Measure Zero:

Definition: Let $X$ be a subset of $\mathbb{R}$, the real number line, $X$ has **measure zero** if and only if $\forall \ \epsilon > 0 \ \exists$ a set of open intervals, $\{I_1, ..., I_k\}$, $1 \leq k \leq \infty$, such that (i) $X \subseteq \bigcup_{k=1}^{\infty} I_k$ and (ii) $\sum_{k=1}^{\infty} |I_k| \leq \epsilon$.

**Theorem 1:** If $X$ is a finite set, $X$ a subset of $\mathbb{R}$, then $X$ has measure zero.

Proof:
Let $X = \{x_1, ..., x_N\}$, $N \geq 1$

Given $\epsilon > 0$ let $I_k = (x_k - \epsilon/2N, x_k + \epsilon/2N)$ be the $k^{th}$ interval.

$\Rightarrow X \subseteq \bigcup_{k=1}^{N} I_k$

and

$\Rightarrow \sum_{k=1}^{N} |I_k| = \sum_{k=1}^{N} |x_k + \epsilon/2N - x_k + \epsilon/2N| = \sum_{k=1}^{N} |\epsilon/N| = \sum_{k=1}^{N} \epsilon/N = N \frac{\epsilon}{N} = \epsilon \leq \epsilon$

Therefore if $X$ is a finite subset of $\mathbb{R}$, then $X$ has measure zero.

**Theorem 2:** If $X$ is a countable subset of $\mathbb{R}$, then $X$ has measure zero.

Proof:
Let $X = \{x_1, x_2, ..., x_n\}$, $x_i \in \mathbb{R}$

Given $\epsilon > 0$ let $I_k = (x_k - \epsilon/2^{n+1}, x_k + \epsilon/2^{n+1})$ be the $k^{th}$ interval

$\Rightarrow X \subseteq \bigcup_{k=1}^{\infty} I_k$

and

$\Rightarrow \sum_{k=1}^{\infty} |I_k| = \sum_{k=1}^{\infty} |x_k + \epsilon/2^{k+1} - x_k + \epsilon/2^{k+1}| = \sum_{k=1}^{\infty} |\epsilon/2^k| = \epsilon \sum_{k=1}^{\infty} 1/2^k = \epsilon (\sum_{k=0}^{\infty} 1/2^k - 1) = \epsilon (2 - 1) = \epsilon \leq \epsilon$

Therefore if $X$ is a countable subset of $\mathbb{R}$, then $X$ has measure zero. A famous example of a set that is not countable but has measure zero is the Cantor Set, which is named after the German mathematician Georg Cantor (1845-1918).
**Cantor Set:**

We start with a closed interval from 0 to 1:

We remove the middle third of the interval, by removing an open interval in the middle of the line:

Now we remove the middle third of the remaining intervals:

And, we remove the middle third of those intervals:

We continue this process of removing the middle third of the remaining intervals. The points that are never removed from the interval [0,1] form the Cantor Set. There are an infinite number of points in the Cantor Set.

\[ C_0 = [0, 1] \]
\[ \bigcup \]
\[ C_1 = [0, 1/3] \cup [2/3, 1] \]
\[ \bigcup \]
\[ C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 9/9] \]
\[ \bigcup \]
\[ C_3 = \bigcup_{k=1}^{8} I_{3,k}, \text{ where } I_{3,k} \text{ are the subintervals that remain in } [0,1] \]
\[ \bigcup \]
\[ C_4 = \bigcup_{k=1}^{16} I_{4,k} \]
\[ \bigcup \]
\[ \ldots \]
\[ \ldots \]
\[ \bigcup \]
\[ C_N = \bigcup_{k=1}^{2^N} I_{N,k} \]

We do this infinitely many times, thus let \( N \rightarrow \infty \), and name the

**Figure 1:** MatLab output for \( n=0,\ldots,4 \)
Cantor Set $C_\infty = \bigcap_{n=0}^{\infty} C_n$: the Cantor set is the intersection of the sets from all of the iterations.

**Measure of the Cantor Set:**

In $\mathbb{R}$, we can use the Euclidean distance to measure the length of the interval from 0 to 1, which has the length 1.

