A detailed proof of the irrationality of $\pi$

The proof is due to Ivan Niven (1947) and essential to the proof are Lemmas 2 and 3 due to Charles Hermite (1800’s).

First let us introduce some definitions.

**Definition.** Let $w \in \mathbb{C}$. Then we define $e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$ which converges for all $w \in \mathbb{C}$.

**Definition.** Let $z \in \mathbb{C}$ then we define $\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$.

**Definition.** $\pi := \inf\{x \in (0, \infty) : \sin(x) = 0\}$

Now recall some basic properties of the exponential function.

(a) $e^{w+z} = e^w e^z$ for all $w, z \in \mathbb{C}$. In particular, $e^0 = 1$.

(b) Let $a \in \mathbb{R}$. Then $\frac{d}{dx} e^{ax} = ae^{ax}$.

Now some properties of $\sin$ and $\cos$.

**Lemma 1.** (a) We have $\sin(0) = 0$ and $\cos(0) = 1$.

(b) For $x \in \mathbb{R}$ we have $\sin^2(x) + \cos^2(x) = 1$.

(c) We have $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.

(d) $0 < \sin(\pi x) \leq 1$ for $0 < x < 1$.

(e) We have $\sin(\pi) = 0$ and $\cos(\pi) = -1$.

**Proof.** (a) $\sin(0) = \frac{e^0 - e^0}{2i} = 0$ and $\cos(0) = \frac{e^0 + e^0}{2} = \frac{2}{2} = 1$.

(b) $\sin^2(x) + \cos^2(x) = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 + \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 = \frac{e^{2ix} - 2e^{ix}e^{-ix} + e^{-2ix}}{4} + \frac{e^{2ix} + 2e^{ix}e^{-ix} + e^{-2ix}}{4} = \frac{-e^{2ix} + 2 - e^{-2ix} + e^{2ix} + 2 + e^{-2ix}}{4} = \frac{4}{4} = 1$.
(c)

\[
\sin'(x) = \frac{d}{dx} \sin(x) \\
= \frac{d}{dx} \left( \frac{e^{ix} - e^{-ix}}{2i} \right) \\
= \frac{ie^{ix} - ie^{-ix}}{2i} \\
= \frac{e^{ix} - e^{-ix}}{2} \\
= \frac{e^{ix} + e^{-ix}}{2} \\
= \cos(x)
\]

\[
\cos'(x) = \frac{d}{dx} \cos(x) \\
= \frac{d}{dx} \left( \frac{e^{ix} + e^{-ix}}{2} \right) \\
= \frac{ie^{ix} - ie^{-ix}}{2i} \\
= \frac{e^{ix} - e^{-ix}}{2} \\
= -\frac{e^{ix} - e^{-ix}}{2i} \\
= -\sin(x)
\]

Thus we have \(\sin'(x) = \cos(x)\) and \(\cos'(x) = -\sin(x)\).

(d) From (b) we know that \(\sin(\pi x) \leq 1\). Now we just need \(\sin(\pi x) > 0\) for \(0 < x < \pi\). But since \(\sin(0) = 0\) and \(\sin'(0) = 1 > 0\) we get that \(\sin(x)\) is increasing to the right of 0. But we know that \(\sin(x)\) is continuous (since \(\sin(x)\) is differentiable), and we know that \(\pi = \inf\{x \in (0, \infty) : \sin(x) = 0\}\). Hence \(\sin(\pi x) > 0\) for \(0 < x < 1\). Thus we have \(0 < \sin(\pi x) \leq 1\). This is key to the proof that we give of the irrationality of \(\pi^2\), since this fact really depends on the definition of \(\pi\).

(e) By definition of \(\pi\) we know \(\sin(\pi) = 0\). Now using part (b) we get \(\cos(\pi) = 1\) or \(\cos(\pi) = -1\). But from (c) we know that \(\cos'(x) = -\sin(x)\). And from (d) we have that \(\sin(x) > 0\) for \(x \in (0, \pi)\) thus \(\cos(x)\) is strictly decreasing for \(x \in (0, \pi)\). But \(\cos(0) = 1\) and \(\cos\) is continuous, hence \(\cos(\pi) < 1\). Therefore \(\cos(\pi) = -1\). \(\square\)

Now consider the following function

\[
f_n(x) = \frac{x^n(1-x)^n}{n!}.
\]

You might wonder why we are considering this function, and the reason is that it works :)

Let us now prove some basic properties of this function.
Lemma 2. \(0 < f_n(x) < \frac{1}{n!}\) for \(0 < x < 1\).

