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Math 402
Homework 2

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(p. 405) 12.2.3, 12.2.4;
(p. 408) 12.3.1;
(p. 411) 12.4.7.

First let's prove the following Lemma:

Lemma 1. *Let (X, d) be a metric space, and let $E \subseteq X$. Then*
(a) $x \in B(x,r)\ \forall r > 0$
(b) $\text{Int}(A) \subseteq A$
(c) (i) $\text{Int}(E) \cap \text{Ext}(E) = \emptyset$
(ii) $\text{Int}(E) \cap \partial E = \emptyset$
(iii) $\text{Ext}(E) \cap \partial E = \emptyset$
(iv) $\text{Int}(E) \cup \text{Ext}(E) \cup \partial E = X$
(d) $\partial E = \partial(X \backslash E)$
(e) $\text{Int}(X \backslash E) = X \backslash \overline{E}$

Proof. **(a)** Obviously true since $d(x,x) = 0 < r$
(b) Let $x \in \text{Int}(E)$ then we know $\exists r > 0$ such that $B(x,r) \subseteq E$, but from (a) we know $x \in B(x,r)$ and hence $x \in E$. Therefore $\text{Int}(E) \subseteq E$.
(c) (i) Assume that the intersection is not empty and let $x \in \text{Int}(E) \subseteq E$ (by (b)) and $x \in \text{Ext}(E)$. Then by definition $\exists r > 0$ such that $B(x,r) \subseteq E$ and $\exists r' > 0$ such that $B(x,r') \cap E = \emptyset$. However, $x \in B(x,r')$ and $x \in E$, a contradiction. Thus $\text{Int}(E) \cap \text{Ext}(E) = \emptyset$

(ii), (iii), and (iv) are all true by definition.

(d)

$$\begin{aligned}
& x \in \partial E \\
& \Leftrightarrow x \notin \text{Int}(E), \text{ and } x \notin \text{Ext}(E) \\
& \Leftrightarrow \forall r > 0\ B(x,r) \cap E \neq \emptyset \text{ and } B(x,r) \not\subseteq E \\
& \Leftrightarrow \forall r > 0\ \exists\ y \text{ such that } y \in B(x,r) \text{ and } y \in E. \text{ And } \exists\ z \text{ such that } z \in B(x,r) \text{ and } z \notin E \\
& \Leftrightarrow \forall r > 0\ z \in B(x,r) \text{ and } z \in X \backslash E. \text{ And } y \in B(x,r) \text{ and } y \notin X \backslash E \\
& \Leftrightarrow \forall r > 0\ B(x,r) \cap (X \backslash E) \neq \emptyset, \text{ and } B(x,r) \not\subseteq X \backslash E \\
& \Leftrightarrow x \in \partial(X \backslash E)
\end{aligned}$$

Hence, $\partial E = \partial(X \backslash E)$.

(e) First note that $X \backslash \overline{E} = X \backslash (\text{Int}(E) \cup \partial E) = \text{Ext}(E)$ (by definition, and Lemma (c:iv)).

Let $x \in \text{Int}(X \backslash E)$ then $\exists r > 0$ such that $B(x,r) \subseteq \text{Int}(X \backslash E) \subseteq X \backslash E$.

Now we will show that $x \in \text{Ext}(E)$ by showing that $\exists r' > 0$ such that $B(x,r') \cap E = \emptyset$.

Let $r' = r$. Then if $y \in B(x,r') = B(x,r) \subseteq X \backslash E$ we get that $y \notin E$. Hence $B(x,r') \cap E = \emptyset$. Thus $y \in \text{Ext}(E)$. Hence, $\text{Int}(X \backslash E) \subseteq \text{Ext}(E)$.

Let $x \in \text{Ext}(E)$. Then $\exists r > 0$ such that $B(x,r) \cap E = \emptyset$. But since $x \in B(x,r)$ we must have that $x \notin E$. Thus $x \in X \backslash E$. Hence $\text{Ext}(E) \subseteq \text{Int}(X \backslash E)$.

Thus $\text{Ext}(E) = \text{Int}(X \backslash E)$, but as we noted earlier: $X \backslash \overline{E} = X \backslash (\text{Int}(E) \cup \partial E) = \text{Ext}(E)$.