Now let's consider the length of what we have remaining in $[0,1]$ at each step:

| Step $n$ | $(k)$ Number of subintervals remaining | Length of one subinterval $|I_{n,k}|$ | total length $\sum_{k=1}^{2^n} |I_{n,k}|$ |
|---------|----------------------------------------|-----------------------------------|---------------------------------|
| 0       | 1                                      | $1$                               | $1$                             |
| 1       | 2                                      | $1/3$                             | $2/3$                           |
| 2       | 4                                      | $1/9$                             | $4/9$                           |
| 3       | 8                                      | $1/27$                            | $8/27$                          |
| 4       | 16                                     | $1/81$                            | $16/81$                         |
| $N$     | $2^N$                                  | $1/3^N$                           | $(2/3)^N$                       |

\[ \lim_{N \to \infty} 2^N \to \infty \quad \lim_{N \to \infty} 1/3^N \to 0 \quad \lim_{N \to \infty} (2/3)^N \to 0 \]

CLAIM: The Cantor Set has measure zero.

**PROOF:** Given $\varepsilon > 0$ (arbitrarily small), choose $N_* = \ln(\varepsilon / 3) / \ln(2/3)$, such that $\forall N \geq N_*$, at the $N$th step we have the closed intervals $I_{N,k} = [a_k, b_k]$, $k = 1, \ldots, 2^N$ where $|I_{N,k}| = |b_k - a_k|$ and $\sum_{k=1}^{2^N} |I_{N,k}| = (2/3)^N$.

Let $J_{N,k} = (a_k - (2/3)^N, b_k + (2/3)^N)$ be the open interval such that $I_{N,k} \subset J_{N,k}$, $\forall k$, and such that $|J_{N,k}| = |b_k + (2/3)^N - a_k - (2/3)^N| = |b_k - a_k + (2/3)^N| - (2/3)^N$.

Since $C_\infty \subset \bigcup_{k=1}^{2^N} I_{N,k}$ it follows that $C_\infty \subset \bigcup_{k=1}^{2^N} J_{N,k}$ (satisfying (i)).

Now let's consider $\sum_{k=1}^{2^N} |J_{N,k}| = \sum_{k=1}^{2^N} |b_k - a_k + (2/3)^N| \leq \sum_{k=1}^{2^N} |I_{N,k}| + \sum_{k=1}^{2^N} \frac{(2/3)^N}{2^N} = (2/3)^N + 2 \frac{2^N(2/3)^N}{2^N} = 3(2/3)^N = 3e^{\ln(2/3)} = 3e^{\ln(3/2)} = 3e^{\ln(3/2)} = 3(\varepsilon/3) = \varepsilon$ (satisfying (ii)).

Therefore, by the definition, the Cantor Set has measure zero. \[ \blacksquare \]

By Corollary 1.5.9 in Book II (Tao) proved in Jocelyn and Michael’s presentation, since the sets $C_0, C_1, C_2, \ldots$ are a sequence of nonempty compact subsets of $\mathbb{R}$ such that $C_0 \supset C_1 \supset C_2 \supset \ldots$, then the intersection $\bigcap_{n=0}^{\infty} C_n$ is non-empty. That is, the Cantor Set is a non-empty set.

The endpoints of each $I_{N,k}$ are in the Cantor Set $C_\infty$:
0, 1, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, ... $\in C$.

All of the points in $C_\infty$ are of the form $a/3^n$ for some integers $a$ and $n$. In fact, each point $x$ in $C_\infty$ can be written $\sum_{n=1}^{\infty} 2a_n/3^n$, where each $a_n$ is either 0 or 1.

For example:

$0 = 0$ (base 3)

$1 = 1.0$ (base 3) = $\sum_{n=1}^{\infty} 2/3^n = 2(\frac{1}{9}) = 2(1/3) = 1$

$\frac{1}{3} = 0.1$ (base 3) = $\sum_{n=2}^{\infty} 2/3^n = 2(\frac{1}{1\cdot3}) = 2(1/6) = 1/3$

$\frac{2}{3} = 0.2$ (base 3)

$\frac{1}{9} = 0.01$ (base 3) = $\sum_{n=3}^{\infty} 2/3^n = 2(\frac{1}{1\cdot3\cdot3}) = 2(1/18) = 1/9$

$\frac{2}{9} = 0.02$ (base 3)

Thus the Cantor Set consists of the points in [0,1] having some base 3 decimal representation that does not include the digit 1 (Carothers, ch 2).

To see that the Cantor Set is uncountable, we will use Cantor's diagonal argument (en.wikipedia.org). First let $x_i \in C_\omega$ for $i = 1,2,3,...$

$x_i = 0.a_1,a_2,a_3,...$ where $a_i = 0,2$.

Now let's try to enumerate:

```
x_1 = 0.0000000000...
x_2 = 0.2222222222...
x_3 = 0.0202020202...
x_4 = 0.2020202020...
x_5 = 0.2202202202...
x_6 = 0.0022022022...
x_7 = 0.2002002002...
x_8 = 0.0022002202...
x_9 = 0.2200220022...
x_{10} = 0.2202202202...
x_{11} = 0.220202002002...
```

By construction $x$ differs from each $x_i$, since their $n^{th}$ digits differ (highlighted).

