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# Non-measurable sets on the real line

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## 1 Formal Measures

We hope to be able to define a function called a measure which will serve as a formalization of the notion of lengths of sets in $\mathbb{R}$. We desire the following properties for our idealized "measure", $\mu$:

Let $E$ be a subset of $\mathbb{R}$

$$\begin{aligned} \mu : P(\mathbb{R}) &\to [0, +\infty] \\ \mu(\emptyset) &= 0 \\ \{E_i\}_{i\in\mathbb{N}} \text{ is a collection of pairwise disjoint sets, then} \\ \mu\left(\bigcup_{i\in\mathbb{N}} E_i\right) &= \sum_{i\in\mathbb{N}} \mu(E_i) \\ \text{if } E_1 \subseteq E_2 \text{ then } \mu(E_1) &\leq \mu(E_2) \end{aligned} \tag{1.1}$$

In English, such a function, a "measure", maps subsets of the reals into the non-negative reals union infinity. It also has the property that the "measure" of a countable collection of pairwise disjoint sets is the sum of the "measures" of the individual sets.

## 2 Constructing a non-measurable set

The existence of non-measurable subsets of the reals was proved by the analyst Giuseppe Vitali. We state the "Vitali theorem" as follows:

There is no non-trivial additive function, $\mu$ on $\mathbb{R}$ that is translation invariant.

In other words, we cannot have a $\mu$ satisfying

$$\begin{aligned} A \subseteq \mathbb{R}, x \in \mathbb{R}, A + x = \{a + x, a \in A\} \\ \mu(A) = \mu(A + x) \end{aligned} \tag{2.1}$$

Proof: let $x, y \in [0,1]$ define an equivalence relation, $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.
The equivalence class given by this relation is written $\bar{x} = \left\{y \in [0,1] : x \sim y\right\}$

Note that the equivalence relation partitions the unit interval:

$$[0,1] = \bigcup_{S \in \mathscr{S}} (S) \text{ where } \mathscr{S} = \{\bar{x} : x \in [0,1]\}$$

We now construct the Vitali set, one which must be non-measurable. We define the set $V$ to be a set of numbers, one from each equivalence class $S_x \in \mathscr{S}$.
We have a one-to-one correspondence, that is, there is one $x \in V$ for each $S_x \in \mathscr{S}$.

To show that $V \cap (V + r) = \emptyset$ for $r \in \mathbb{Q}, r \neq 0$, that is, the Vitali set does not intersect with the shifted version of itself by a non-zero, rational distance, suppose for contradcition that:

$$\exists p \in V \cap (V + r) p \in V \text{and} p \in (V + r) \text{so} p = q + r, q \in V p - q = r, \text{so} p \sim q, \text{by definition} S_p = S_q$$

However, we specifically chose only a single element, $x$, from each equivalence class, so $p = q \Rightarrow r = 0$, a contradiction. Therefore the non-zero, rational translation of $V$ and $V$ are disjoint.

Observe the following containment:

$$[0,1] \subseteq \bigcup_{r \in \mathbb{Q}, |r| < 1} (V + r) \subseteq [-1,2]$$

We have this because, if $x \in [0,1]$ then $x$ falls into one of the equivalence classes, $S_y$ where $y$ is the representative in the Vitali set $V$. In other words, $x - y = r \in \mathbb{Q}$ so $x \in V + r$. Since the center term is a disjoint union, and by the desired property of measures of subsets

$$\mu([0,1]) \leq \sum_{r \in \mathbb{Q}} \mu(V + r) \leq \mu([-1,2])$$

Because we assume the "measure" is translation invariant:

$$\mu([0,1]) \leq \sum_{r \in \mathbb{Q}} \mu(V) \leq \mu([-1,2])$$

The left side of the inequality implies $\mu(V) > 0$, or else $\mu([0,1]) = 0$, which gives the trivial measure. However, this forces the center term to be an infinite sum of positive numbers which forces $\mu([-1,2]) = \infty$, which clearly violates our assumption of additivity. Thus, we are forced to conclude that there is no general, non-trivial additive function on $\mathbb{R}$ which satisfies the desired properties of a "measure" because $\mathbb{R}$ contains non-measurable subsets. Therefore, a properly defined measure is a function whose domain is only those suitable, measurable subsets.