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Math 402
Midterm

A detailed proof of the irrationality of π

The proof is due to Ivan Niven (1947) and essential to the proof are Lemmas 2 and 3 due to Charles Hermite (1800's).

First let us introduce some definitions.

Definition. Let $w \in \mathbb{C}$. Then we define $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$ which converges for all $w \in \mathbb{C}$.

Definition. Let $z \in \mathbb{C}$ then we define

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}$$

and

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

Definition. $\pi := \inf\{x \in (0, \infty) : \sin(x) = 0\}$

Now recall some basic properties of the exponential function.

(a) $e^{w+z} = e^w e^z$ *for all* $w, z \in \mathbb{C}$. *In particular,* $e^0 = 1$.

(b) *Let* $a \in \mathbb{R}$. *Then* $\frac{d}{dx} e^{ax} = a e^{ax}$.

Now some properties of sin and cos.

Lemma 1. (a) *We have* $\sin(0) = 0$ *and* $\cos(0) = 1$.

(b) *For* $x \in \mathbb{R}$ *we have* $\sin^2(x) + \cos^2(x) = 1$.

(c) *We have* $\sin'(x) = \cos(x)$ *and* $\cos'(x) = -\sin(x)$.

(d) $0 < \sin(\pi x) \leq 1$ *for* $0 < x < 1$.

(e) *We have* $\sin(\pi) = 0$ *and* $\cos(\pi) = -1$.

Proof. (a) $\sin(0) = \frac{e^0 - e^0}{2i} = 0$ and $\cos(0) = \frac{e^0 + e^0}{2} = \frac{2}{2} = 1$.

(b)

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 + \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 \\
&= \frac{e^{2ix} - 2e^{ix}e^{-ix} + e^{-2ix}}{-4} + \frac{e^{2ix} + 2e^{ix}e^{-ix} + e^{-2ix}}{4} \\
&= \frac{-e^{2ix} + 2 - e^{-2ix} + e^{2ix} + 2 + e^{-2ix}}{4} \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

(c)

$$\begin{aligned}
\sin'(x) &= \frac{d}{dx}\sin(x) \\
&= \frac{d}{dx}\left(\frac{e^{ix}-e^{-ix}}{2i}\right) \\
&= \frac{ie^{ix}}{2i} - \frac{-ie^{-ix}}{2i} \\
&= \frac{e^{ix}}{2} + \frac{e^{-ix}}{2} \\
&= \frac{e^{ix}+e^{-ix}}{2} \\
&= \cos(x)
\end{aligned}$$

$$\begin{aligned}
\cos'(x) &= \frac{d}{dx}\cos(x) \\
&= \frac{d}{dx}\left(\frac{e^{ix}+e^{-ix}}{2}\right) \\
&= \frac{ie^{ix}}{2} + \frac{-ie^{-ix}}{2} \\
&= \frac{i^2e^{ix}}{2i} - \frac{i^2e^{-ix}}{2i} \\
&= \frac{-e^{ix}+e^{-ix}}{2i} \\
&= -\frac{e^{ix}-e^{-ix}}{2i} \\
&= -\sin(x)
\end{aligned}$$

Thus we have $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.

(d) From (b) we know that $\sin(\pi x) \leq 1$. Now we just need $\sin(\pi x) > 0$ for $0 < x < \pi$. But since $\sin(0) = 0$ and $\sin'(0) = 1 > 0$ we get that $\sin(x)$ is increasing to the right of 0. But we know that $\sin(x)$ is continuous (since $\sin(x)$ is differentiable), and we know that $\pi = \inf\{x \in (0,\infty) : \sin(x) = 0\}$. Hence $\sin(\pi x) > 0$ for $0 < x < 1$. Thus we have $0 < \sin(\pi x) \leq 1$. This is key to the proof that we give of the irrationality of π^2, since this fact really depends on the definition of π.

(e) By definition of π we know $\sin(\pi) = 0$. Now using part (b) we get $\cos(\pi) = 1$ or $\cos(\pi) = -1$. But from (c) we know that $\cos'(x) = -\sin(x)$. And from (d) we have that $\sin(x) > 0$ for $x \in (0,\pi)$ thus $\cos(x)$ is strictly decreasing for $x \in (0,\pi)$. But $\cos(0) = 1$ and cos is continuous, hence $\cos(\pi) < 1$. Therefore $\cos(\pi) = -1$. □

Now consider the following function

$$f_n(x) = \frac{x^n(1-x)^n}{n!}.$$

You might wonder why we are considering this function, and the reason is that it works :)

Let us now prove some basic properties of this function.

