequence $\{x_n\}$ if for all $\epsilon > 0$ there are "infinitely many" terms of the sequence in the interval $[c-\epsilon, c+\epsilon]$. More precisely, for all $\epsilon, N > 0$ there is an $n_N \geq N$ such that $|x_{n_N} - c| \leq \epsilon$ (necessarily the set of labels $\{n_N\}_{N\geq 0}$ is an infinite set!).

- Be able to identify the "limit points" (or "subsequencial limits") of a concrete sequence e.g: $a_n = 3$ for all $n \geq 0$, $b_n = (-1)^n$ for all $n \geq 0$, $c_n = (-1)^n n$ for all $n \geq 0$.

- Know that bounded sequences in $\mathbb{R}$ have limit superior/inferior in $\mathbb{R}$, defined as $\limsup\{x_n\} := \lim_{N\to\infty} \sup_{n\geq N} x_n$ and $\liminf\{x_n\} := \lim_{N\to\infty} \inf_{n\geq N} x_n$.

- Be aware of the $\epsilon$ characterization of limsup (similarly liminf): for all $\epsilon > 0$
  (i) Finitely many terms of the sequence $\{x_n\}$ are larger than $\limsup\{x_n\} + \epsilon$. More precisely for all $\epsilon > 0$ there is $N > 0$ such that for all $n \geq N$ we have $x_n \leq \limsup\{x_n\} + \epsilon$.
  (ii) Infinitely many terms of the sequence $\{x_n\}$ are in between $\limsup\{x_n\} - \epsilon$ and $\limsup\{x_n\} + \epsilon$. More precisely, for all $\epsilon > 0$ and $N > 0$ there is $n_N \geq N$ such that $|x_{n_N} - \limsup\{x_n\}| \leq \epsilon$.

  And its consequences:
  - $\limsup\{x_n\}$ and $\liminf\{x_n\}$ are limit points (subsequencial limits) of the sequence.
  - A sequence of real numbers converges if and only if the limsup and the liminf coincide.
  - The limsup is the "largest limit point" (or "largest subsequential limit") of the sequence, and liminf is the "smallest limit point" (or "smallest subsequential limit") of the sequence.
  - A sequence converges to $L$ iff all its subsequences converge to $L$ iff the unique limit point of the sequence is $L$.
  - Every bounded sequence has a at least one convergent subsequence or equivalently at least one "limit point" (Bolzano-Weierstrass theorem).

- Be able to identify the $\limsup$ and $\liminf$ of a given sequence. Use this knowledge to conclude that if $\limsup a_n = \liminf a_n = L$ then the sequence $\{a_n\}$ converges AND $\lim_{n\to\infty} a_n = L$.

- Be able to use the squeeze theorem to deduce convergence of the sequence being squeezed.

**Series**

- Understand that convergence of a series is by definition convergence of the sequence of partial sums.

- Be able to deduce from the theory of sequences basic convergence tests: Cauchy test, divergence test, absolute convergence test, comparison test.

- Be familiar with other useful tests such as: alternating series test, p-test, root test, and ratio test. Be able to use these tests to deduce convergence or divergence of specific series.

- Be able to exploit convergence properties of geometric series: $\sum_{n=0}^{\infty} r^n$ converges to $1/(1-r)$ if $|r| < 1$, diverges otherwise.

**Limits and continuity of functions $f : E \to \mathbb{R}$, $E \subset \mathbb{R}$**

- Know definition of a bounded function: $\exists M > 0$ such that $|f(x)| \leq M$ for all $x \in E$.
- Know the equivalent "definitions" of $\lim_{x\to x_0} f(x) = L$. Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$, $x_0$ is an *adherent point*[1] of $E$, then $\lim_{x\to x_0, x\in E} f(x) = L$ if and only if
  - ($\epsilon - \delta$ definition) $\forall\epsilon > 0\ \exists\delta > 0$ such that $|f(x) - L| \leq \epsilon\ \forall x \in E$ such that $|x - x_0| \leq \delta$.
  - (Sequential definition) For all sequences $\{x_n\}_{n\geq 0}$ in $E$ if $\lim_{n\to\infty} x_n = x_0$ then $\lim_{n\to\infty} f(x_n) = L$.
- Know the equivalent "definitions" of continuity at a point $x_0$. Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$, $x_0 \in E$, then $f$ is continuous at $x_0$ if and only if
  - (Limit definition) $\lim_{x\to x_0, x\in E} f(x) = f(x_0)$.
  - ($\epsilon - \delta$ definition) $\forall\epsilon > 0\ \exists\delta > 0$ such that $|f(x) - f(x_0)| \leq \epsilon\ \forall x \in E$ such that $|x - x_0| \leq \delta$.
  - (Sequential definition) For all sequences $\{x_n\}_{n\geq 0}$ in $E$ if $\lim_{n\to\infty} x_n = x_0$ then $\lim_{n\to\infty} f(x_n) = f(x_0)$.
- Be able to decide whether a function is bounded or not and whether a function is continuous or not.
- Know that basic functions are continuous such us: constant function ($f(x) = c$), identity function ($f(x) = x$), absolute value function ($f(x) = |x|$ for $x \in \mathbb{R}$), and exponential functions ($f(x) = x^p$ for $x > 0$, and $g(x) = a^x$ for $a > 0$ and $x \in R$).
- Know the limit laws for functions and be able to prove them and use them to compute limits.
- Know and be able to prove that composition and arithmetic operations preserve continuity. Use these properties to conclude that more complex functions are continuous such us: polynomials ($p(x) = a_0 + a_1 x + \cdots + a_n x^n$), rational functions (quotients of polynomials, wherever the denominator is non-zero), exponentials with continuous base or exponent ($f(x) = a^{p(x)}$ or $g(x) = f(x)^q$ where $f$ is a positive and continuous function and $q \in \mathbb{R}$.
- Use your knowledge of continuous functions to compute limits for example:
  - $\lim_{x\to x_0} x^q = x_0^q$ for $x, x_0 > 0$ and $q \in \mathbb{R}$.
  - $\lim_{x\to x_0} a^x = a^{x_0}$ for $x, x_0 \in \mathbb{R}$ and $a > 0$.

**Practice Problems for Midterm #2**

1. If the real number $x$ is not rational we say $x$ is "irrational".
   (a) Show that if $p \in \mathbb{Q}$, $p \neq 0$, and $x$ is irrational then $px$ is irrational.
   (b) Show that if $x, y \in \mathbb{R}$ and $x < y$ then there is an irrational number $w$ such that $x < w < y$ (density of the irrational numbers).
2. For each subset $A$ of real numbers decide whether is bounded (above, below or both), find supremum and infimum: (a) $A = \{1, -1/2, 3\}$, (b) $A = \{n/(n+1) :\ n \in \mathbb{N},\ n \geq 1\}$, (c) $A = \{r \in \mathbb{Q} : r < 5\}$.
3. If $A$ and $B$ are nonempty and bounded subsets of $\mathbb{R}$ such that $A \subset B$ show that $\inf(B) \leq \inf(A)$.
4. Let $E$ be a nonempty and bounded subset of $\mathbb{R}$, let $\lambda \in \mathbb{R}$ and $\lambda > 0$. Define $\lambda E = \{\lambda x : x \in E\}$ a subset of $\mathbb{R}$. Prove that If $\lambda \geq 0$ then $\sup(\lambda E) = \lambda \sup(E)$. What is $\inf(\lambda E)$? What if $\lambda < 0$?
5. Given $\lambda > 0$ and $\{s_n\}_{n\geq 0}$ is a bounded sequence. Show that $\limsup\{\lambda s_n\} = \lambda \limsup\{s_n\}$. What can you say when $\lambda < 0$? (Hint use previous exercise).

[1] A point $x_0 \in E \subset \mathbb{R}$ is adherent iff for all $\delta > 0$ there is $x \in E$ such that $|x - x_0| \leq \delta$ (in words, we can get arbitrarily close to $x_0$ with points $x$ in $E$).

6. For each of the following, prove or give a counterexample.
   (a) If $\{x_n\}_{n\geq 0}$ converges to $x$ then $\{|x_n|\}_{n\geq 0}$ converges to $|x|$.
   (b) If $\{|x_n|\}_{n\geq 0}$ is convergent then $\{x_n\}_{n\geq 0}$ is convergent .
7. We say the sequence $\{x_n\}_{n\geq 0}$ diverges to $+\infty$ and we write $\lim_{n\to\infty} x_n = +\infty$ iff for all $M > 0$ there is $N > 0$ such that for all $n \geq N$ we have $x_n \geq M$.
   (a) Write down a definition for a sequence $\{y_n\}_{n\geq 0}$ to diverge to $-\infty$.
   (b) Show that if $x_n \leq z_n$ for all $n \geq 0$ and $\{x_n\}$ diverges to $+\infty$ then $\{z_n\}$ diverges to $+\infty$.
   (c) Let $\{x_n\}$ sequence in $\mathbb{R}$, $x_n > 0$. Show that $\lim_{n\to\infty} x_n = +\infty$ if and only if $\lim_{n\to\infty}(1/x_n) = 0$.
8. The sequence of positive real numbers $\{t_n\}_{n\geq 0}$ converges to $t$. Decide whether the following sequences are convergent or not. If convergent explain why and identify the limit, if not convergent explain why.
   (a) $a_n = \sqrt{t_n}$, (b) $b_n = 5t_n^3 - t_n^2 + 7$, (c) $c_n = \dfrac{n}{2^n}(-1)^n$, (c) $d_n = n + t_n$.
9. Use squeeze theorem and properties of sine function to show $\lim_{n\to\infty} \dfrac{\sin n}{n} = 0$.
10. Show that the sequence defined by $x_1 = 1$ and $x_{n+1} = \sqrt{1 + x_n}$ for $n \geq 1$ is convergent (hint: show that it is increasing and bounded by 2). Find the limit.
11. Let $x_n = n \sin^2(n\pi/2)$. Find the set $S$ of limit points (subsequencial limits), find $\limsup x_n$ and $\liminf x_n$. (Assume known properties about sine function.)
12. Show that the sequence of partial sums: $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ defined for $n \geq 1$ is not Cauchy (hint: show that $S_{2n} - S_n \geq 1/2$). Conclude that the harmonic series is divergent.
13. A sequence $\{s_n\}_{n\geq 0}$ is contractive if there is a constant $r$ with $0 < r < 1$ such that $|s_{n+2} - s_{n+1}| \leq r|s_{n+1} - s_n|$ for all $n \geq 0$. Show that a contractive sequence is a Cauchy sequence and hence a convergent sequence (hint: recall convergent geometric series).
14. Show that if a series converges absolutely then it converges.
15. Assume that $|a_n| \leq 2b_n + 3^{-n}$ for all $n \geq 0$ and $\sum_{n=0}^{\infty} b_n$ converges. Show that $\sum_{n=0}^{\infty} a_n$ converges.
16. Determine for each $x \in \mathbb{R}$ whether the series $\sum_{n=1}^{\infty} \dfrac{2^n x^n}{n}$ is convergent or divergent.
17. Let $a_n > 0$ for all $n \geq 1$. Show that $\limsup a_n^{1/n} \leq \limsup \frac{a_{n+1}}{a_n}$ and $\liminf a_n^{1/n} \geq \liminf \frac{a_{n+1}}{a_n}$. Deduce that $\lim_{n\to\infty} n^{1/n} = 1$ (hint: choose $a_n = n$).
18. Let $E \subset \mathbb{R}$, $f, g : E \to \mathbb{R}$ be functions, $x_0$ an adherent point of $E$. Assume $f$ has limit $L$ at $x_0$ in $E$ and $g$ has limit $M$ at $x_0$ in $E$. Show that $\lim_{x\to x_0, x\in E} f(x)g(x) = LM$. Deduce that the product of two continuous functions at $x_0$ is continuous at $x_0$.
19. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined to be $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = -1$ if $x \notin \mathbb{Q}$ is nowhere continuous.
20. Let $p \in \mathbb{R}$. Show that the function $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = x^p$ is continuous on $(0, \infty)$. Hint: use that whenever $a_n > 0$ and $\lim_{n\to\infty} a_n = 1$ then $\lim_{n\to\infty} (a_n)^q = 1$.
21. Show that the function $f(x) = |x|$ is continuous on $\mathbb{R}$.
22. Study the continuity properties of the function $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = \begin{cases} x^2 & \text{if } -1 \leq x < 0 \\ x + 1 & \text{if } 0 \leq x \leq 1. \end{cases}$
23. Exercise 9.5.1 in the book (third edition). Compare to or use Exercise 7.