Therefore we have $\text{Int}(X \backslash E) = X \backslash \overline{E}$. □

Exercise 12.2.3: Prove Proposition 12.2.15.
Proposition 12.2.15 (Basic properties of open and closed sets). Let (X, d) be a metric space.

(a) Let E be a subset of X. Then E is open if and only if $E = \text{Int}(E)$. In other words, E is open if and only if for every $x \in E$, there exists an $r > 0$ such that $B(x,r) \subseteq E$.

(b) Let E be a subset of X. Then E is closed if and only if E contains all of its adherent points. In other words, E is closed if and only if for every convergent sequence $(x_n)_{n=m}^{\infty}$ in E, the limit $\lim_{n\to\infty} x_n$ of that sequence also lies in E.

(c) For any $x_0 \in X$ and $r > 0$, the ball $B(x_0, r)$ is an open set. The set $C = \{x \in X : d(x, x_0) \leq r\}$ is a closed set.

(d) Any singleton set $\{x_0\}$, where $x_0 \in X$, is automatically closed.

(e) If E is a subset of X, then E is open if and only if the complement $X \backslash E$ is closed.

(f) If $E_1, E_2, \ldots, E_n$ is a finite collection of opens sets in X, then $E_1 \cap E_2 \cap \cdots \cap E_n$ is also open. If $F_1, F_2, \ldots, F_n$ is a finite collection of closed sets in X, then $F_1 \cup F_2 \cup \cdots \cup F_n$ is also closed.

(g) If $\{E_\alpha\}_{\alpha \in I}$ is a collection of open sets in X (where the index set I could be finite, countable, or uncountable), then the union $\bigcup_{\alpha \in I} E_\alpha$ is also open. If $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets in X, then the intersection $\bigcap_{\alpha \in I} F_\alpha$ is also closed.

(h) If E is any subset of X, then $\text{Int}(E)$ is the largest open set which is contained in E; in other words, $\text{Int}(E)$ is open, and given any other open set $X \subseteq E$, we have $V \subseteq \text{Int}(E)$. Similarly $\overline{E}$ is the smallest closed set which contains E; in other words, $\overline{E}$ is closed, and given any other closed set $K \supset E, K \supset \overline{E}$.

Proof. **(a)** First note that if $E = \emptyset$ then we are done. Since then we would have $E = \emptyset \subseteq \text{Int}(E)$. And from Lemma 1(b) we would then $E = \text{Int}(E)$. So we can assume E is not empty.

($\Rightarrow$) Suppose E is open. Then $\partial E \cap E = \emptyset$.

Let $x \in E$, so $x \notin \partial E$. Assume that $x \in \text{ext}(E)$. Then $\exists r > 0$ such that $B(x,r) \cap E = \emptyset$. But $x \in E$ and $x \in B(x,r) \Rightarrow B(x,r) \cap E \neq \emptyset$ a contradiction. Thus, $x \notin \text{Ext}(E)$, hence $x \in \text{Int}(E)$. That is, $E \subseteq \text{Int}(E)$. And from Lemma we have $\text{Int}(E) \subseteq E$. Thus $E = \text{Int}(E)$.

($\Leftarrow$) Suppose $E = \text{Int}(E)$. Assume $\partial E \cap E \neq \emptyset$ then $\exists y \in \partial E \cap E \Rightarrow y \in \partial E, y \in E \Rightarrow y \in \text{Int}(E)$ a contradiction, since $\partial E \cap \text{Int}(E) = \emptyset$. Thus $\partial E \cap E = \emptyset$. Therefore E is open.

(b) ($\Rightarrow$) Suppose E is closed. Then $\partial E \subseteq E$, and from Lemma 1(b) $\text{Int}(E) \subseteq E$, thus $\partial E \cup \text{Int}(E) \subseteq E$.

($\Leftarrow$). Suppose $\partial E \cup$ Int$(E) \subseteq E$ but then $\partial E \subseteq E$, hence E is closed.

(c) First we will prove that for any x_0 and $r > 0$ that $B(x_0, r)$ is open. We will do this by showing that $B(x_0, r) = \text{Int}(B(x_0, r))$.

From Lemma 1(b) we know that $\text{Int}(B(x_0, r)) \subseteq B(x_0, r)$.