Lemma 2. $0 < f_n(x) < \dfrac{1}{n!}$ *for* $0 < x < 1$.

Proof. Let $0 < x < 1$, then we have $0 < 1 - x < 1$. Now raising to the $n-$th power we get $0^n < x^n < 1^n$ and $0^n < (1-x)^n < 1^n$. Thus we have $0 < x^n < 1$ and $0 < (1-x)^n < 1$. Hence $0 < x^n(1-x)^n < 1$.

Therefore $0 < f_n(x) = \dfrac{x^n(1-x)^n}{n!} < \dfrac{1}{n!}$ as desired. □

Now note that if we expand $x^n(1-x)^n$ we get the following formula

$$f_n(x) = \frac{1}{n!}\sum_{i=n}^{2n} c_i x^i$$

where c_i are **integer** coefficients; in fact, c_i are the combinatorial numbers.

Lemma 3. *Let* $f_n^{(k)}$ *denote the* $k-$*th derivative of* f_n. *Then*

(a) $f_n^{(k)}(0)$ *is an integer for all* k. *In particular,* $f_n^{(k)}(0) = 0$ *for* $k < n$ *and for all* $x \in \mathbb{R}$ *we have* $f_n^{(k)}(x) = 0$ *for* $k > 2n$.

(b) $f_n^{(k)}(1)$ *is also an integer for all* k. *In particular,* $f_n^{(k)}(1) = 0$ *for* $k < n$.

Proof. Now you will see why we rewrote $f_n(x)$ as a sum.

$$f_n^{(1)}(x) = \frac{1}{n!}\Big[nc_nx^{n-1} + (n+1)c_{n+1}x^n + \cdots + 2nc_{2n}x^{2n-1}\Big]$$

$$f_n^{(2)}(x) = \frac{1}{n!}\Big[n(n-1)c_nx^{n-2} + (n+1)nc_{n+1}x^{n-1} + \cdots + (2n)(2n-1)c_{2n}x^{2n-2}\Big]$$

$$\vdots$$

$$f_n^{(i)}(x) = \frac{1}{n!}\Big[n(n-1)\cdots(n-(i-1))c_nx^{n-i} + (n+1)(n)\cdots(n+1-(i-1))c_{n+1}x^{n+1-i} + \cdots + (2n)(2n-1)\cdots(2n-(i-1))c_{2n}x^{2n-i}\Big]$$

$$\vdots$$

$$f_n^{(n-1)}(x) = \frac{1}{n!}\Big[n(n-1)\cdots(n-((n-1)-1))c_nx^{n-(n-1)} + (n+1)(n)\cdots(n+1-((n-1)-1))c_{n+1}x^{n+1-(n-1)} + \cdots + (2n)(2n-1)\cdots(2n-((n-1)-1))c_{2n}x^{2n-(n-1)}\Big]$$

So if we look at each of these at $x = 0$ we clearly get $f_n^{(k)}(0) = 0$ for $k < n$.

Now let us look at the derivatives for $n \leq k \leq 2n$.

$$
\begin{aligned}
f_n^{(n)}(x) &= \frac{1}{n!}\Big[n(n-1)\cdots(n-(n-1))c_n x^{n-n} + (n+1)(n)\cdots(n+1-(n-1))c_{n+1}x^{n+1-n} \\
&\quad + \cdots + (2n)(2n-1)\cdots(2n-(n-1))c_{2n}x^{2n-n}\Big] \\
&= \frac{1}{n!}\Big[n!c_n + (n+1)(n)\cdots(2)c_{n+1}x^1 + \cdots + (2n)(2n-1)\cdots(n+1))c_{2n}x^n\Big] \\
f_n^{(n+1)}(x) &= \frac{1}{n!}\Big[0 + (n+1)!c_{n+1} + \cdots + (2n)(2n-1)\cdots(2n-((n+1)-1))c_{2n}x^{2n-(n+1)}\Big] \\
&\vdots \\
f_n^{(2n)}(x) &= \frac{1}{n!}\Big[0 + 0 + \cdots + 0 + (2n)!c_{2n}\Big]
\end{aligned}
$$

Again if we look at $x = 0$, we get that all the terms with x's are 0, and we are left with terms that have a factor of $n!$ in them, and hence we get that $f_n^{(k)}(0)$ is an integer for $n \leq k \leq 2n$.