Proof. Let \(0 < x < 1\), then we have \(0 < 1 - x < 1\). Now raising to the \(n\)–th power we get \(0^n < x^n < 1^n\) and \(0^n < (1 - x)^n < 1^n\). Thus we have \(0 < x^n < 1\) and \(0 < (1 - x)^n < 1\). Hence \(0 < x^n(1 - x)^n < 1\).

Therefore \(0 < f_n(x) = \frac{x^n(1 - x)^n}{n!} \leq \frac{1}{n!}\) as desired. \(\square\)

Now note that if we expand \(x^n(1 - x)^n\) we get the following formula

\[ f_n(x) = \frac{1}{n!} \sum_{i=0}^{2n} c_i x^i \]

where \(c_i\) are integer coefficients; in fact, \(c_i\) are the combinatorial numbers.

Lemma 3. Let \(f_n^{(k)}\) denote the \(k\)–th derivative of \(f_n\). Then

(a) \(f_n^{(k)}(0)\) is an integer for all \(k\). In particular, \(f_n^{(k)}(0) = 0\) for \(k < n\) and for all \(x \in \mathbb{R}\) we have \(f_n^{(k)}(x) = 0\) for \(k > 2n\).

(b) \(f_n^{(k)}(1)\) is also an integer for all \(k\). In particular, \(f_n^{(k)}(1) = 0\) for \(k < n\).

Proof. Now you will see why we rewrote \(f_n(x)\) as a sum.

\[
\begin{align*}
f_n^{(1)}(x) &= \frac{1}{n!} \left[ n c_n x^n - 1 + (n + 1) c_{n+1} x^n + \cdots + 2 n c_{2n} x^{2n-1} \right] \\
f_n^{(2)}(x) &= \frac{1}{n!} \left[ n (n - 1) c_n x^{n-2} + (n + 1) n c_{n+1} x^{n-1} + \cdots + (2n)(2n-1) c_{2n} x^{2n-2} \right] \\
\vdots \\
f_n^{(i)}(x) &= \frac{1}{n!} \left[ n(n-1) \cdots (n-(i-1)) c_n x^{n-i} + (n+1)(n) \cdots (n+1-(i-1)) c_{n+1} x^{n+1-i} + \cdots + (2n)(2n-1) \cdots (2n-(i-1)) c_{2n} x^{2n-i} \right] \\
\vdots \\
f_n^{(n-1)}(x) &= \frac{1}{n!} \left[ n(n-1) \cdots (n-(n-1)) c_n x^{n-(n-1)} + (n+1)(n) \cdots (n+1-(n-1)) c_{n+1} x^{n+1-(n-1)} + \cdots + (2n)(2n-1) \cdots (2n-(n-1)) c_{2n} x^{2n-(n-1)} \right]
\end{align*}
\]

So if we look at each of these at \(x = 0\) we clearly get \(f_n^{(k)}(0) = 0\) for \(k < n\).
Now let us look at the derivatives for \( n \leq k \leq 2n \).

\[
\begin{align*}
f^{(n)}(x) &= \frac{1}{n!} \left[ n(n-1) \cdots (n-(n-1))c_n x^{n-n} + (n+1)(n) \cdots (n+1-(n-1))c_{n+1} x^{n+1-n} \
& \quad + \cdots + (2n)(2n-1) \cdots (2n-(n-1))c_{2n} x^{2n-n} \right] \\
&= \frac{1}{n!} \left[ n!c_n + (n+1)(n) \cdots (2)c_{n+1} x^1 + \cdots + (2n)(2n-1) \cdots (n+1)c_{2n} x^n \right] \\
f^{(n+1)}(x) &= \frac{1}{n!} \left[ 0 + (n+1)!c_{n+1} + \cdots + (2n)(2n-1) \cdots (2n-(n+1))c_{2n} x^{2n-(n+1)} \right] \\
& \vdots \\
f^{(2n)}(x) &= \frac{1}{n!} \left[ 0 + 0 + \cdots + 0 + (2n)!c_{2n} \right]
\end{align*}
\]

Again if we look at \( x = 0 \), we get that all the terms with \( x \)'s are 0, and we are left with terms that have a factor of \( n! \) in them, and hence we get that \( f^{(k)}(0) \) is an integer for \( n \leq k \leq 2n \).

Now for \( k > 2n \).

\[
\begin{align*}
f^{(2n+1)}(x) &= \frac{1}{n!} \left[ 0 + 0 + \cdots \right] = 0 \\
f^{(2n+2)}(x) &= 0 \\
& \vdots
\end{align*}
\]

So we get that for all \( x \in \mathbb{R} \) we have \( f^{(k)}(x) = 0 \) for \( k > 2n \).

Thus we have proved that \( f^{(k)}(0) \) is an integer for all \( k \). And, in particular, that \( f^{(k)}(0) = 0 \) for \( k < n \) and for all \( x \in \mathbb{R} \) we have \( f^{(k)}(x) = 0 \) for \( k > 2n \).

Now for part (b) note the following

\[ f_n(x) = f_n(1-x). \]

Differentiating both sides \( k \)-times gives us

\[ f^{(k)}_n(x) = (-1)^k f^{(k)}_n(1-x). \]

So for \( x = 1 \) we get

\[ f^{(k)}_n(1) = (-1)^k f^{(k)}_n(1-1) = (-1)^k f^{(k)}_n(0). \]

But from part (a) we know that \( f^{(k)}_n(0) \) is always an integer and that \( f^{(k)}_n(0) = 0 \) for \( k < n \). Thus we get the same conclusion for \( f^{(k)}_n(1) \).

That is, \( f^{(k)}_n(1) \) is an integer for all \( k \), and in particular, \( f^{(k)}_n(1) = 0 \) for \( k < n \).
Lemma 4. Let \( a \in \mathbb{R} \). Given \( \epsilon > 0 \) there exists \( N > 0 \) such that \( \frac{a^n}{n!} < \epsilon \) for all \( n > N \). Note that \( N \) depends on \( \epsilon \) and on \( a \).

Proof. Note that this is equivalent to showing that \( \lim_{n \to \infty} \frac{a^n}{n!} = 0 \).

Let \( a \in \mathbb{R} \). Then we know that \( e^a \) is a real number.

But \( e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!} \) which is convergent.

So from the divergence test we get that \( \lim_{n \to \infty} \frac{a^n}{n!} = 0 \). \( \square \)

Now let us give a definition that we all should know.

Definition. A real number is called rational if it can be written in the form \( \frac{a}{b} \) for some integers \( a, b \) with \( b \neq 0 \). A real number is irrational if it is not rational. That is, a real number is irrational if it cannot be written in the form \( \frac{a}{b} \) for any two integers \( a, b \) with \( b \neq 0 \).

Lemma 5. If \( n^2 \) is irrational, then \( n \) is irrational.

Proof. Let us prove the contrapositive. That is, if \( n \) is a rational number, then \( n^2 \) is also a rational number.

Let \( n \) be a rational number, then \( n = \frac{a}{b} \) for some integers \( a, b \) with \( b \neq 0 \). Squaring both sides gives us \( n^2 = \frac{a^2}{b^2} \). But \( a^2, b^2 \) are integers, and since \( b \neq 0 \) we also have that \( b^2 \neq 0 \). Hence \( n^2 \) is rational. \( \square \)

Now we will actually prove the stronger statement that \( \pi^2 \) is irrational, and conclude from the Lemma that \( \pi \) is irrational.

Theorem. \( \pi^2 \) is irrational.

Proof. Suppose \( \pi^2 \) is rational. That is, \( \pi^2 = \frac{a}{b} \) for some positives integers \( a, b \) (we know that \( \pi > 0 \) by definition).

Consider the following function

\[
G(x) = b^n \left[ \pi^{2n} f_n(x) - \pi^{2n-2} f_n^{(2)}(x) + \pi^{2n-4} f_n^{(4)}(x) - \cdots + (-1)^n f_n^{(2n)}(x) \right].
\]

Note that each factor

\[
b^n \pi^{2n-2k} = b^n (\pi^2)^{n-k} = b^n \left( \frac{a}{b} \right)^{n-k} = b^n a^{n-k} b^{-n+k} = a^{n-k} b^k
\]

is an integer.

Also, from Lemma 3 we know \( f_n^{(k)}(0) \) and \( f_n^{(k)}(1) \) are integers for all \( k \). Thus \( G(0) \) and \( G(1) \) are also integers for all \( k \).