Let $x \in B(x_0, r)$ so $d(x, x_0) < r$. Now we will now show that x has to be in the interior of $B(x_0, r)$ by showing that there is an r' such that $B(x, r') \subseteq B(x_0, r)$

Let $r' = r - d(x, x_0) > 0$. Let $y \in B(x, r')$ then $d(y, x) < r'$. From the triangle inequality we get: $d(y, x_0) \leq d(y, x) + d(x, x_0) < r' + d(x, x_0) = r - d(x, x_0) + d(x, x_0) = r$. Hence $d(y, x_0) < r \Rightarrow y \in B(x_0, r)$. Therefore $B(x, r') \subseteq B(x_0, r)$. Hence x is an interior point of $B(x_0, r)$. Thus, $B(x_0, r) \subseteq \text{Int}(B(x_0, r))$. Hence, $B(x_0, r) = \text{Int}(B(x_0, r))$. Therefore $B(x_0, r)$ is open.

Now let us show that $C = \{x \in X : d(x, x_0) \leq r\}$ is a closed set.

Let (x_n) be a sequence in C that converges to $x' \in X$. That is, $\forall \epsilon > 0\ \exists N > 0$ such that $d(x_n, x') < \epsilon\ \forall n \geq N$.

From the triangle inequality we have $d(x', x_0) \leq d(x', x_n) + d(x_n, x_0) \leq \epsilon + r$. Letting ϵ go to zero we then get that $d(x', x_0) \leq r$ hence $x' \in C$. Therefore C is closed.

(d) There is only one convergent sequence in $\{x_0\}$ and that is $(x_0, x_0, \ldots)$ and this sequence converges to $x_0 \in \{x_0\}$. Thus $\{x_0\}$ is closed.

(e) ($\Rightarrow$) Suppose E is open, then $E = \text{Int}(E)$.

But,

$$\begin{aligned}
&X = \text{Int}(E) \cup \text{Ext}(E) \cup \partial E && \text{(From Lemma 1(c))} \\
&\Rightarrow X \backslash E = \text{Ext}(E) \cup \partial E && \text{(Since } E = \text{Int}(E)) \\
&\Rightarrow X \backslash E = \text{Ext}(E) \cup \partial(X \backslash E) && \text{(From Lemma 1(d))} \\
&\Rightarrow \partial(X \backslash E) \subseteq X \backslash E
\end{aligned}$$

Hence $X \backslash E$ is closed.

($\Leftarrow$) Suppose $X \backslash E$ is closed. Then we have $\partial(X \backslash E) \subseteq X \backslash E$. But from Lemma 1(d) $\partial(X \backslash E) = \partial(E)$. Hence $\partial(E) \subseteq X \backslash E$. So if $x \in \partial E$ then $x \in X \backslash E \Rightarrow x \in X$ and $x \notin E$. Hence $\partial E \cap E = \emptyset$. Therefore E is open.

(f) (i) Let $x \in E_1 \cap \cdots \cap E_n$. Thus $x \in E_i\ \forall i = 1, \ldots, n$. But E_i is open for all i, so $\exists r_i > 0$ such that $B(x, r_i) \subseteq E_i$. Let $r' = \min\{r_1, r_2, \ldots, r_n\}$. Then $B(x, r') \subseteq E_i\ \forall i = 1, \ldots, n$. Hence $B(x, r') \subseteq E_1 \cap \cdots \cap E_n$. Hence $E_1 \cap \cdots \cap E_n$ is open.

(ii) From part (e) we know that $X \backslash F_1, \ldots, X \backslash F_n$ are each open. And so from part (i) of this exercise we have that $(X \backslash F_1) \cap \cdots \cap (X \backslash F_n)$ is also open. But $(X \backslash F_1) \cap \cdots \cap (X \backslash F_n) = X \backslash (F_1 \cup \cdots \cup F_n)$ from De Morgan's Law, and hence is also open. Thus we get that $F_1 \cup \cdots \cup F_n$ is closed (from part (e) again).

(g) (i) Let $x \in \bigcup_{\alpha \in I} E_\alpha$. That is, $x \in E_\alpha$ for some $\alpha \in I$. But E_α is open $\forall \alpha \in I$, hence $\exists r > 0$ such that $B(x, r) \subseteq E_\alpha \subseteq \bigcup_{\alpha \in I} E_\alpha$. Thus, $\bigcup_{\alpha \in I} E_\alpha$ is open.

