

Homework 3

(p. 417-419) 12.5.4, 12.5.5, 12.5.12, 12.5.13

Exercise 12.5.4. Let $(\mathbb{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$, and an open set $V \subseteq \mathbb{R}$ such that the image, $f(V)$, of V is not open.

Take $f(x) = 0 \forall x$. Clearly f is continuous, and if we look at the open set $(0, 1)$ it gets mapped into $\{0\}$ which is clearly not open (with respect to $(\mathbb{R}, d)$).

Exercise 12.5.5. Let $(\mathbb{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$, and a closed set $F \subseteq \mathbb{R}$ such that $f(F)$ is not closed.

Take $f(x) = e^x \forall x$. f is continuous, and if we take the closed set $(-\infty, 0]$ it gets mapped into $(0, 1]$ which is not closed.

Exercise 12.5.12. Let (X, d_{disc}) be a metric space with the discrete metric d_{disc} .

(a) Show that X is always complete.

(b) When is X compact, and when is X not compact?

Proof. **(a)** Let $(x^{(n)})_{n=m}^{\infty}$ be a Cauchy sequence in X with respect to d_{disc} . Then we know for every $\epsilon > 0$ there exists an $N \geq m$ such that $d(x^{(j)}, x^{(k)}) < \epsilon$ for all $j, k \geq N$. However, since we are in the discrete metric, for this to be true, we must eventually have N large enough so that $d(x^{(j)}, x^{(k)}) = 0 \forall j, k \geq N$. So $x^{(j)} = x^{(k)}$ (that is, eventually we are tapping on the same spot over and over again), but each of the $x^{(j)}$'s are in X , thus we are convergent in X . Therefore X is complete (with respect to d_{disc}).

(b) X is compact whenever X is finite (this is clear since any sequence must contain a limit point (as there are only a finite number of places to go), and hence a convergent subsequence). X is not compact whenever X is infinite. Take any point in X make it x_1 , then a different point x_2 , and we can do this forever without repeating points, and since each point is different the distance between each point will be 1, hence we have no limit points, and thus no convergent subsequences. $\square$

Exercise 12.5.13. Let E and F be two compact subsets of $\mathbb{R}$ (with the standard metric $d(x, y) = |x - y|$). Show that the Cartesian product $E \times F$ is a compact subset of $\mathbb{R}^2$ (with the Euclidean metric d_{l^2}).

Proof. Consider the sequence $\{(x_n, y_n)\} \in E \times F$, our goal is to find a convergent subsequence in $E \times F$.

Let $(x_n) \subset E$. Since E is compact, we know there is a convergent subsequence in E . That is, $x_{n_j} \rightarrow x \in E$.

Let $(y_{n_j}) \subset F$. Since F is compact, we know there is a convergent subsequence in F . That is, $y_{n_{j_k}} \rightarrow y \in F$.

Now we also have that $x_{n_{j_k}} \rightarrow x \in E$. (Lemma 12.4.3)

Now we can use Proposition 12.1.18 to get that $(x_{n_{j_k}}, y_{n_{j_k}}) \rightarrow (x, y) \in E \times F$ $\square$