Now for $k > 2n$.

$$
\begin{aligned}
f_n^{(2n+1)}(x) &= \frac{1}{n!}\Big[0 + 0 + \cdots 0\Big] = 0 \\
f_n^{(2n+2)}(x) &= 0 \\
&\vdots
\end{aligned}
$$

So we get that for all $x \in \mathbb{R}$ we have $f_n^{(k)}(x) = 0$ for $k > 2n$.
Thus we have proved that $f_n^{(k)}(0)$ is an integer for all k. And, in particular, that $f_n^{(k)}(0) = 0$ for $k < n$ and for all $x \in \mathbb{R}$ we have $f_n^{(k)}(x) = 0$ for $k > 2n$.

Now for part (b) note the following

$$f_n(x) = f_n(1-x).$$

Differentiating both sides $k-$times gives us

$$f_n^{(k)}(x) = (-1)^k f_n^{(k)}(1-x).$$

So for $x = 1$ we get

$$f_n^{(k)}(1) = (-1)^k f_n^{(k)}(1-1) = (-1)^k f_n^{(k)}(0).$$

But from part (a) we know that $f_n^{(k)}(0)$ is always an integer and that $f_n^{(k)}(0) = 0$ for $k < n$. Thus we get the same conclusion for $f_n^{(k)}(1)$.

That is, $f_n^{(k)}(1)$ is an integer for all k, and in particular, $f_n^{(k)}(1) = 0$ for $k < n$. $\square$

Lemma 4. *Let $a \in \mathbb{R}$. Given $\epsilon > 0$ there exists $N > 0$ such that $\dfrac{a^n}{n!} < \epsilon$ for all $n > N$. Note that N depends on ϵ and on a.*

Proof. Note that this is equivalent to showing that $\lim_{n \to \infty} \dfrac{a^n}{n!} = 0$.

Let $a \in \mathbb{R}$. Then we know that e^a is a real number.

But $e^a = \displaystyle\sum_{n=0}^{\infty} \frac{a^n}{n!}$ which is convergent.

So from the divergence test we get that $\lim_{n \to \infty} \dfrac{a^n}{n!} = 0$. □

Now let us give a definition that we all should know.

Definition. A real number is called *rational* if it can be written in the form a/b for some integers a, b with $b \neq 0$. A real number is *irrational* if it is not rational. That is, a real number is *irrational* if it cannot be written in the form a/b for any two integers a, b with $b \neq 0$.

Lemma 5. *If n^2 is irrational, then n is irrational.*

Proof. Let us prove the contrapositive. That is, if n is a rational number, then n^2 is also a rational number.

Let n be a rational number, then $n = a/b$ for some integers a, b with $b \neq 0$. Squaring both sides gives us $n^2 = a^2/b^2$. But a^2, b^2 are integers, and since $b \neq 0$ we also have that $b^2 \neq 0$. Hence n^2 is rational. □

Now we will actually prove the stronger statement that π^2 is irrational, and conclude from the Lemma that π is irrational.

Theorem. π^2 is irrational.

Proof. Suppose π^2 is rational. That is, $\pi^2 = \dfrac{a}{b}$ for some positives integers a, b (we know that $\pi > 0$ by definition).

Consider the following function

$$G(x) = b^n \left[\pi^{2n} f_n(x) - \pi^{2n-2} f_n^{(2)}(x) + \pi^{2n-4} f_n^{(4)}(x) - \cdots + (-1)^n f_n^{(2n)}(x) \right].$$

Note that each factor

$$b^n \pi^{2n-2k} = b^n (\pi^2)^{n-k} = b^n \left(\frac{a}{b}\right)^{n-k} = b^n a^{n-k} b^{-n+k} = a^{n-k} b^k$$

is an integer.

Also, from Lemma 3 we know $f_n^{(k)}(0)$ and $f_n^{(k)}(1)$ are integers for all k. Thus $G(0)$ and $G(1)$ are also integers for all k.