(ii). Let (x_n) be a sequence of elements in $\bigcap_{\alpha \in I} F_\alpha$ that converges to $x \in X$. Since $\forall j \; x_j \in F_\alpha \; \forall \alpha \in I$ and that F_α is closed for all $\alpha \in I$ we know that (x_n) converges in $F_\alpha \; \forall \alpha \in I$. That is, $x \in F_\alpha \; \forall \alpha \in I$. Hence $x \in \bigcap_{\alpha \in I} F_\alpha$. Thus, $\bigcap_{\alpha \in I} F_\alpha$ is closed.

Note that you can also prove this by using DeMorgan's laws and part (i).

(h) (i) First we will show that for $V \subseteq E, V$ open, that $V \subseteq \text{Int}(E)$. Suppose V is open, then $V = \text{Int}(V)$. Let $x \in V = \text{Int}(V)$, then $\exists r' > 0$ such that $B(x, r') \subseteq V \subseteq E$. But this means that $x \in \text{Int}(E)$. Hence $V \subseteq \text{Int}(E)$.

Now we will show that $\text{Int}(E)$ is open. Let $x \in \text{Int}(E)$. Then $\exists r > 0$ such that $B(x, r) \subseteq E$. But from part (c) we know that $B(x, r)$ is open $\forall x, r > 0$. But that means that $B(x, r) \subseteq \text{Int}(E)$. Thus $\text{Int}(E)$ is open.

(ii) From Lemma 1 (e) we know that $\text{Int}(X \backslash E) = X \backslash \overline{E}$. From part (i) we know that $\text{Int}(X \backslash E)$ is open, and now if we take the complement, we can use the result from part (e) and we get that $\overline{E}$ is closed.

Let $V \supseteq E$, V be a closed set. Hence $X \backslash V$ is open, and $X \backslash V \subseteq X \backslash E$. Now using part (i) of this exercise we know that $X \backslash V \subseteq \text{Int}(X \backslash E)$. But from Lemma 1 (e) we know $\text{Int}(X \backslash E) = X \backslash \overline{E}$. Hence we get $X \backslash V \subseteq X \backslash \overline{E}$. Therefore, $\overline{E} \subseteq V$. □

Exercise 12.2.4: Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ and let C be the closed ball $C := \{x \in X : d(x, x_0) \leq r\}$.

(a) Show that $\overline{B} \subset C$.

Proof. Let $x \in \overline{B} = \text{int}(B) \cup \partial B$. Then we have two cases (since by definition a point cannot be both a interior point and boundary point of the same set).

Case 1: ($x \in \text{int}(B)$). This means that $x \in B$ so $d(x, x_0) < r$ but then we also have $d(x, x_0) \leq r$ so $x \in C$.

Case 2: ($x \in \partial B$). Here is a rough outline of what we will do: We will assume $x \notin C$. Then we will show that this leads to $x \in \text{ext}(B)$ (which we will do by finding r' such that $B(x, r') \cap B(x_0, r) = \emptyset$) which contradicts that $x \in \partial B$

Assume, for sake of contradiction, that $x \notin C$. That is, $d(x, x_0) > r$.

Now, let $r' = d(x, x_0) - r > 0$.

Assume $y \in B(x, r') \cap B(x_0, r)$. That is, $y \in B(x, r')$ and $y \in B(x_0, r)$. This means that $d(y, x) < r'$ and $d(y, x_0) < r$.

From the triangle inequality we have:

$d(x, x_0) \leq d(x, y) + d(y, x_0) < r' + r = r - d(x, x_0) + r = d(x, x_0)$.

So we have $d(x, x_0) < d(x, x_0)$. Thus there is no element $y \in B(x, r') \cap B(x_0, r)$. Hence, $B(x, r') \cap B(x_0, r) = \emptyset$. Thus x is an exterior point of B, but this contradicts that $x \in \partial B$, so our original assumption that $x \notin C$ was wrong. And therefore $x \in C$.

In both cases we get that $x \in C$. Thus $x \in \overline{B} \Rightarrow x \in C$, hence $\overline{B} \subset C$ □

(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B}$ is not equal to C.

Solution.

Let $X = \mathbb{R}^2$ and d be the discrete metric. Let $x_0 = (0, 0)$ and take $r = 1$.

Then we have $B_{(\mathbb{R}^2, d_{disc})}((0,0), 1) = \{(0,0)\}$ And so $\overline{B} = \{(0,0)\}$

But $C = \mathbb{R}^2$ since $\forall\ x, y \in \mathbb{R}^2, d(x, y) \leq 1$.