Since \( f_n \in C^\infty \) we get that \( G \in C^\infty \) and if we differentiate \( G \) twice we get
\[ G^{(2)}(x) = b^n \left[ \pi^{2n} f_n^{(2)}(x) - \pi^{2n-2} f_n^{(4)}(x) + \pi^{2n-4} f_n^{(6)}(x) - \cdots + (-1)^n f_n^{(2n+2)}(x) \right]. \]

Now consider the following sum

\[ \pi^2 G(x) + G^{(2)}(x) = b^n \left[ \pi^{2n+2} f_n(x) - \pi^{2n} f_n^{(2)}(x) + \pi^{2n-2} f_n^{(4)}(x) - \cdots + (-1)^n \pi^2 f_n^{(2n+2)}(x) \right] \]

Looking at the signs of the terms, notice that everything cancels out except for the first and last terms. So we get

\[ \pi^2 G(x) + G^{(2)}(x) = b^n \left[ \pi^{2n+2} f_n(x) + (-1)^n f_n^{(2n+2)}(x) \right]. \]

But recall from Lemma 3 that \( f_n^{(2n+2)}(x) = 0 \) since \( 2n + 2 > 2n \). Thus we have

\[ \pi^2 G(x) + G^{(2)}(x) = b^n \pi^{2n+2} f_n(x). \]

However, \( \pi^2 = \frac{a}{b} \). This means that \( b^n = \frac{a^n}{(\pi^2)^n} \).

Hence,

\[ \pi^2 G(x) + G^{(2)}(x) = b^n \pi^{2n+2} f_n(x) = \frac{a^n}{\pi^{2n}} \pi^{2n+2} f_n(x) = \pi^2 a^n f_n(x). \]

Let

\[ H(x) = G^{(1)}(x) \sin \pi x - \pi G(x) \cos \pi x. \]

Hence

\[ H^{(1)}(x) = G^{(2)}(x) \sin \pi x + \pi G^{(1)}(x) \cos \pi x - \pi G^{(1)}(x) \cos \pi x + \pi^2 G(x) \sin \pi x = G^{(2)}(x) \sin \pi x + \pi^2 G(x) \sin \pi x = \left[ G^{(2)}(x) + \pi^2 G(x) \right] \sin \pi x = \pi^2 a^n f_n(x) \sin \pi x. \]

Using the second fundamental theorem of calculus we get

\[ \pi^2 \int_0^1 a^n f_n(x) \sin \pi x = H(1) - H(0) = (G^{(1)}(1) \sin \pi - \pi G^{(1)}(1) \cos \pi) - (G^{(1)}(0) \sin 0 - \pi G(0) \cos 0) = 0 + \pi G(1) - (0 - \pi G(0)) \quad \text{By Lemma 1} = \pi (G(1) + G(0)) \]

Dividing both sides by \( \pi \) we get that

\[ \pi \int_0^1 a^n f_n(x) \sin \pi x = G(1) + G(0). \]
And from above we know that $G(1), G(0)$ are integers, hence

$$\pi \int_0^1 a^n f_n(x) \sin \pi x$$

is an integer.

Now recall from Lemma 2 that $0 < f_n(x) < \frac{1}{n!}$ for $0 < x < 1$. Multiplying this by $\pi a^n$ gives us

$$0 < \pi a^n f_n(x) < \frac{\pi a^n}{n!}$$

But we also have from Lemma 1(d) that $0 < \sin \pi x \leq 1$ for $0 < x < 1$.

Hence,

$$0 < \pi a^n f_n(x) \sin \pi x < \frac{\pi a^n}{n!}$$

Now if we take the integral from 0 to 1 with respect to $x$ we get

$$\int_0^1 0dx < \pi \int_0^1 a^n f_n(x) \sin \pi x dx < \frac{\pi a^n}{n!} \int_0^1 dx$$

Simplifying yields

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x < \frac{\pi a^n}{n!}$$

But from Lemma 4 we can make $\frac{a^n}{n!}$ as small as we want. That is, we can make $\frac{a^n}{n!} < \frac{1}{\pi}$. And this would give us for large enough $n$ that

$$0 < \pi \int_0^1 a^n f_n(x) \sin \pi x < \frac{\pi a^n}{n!} < 1.$$ 

However,

$$\pi \int_0^1 a^n f_n(x) \sin \pi x$$

is an integer. And so we have an integer between 0 and 1, a contradiction.

Thus our original assumption was wrong. And therefore $\pi^2$ is irrational. Thus from Lemma 5 we get that $\pi$ is irrational. 

\[\square\]

References