Since $f_n \in C^\infty$ we get that $G \in C^\infty$ and if we differentiate G twice we get

$$G^{(2)}(x) = b^n\Big[\pi^{2n} f_n^{(2)}(x) - \pi^{2n-2} f_n^{(4)}(x) + \pi^{2n-4} f_n^{(6)}(x) - \cdots + (-1)^n f_n^{(2n+2)}(x)\Big].$$

Now consider the following sum

$$\begin{aligned}\pi^2 G(x) + G^{(2)}(x) = b^n\Big[\pi^{2n+2} f_n(x) - \pi^{2n} f_n^{(2)}(x) + \pi^{2n-2} f_n^{(4)}(x) - \cdots + (-1)^n \pi^2 f_n^{(2n)}(x)\Big] \\ + b^n\Big[\pi^{2n} f_n^{(2)}(x) - \pi^{2n-2} f_n^{(4)}(x) + \pi^{2n-4} f_n^{(6)}(x) - \cdots + (-1)^n f_n^{(2n+2)}(x)\Big].\end{aligned}$$

Looking at the signs of the terms, notice that everything cancels out except for the first and last terms. So we get

$$\pi^2 G(x) + G^{(2)}(x) = b^n\Big[\pi^{2n+2} f_n(x) + (-1)^n f_n^{(2n+2)}(x)\Big].$$

But recall from Lemma 3 that $f_n^{(2n+2)}(x) = 0$ since $2n+2 > 2n$. Thus we have

$$\pi^2 G(x) + G^{(2)}(x) = b^n \pi^{2n+2} f_n(x).$$

However, $\pi^2 = \dfrac{a}{b}$. This means that $b^n = \dfrac{a^n}{(\pi^2)^n}$.

Hence,

$$\pi^2 G(x) + G^{(2)}(x) = b^n \pi^{2n+2} f_n(x) = \frac{a^n}{\pi^{2n}} \pi^{2n+2} f_n(x) = \pi^2 a^n f_n(x).$$

Let

$$H(x) = G^{(1)}(x) \sin \pi x - \pi G(x) \cos \pi x.$$

Hence

$$\begin{aligned} H^{(1)}(x) &= G^{(2)}(x) \sin \pi x + \pi G^{(1)}(x) \cos \pi x - \pi G^{(1)}(x) \cos \pi x + \pi^2 G(x) \sin \pi x \\ &= G^{(2)}(x) \sin \pi x + \pi^2 G(x) \sin \pi x \\ &= \Big[G^{(2)}(x) + \pi^2 G(x)\Big] \sin \pi x \\ &= \pi^2 a^n f_n(x) \sin \pi x \end{aligned}$$

Using the second fundamental theorem of calculus we get

$$\begin{aligned} \pi^2 \int_0^1 a^n f_n(x) \sin \pi x &= H(1) - H(0) \\ &= (G^{(1)}(1) \sin \pi - \pi G(1) \cos \pi) - (G^{(1)}(0) \sin 0 - \pi G(0) \cos 0) \\ &= 0 + \pi G(1) - (0 - \pi G(0)) \quad \text{By Lemma 1} \\ &= \pi(G(1) + G(0)) \end{aligned}$$

Dividing both sides by π we get that

$$\pi \int_0^1 a^n f_n(x) \sin \pi x = G(1) + G(0).$$

And from above we know that $G(1), G(0)$ are integers, hence

$$\pi \int_0^1 a^n f_n(x) \sin \pi x$$

is an integer.

Now recall from Lemma 2 that $0 < f_n(x) < \frac{1}{n!}$ for $0 < x < 1$. Multiplying this by πa^n gives us

$$0 < \pi a^n f_n(x) < \frac{\pi a^n}{n!}$$

But we also have from Lemma 1(d) that $0 < \sin \pi x \leq 1$ for $0 < x < 1$.

Hence,

$$0 < \pi a^n f_n(x) \sin \pi x < \frac{\pi a^n}{n!}$$

Now if we take the integral from 0 to 1 with respect to x we get

$$\int_0^1 0 dx < \pi \int_0^1 a^n f_n(x) \sin \pi x dx < \frac{\pi a^n}{n!} \int_0^1 dx$$

Simplifying yields

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x < \frac{\pi a^n}{n!}$$

But from Lemma 4 we can make $\frac{a^n}{n!}$ as small as we want. That is, we can make $\frac{a^n}{n!} < \frac{1}{\pi}$. And this would give us for large enough n that

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x < \frac{\pi a^n}{n!} < 1.$$

However,

$$\pi \int_0^1 a^n f_n(x) \sin \pi x$$

is an integer. And so we have an integer between 0 and 1, a contradiction.

Thus our original assumption was wrong. And therefore π^2 is irrational. Thus from Lemma 5 we get that π is irrational. □

References

[1] Aigner, M., Ziegler, G.M., Hofmann, K.H., *Proofs from THE BOOK*, Third Edition (2004), pg 27-29.

[2] Herstein, I.N., *Abstract Algebra*, Second Edition (1990), pg 285-287.

[3] Spivak, M., *Calculus*, Third Edition (1994), pg. 321-324.