$x = 0.2022020022...$

Hence, $x$ cannot occur in the enumeration. Consequently, it is impossible to enumerate the Cantor Set points; they are uncountable.
We can now see that the **Cantor Set is not countable but has measure zero**, that is, it is simultaneously large and meager.

**Cantor Set 2D example:**

The Sierpinski Carpet is a Cantor Set in 2D. The square $[0,1] \times [0,1]$ is viewed as having 9 sub-squares. We remove the middle most square (open set). To recurse, we remove the middle square of each of the remaining 8 sub-squares. And, repeat. (See Figure 2.)

**Area:**

We can use the Euclidean distance to see the area of the square from $[0,1] \times [0,1]$ is 1. This time we will measure the area of the original square (Area=1) and subtract the area of the middle square.

Now let’s look at the area of what we have removed:

<table>
<thead>
<tr>
<th>Step n</th>
<th>Number of sub-squares removed</th>
<th>Length of one sub-square</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$(1/3)^2$</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>$(1/9)^2$</td>
</tr>
<tr>
<td>3</td>
<td>$8^2$</td>
<td>$(1/27)^2$</td>
</tr>
<tr>
<td>4</td>
<td>$8^3$</td>
<td>$(1/81)^2$</td>
</tr>
<tr>
<td>i</td>
<td>$8^{i-1}$</td>
<td>$(1/3^i)^2$</td>
</tr>
</tbody>
</table>

To find the total area of the removed sub-squares we add up the number of sub-squares removed per step multiplied by the area of one such sub-square:

$$
\sum_{i=1}^{\infty} 8^{i-1} \times \left(\frac{1}{3^i}\right)^2 = \sum_{i=1}^{\infty} \frac{8^{i-1}}{3^{2i}} = \frac{1}{9} \sum_{j=0}^{\infty} \left(\frac{8}{9}\right)^j \quad ; \text{we have a convergent geometric series.}
$$

$$
\frac{1}{9} \sum_{j=0}^{\infty} \left(\frac{8}{9}\right)^j = \frac{1}{9} \left(\frac{1}{1-\frac{8}{9}}\right) = \frac{1}{9} \times 9 = 1 \quad ; \text{we can see that the sum of the areas of the sub-squares that we removed total to area 1, the area of the original square. Therefore, the area of the Sierpinski Carpet is zero. This Cantor Set in 2D (which has infinitely many points) is another example of a set with measure zero.}
$$
In the long run the white part of the square will be so thin that it will not take up any area. There will be individual white strings in the carpet, but their area will equal 0. A line in 2D has no area, as a point in 1D had no length.

The Sierpinski Triangle is another 2D version of a set with measure zero.
Cantor Function:

The Cantor function is a function which is continuous everywhere, differentiable almost everywhere, with a derivative of 0 where it exists, and increases from 0 to 1 on the interval [0,1]. Its construction and a proof of its continuity is provided below.

**Construction of the Cantor Function:**

To construct the Cantor function, we create a uniformly Cauchy sequence of continuous increasing functions, where $F_m$ is a piecewise function which is always 0 at 0 and 1 at 1, linear on the cantor set, and has values of $F_m(x) = (2k - 1)/2^n$, $n \leq m$, where $x \in I_{n,k}$ and $I_{n,k}$ is the $k^{th}$ interval removed from [0,1] to create $C_n$. A visual of the first 3 iterations is shown below, as well as another visual of the Cantor function, also known as the Cantor-Lebesgue function, or the devil’s staircase.

![Figure 4: The first three Cantor function iterations.](http://upload.wikimedia.org/wikipedia/commons/thumb/e/e1/Cantor_function_sequence.png/250px-Cantor_function_sequence.png)

![Figure 5: The Cantor function.](http://www.math.harvard.edu/~ctm/home/text/class/harvard/114/07/html/cantor.gif)
Proof that $F_m$ is Cauchy: adapted from <http://www.math.wustl.edu/~luthy/2013_F/cantor.pdf>

By construction, for $|F_{n+1}(x) - F_n(x)|$, it suffices to, without loss of generality, consider only the interval $[0, \frac{1}{3^n}]$. This is caused by the symmetry of $F_n$: where $F_n$ is larger than $F_{n+1}$ on the first interval, it is less than $F_{n+1}$ by the same amount on the second interval, and this pattern is repeated. This may be seen in Figure 4.