Thus $\overline{B} \neq C$.

Exercise 12.3.1: Prove Proposition 12.3.4(b).
Proposition 12.3.4. Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y.
(a) E is relatively open with respect to Y if and only if $E = V \cap Y$ for some set $V \subseteq X$ which is open in X.

(b) E is relatively closed with respect to Y if and only if $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X.

Proof. First note that for $Y \subseteq X$ we have $(X \backslash V) \cap Y = Y \backslash (V \cap Y)$. Here is a proof for those that are interested: If $x \in (X \backslash V) \cap Y$, then $x \in X, x \notin V, x \in Y$. Thus $x \in Y$, and $x \notin V \cap Y$, hence $x \in Y \backslash (V \cap Y)$. If $x \in Y \backslash (V \cap Y)$, then $x \in Y, x \notin V$. But $Y \subseteq X$, thus $x \in X$, hence $x \in (X \backslash V) \cap Y$.

($\Rightarrow$) Suppose E is relatively closed with respect to Y. Taking the complement in Y we get that $Y \backslash E$ is relatively open with respect to Y. Now using the results of part (a) we know that there is some $V \subseteq X$, V open, such that $Y \backslash E = V \cap Y$. Now take $K = X \backslash V$ which we know is closed in X, and clearly $K \subseteq X$. But $K \cap Y = (X \backslash V) \cap Y = Y \backslash (V \cap Y) = Y \backslash (Y \backslash E) = E$

($\Leftarrow$) Suppose $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X. Thus $X \backslash K$ is open. But, from what we noted above, we know $(X \backslash K) \cap Y = Y \backslash (K \cap Y)$, and since $X \backslash K$ is open, we know that $Y \backslash (K \cap Y)$ is relatively open with respect to Y. But $Y \backslash (K \cap Y) = Y \backslash E$, thus E is relatively closed with respect to Y. □

Here is a direct proof of the first implication (Due to Jorge):

Proof. ($\Rightarrow$) Suppose E is relatively closed in Y, thus every convergent sequence $\{x_n\} \subseteq E$ converges in E. That is, $x_n \to x \in E$.

Let K = closure of E in $(X, d) = \{x \in X : x \text{ is an adherent point of } E \text{ w.r.t. } (X, d)\}$. So K is closed in (X, d)

Now we need to show that $E = K \cap Y$.

If $x \in E$, then clearly $x \in K$ and $x \in Y$. Hence $E \subseteq K \cap Y$.

If $x \in K \cap Y$ then $x \in K$ and $x \in Y$. Thus there is $\{x_n\} \subseteq E$ such that $x_n \to x \in K$ with respect to (X, d). But $\{x_n\} \subseteq Y$ and $x \in Y$, then $x_n \to x$ with respect to $(Y, d|_{Y \times Y})$. But since E is relatively closed with respect to $(Y, d|_{Y \times Y})$, we get $x_n \to x \in E$. Hence $K \cap Y \subseteq E$.

Thus we have $E = K \cap Y$ as desired. □

Exercise 12.4.7: Prove Proposition 12.4.12.
Proposition 12.4.12. (a) Let (X, d) be a metric space, and let $(Y, d|_{Y \times Y})$ is complete, then Y must be closed in X.

(b) Conversely, suppose that (X, d) is a complete metric space, and Y is a closed subset of X. Then the subspace $(Y, d|_{Y \times Y})$ is also complete.

Proof. (a) If $(Y, d|_{Y \times Y})$ is complete then every Cauchy sequence in Y also converges in Y. Given any convergent sequence of elements of Y that converges to say x, we know that it is also Cauchy in Y (Lemma 12.4.7) and hence convergent in Y, that is, $x \in Y$. Thus Y is closed in X (by Proposition 12.2.15(b)).

(b) Let (y_n) be a Cauchy sequence of elements in Y. Since X is complete we know that (y_n) converges to some $x \in X$. From Proposition 12.2.10 we can then conclude that $x \in \text{Int}(Y)$ or $x \in \partial Y$. If $x \in \text{Int}(Y)$ then by Lemma 1 (b) we know $x \in Y$. If $x \in \partial Y$ then we know that $x \in Y$ as Y is closed. In both cases we get $x \in Y$. Thus every Cauchy sequence in Y also converges in Y, and hence Y is complete. □