On $[0, \frac{1}{3^n}]$, $F_n(x) = \frac{3^n}{2^n} x$, and $F_{n+1}(x)$ on $[0, \frac{1}{3^n}]$ is given by

$$F_{n+1}(x) = \begin{cases} \frac{3^{n+1}}{2^{n+1}} x & \text{if } x \in [0, \frac{1}{3^{n+1}}] \\ \frac{1}{2^{n+1}} & \text{if } x \in (\frac{1}{3^{n+1}}, \frac{2}{3^{n+1}}] \\ -\frac{1}{2^{n+1}} + \frac{3^{n+1}}{2^{n+1}} x & \text{if } x \in (\frac{2}{3^{n+1}}, \frac{3}{3^{n+1}}] \end{cases}$$

So $|F_{n+1}(x) - F_n(x)| \leq \frac{1}{2^{n+1}}$ for all $x \in [0, \frac{1}{3^n}]$, as the difference on each sub-interval obeys such.

So $\sup_{x \in [0,1]} |F_{n+1}(x) - F_n(x)| \leq \frac{1}{2^{n+1}}$.

So for all $k > 0$ and $x \in [0,1]$,

$$|F_{n+k}(x) - F_n(x)| \leq |F_{n+k}(x) - F_{n+k-1}(x)| + |F_{n+k-1}(x) - F_{n+k-2}(x)|$$

$$\leq 2^{-(n+k+1)} + |F_{n+k-1}(x) - F_n(x)|$$

$$\leq 2^{-(n+k+1)} + |F_{n+k-1}(x) - F_{n+k-2}(x)| + |F_{n+k-2}(x) - F_n(x)|$$

$$\leq 2^{-(n+k+1)} + 2^{-(n+k)} + |F_{n+k-2}(x) - F_n(x)|$$

$$\leq \ldots$$

$$\leq \sum_{j=n-1}^{n+k+1} 2^{-j}$$

$$\leq \sum_{j=n-1}^{\infty} 2^{-j}$$

$$= 2^{-n}$$

Thus $|F_m(x) - F_n(x)| \leq 2^{-\min\{m,n\}}$.

So $\forall \varepsilon \exists M$ s.t. $\forall n, m \geq M, |F_m(x) - F_n(x)| \leq \varepsilon$, namely $M = \log_2 \varepsilon$.

So $F_m$ is Cauchy, and the Cantor-Lebesgue function is the limit of a convergent sequence of functions.
Proof of Continuity:

Proof of continuity of Cantor – Lebesgue function:

By cases:

1) Let \( x, x' \in \mathbb{C} \).

Then \( x \) may be written as \( \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \), where \( a_n = 0.1 \)

and \( x' \) may be written as \( \sum_{n=1}^{\infty} \frac{2b_n}{3^n} \), where \( b_n = 0.1 \).

Let \( |f(x) - f(x')| < \varepsilon \). Then

\[
\left| \sum_{n=1}^{\infty} \frac{a_n}{2^n} - \sum_{n=1}^{\infty} \frac{b_n}{2^n} \right| < \varepsilon, \text{ where } a_n, b_n = 0.1.
\]

So \( \exists M > 0 \) such that \( \forall_{n \leq M} a_n = b_n \), and

\[
\left| \sum_{n=M}^{\infty} \frac{a_n}{2^n} - \sum_{n=M}^{\infty} \frac{b_n}{2^n} \right| = \sum_{n=M}^{\infty} \left| \frac{1}{2^n} \right| = \frac{1}{2^{M-1}} < \varepsilon, \text{ so } 1 - M < \log_2 \varepsilon.
\]

Which is equivalent to

\[
|x - x'| = \left| \sum_{n=M}^{\infty} \frac{2a_n}{3^n} - \sum_{n=M}^{\infty} \frac{2b_n}{3^n} \right| \leq \sum_{n=M}^{\infty} \left| \frac{2}{3^n} \right| = \frac{1}{3^{M-1}} = 3^{1-M} < 3^{\log_2 \varepsilon} = \delta.
\]

So \( \forall_{x, x' \in \mathbb{C}, \varepsilon > 0} \exists \delta \) such that if \( |x - x'| < \delta \), then \( |f(x) - f(x')| < \varepsilon \), namely \( \delta = 3^{\log_2 \varepsilon} \).

So \( f \) continuous on \( \mathbb{C} \).

2) Let \( x \in [0,1] \setminus \mathbb{C} \). Then \( x \in I = (c, d) \) and \( f(x) = f(x') \forall_{x, x' \in (c, d)} \) by construction.

So \( f \) is continuous on \( [0,1] \setminus \mathbb{C} \).

Also,

\( f(c) = f(d) = f(x) \forall_{x \in (c, d)} \) by construction, so the endpoints match up and \( f \) is continuous on \([0,1]\).
Sources: