MATH 402

FINAL YEAR PROJECT

Lebesgue Measure and The Cantor Set

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1 Overview

- Define a measure.
- Define when a set has measure zero.
- Find the measure of $[0, 1]$, $\mathbb{I}$ and $\mathbb{Q}$.
- Construct the Cantor set.
- Find the measure of the Cantor set.
- Show the Cantor Set is Uncountable.

2 Measure

2.1 Definition of Measurable set

A set Ω is measurable if it belongs to a sigma algebra S of subsets of $\mathbb{R}$.

A sigma algebra S is a collections of subsets of $\mathbb{R}$ such that

1. The empty set is in S.
2. S is closed under complements, that is if Ω is in S then its complement Ω^c is in S.
3. S is closed under countable unions, that is if Ω_n is in S for all n in N, then their union is in S.

2.2 Definition of Measure Function

Measure is a function $m : S \to \mathbb{R}_{\geq 0}$ such that the following three properties hold:

1. Non-negativity: $\forall X \subset S, m(X) \geq 0$
2. The measure of the empty set is 0, $m(\emptyset) = 0$
3. Countable additive: If $(\Omega_j)_{j \in J}$ is a countable collection of pairwise disjoint measurable sets, then $m(\cup_{j \in J} \Omega_j) = \Sigma_{j \in J} m(\Omega_j)$

2.2.1 Note

the Lebesgue measure on $\mathbb{R}$ is defined on the Lebesgue measurable sets and it assigns to each interval I its length as its measure, that is, $m(I) = |I|$.

2.3 Properties of Measure

Consider measurable sets, A and B, which have a measure, $m(A)$ and $m(B)$ respectively, then we have the following properties:

1. If $A \subset B$, then $0 \leq m(A) \leq m(B) \leq +\infty$.
2. If $(A_j)_{j \in J}$ is a countable collection of measurable sets, then $m(\cup_{j \in J} A_j) \leq \Sigma_{j \in J} m(A_j)$.
3. Measure is translation invariant.

3 Measure Zero

3.1 Note

From here on measure will mean outer measure, we will use the same notation, that is the outer measure of a set, A, will be denoted $m(A)$. Outer measure and measure coincide for measurable sets, the only difference being outer measure is sub additive, not additive so, $m(A \cup B) \leq m(A) + m(B)$. The definition of outer measure is as follows, if $A \subseteq \cup_{n=1}^{\infty} I_n$, then $m(A) = inf(\Sigma_{n=1}^{\infty} |I_n|)$, where the infimum is taken over $\{I_n\}_{n=1}^{\infty}$ a cover of A by intervals, and $|I_n|$ denotes the length of the intervals.

3.2 Measure of an Interval

Theorem: The measure of an interval, I, is its length $|I|$.

Proof: If I is unbounded, then it is clear that it cannot be covered by intervals with finite total length so, $m(I) = \infty$ and therefore $m(I) = |I| = \infty$. For any bounded interval we know by definition $m(I) = inf(\Sigma_{n=1}^{\infty} |I_n|)$, since I covers itself we clearly have $I \subseteq I$ so $m(I) \leq |I|$. Also we have that $I \subseteq \cup_{n=1}^{\infty} I_n$ so $m(I)$ cannot be strictly less than $|I|$, which gives the result $m(I) = |I|$.

3.3 Definition of Measure Zero

A set, A, has measure zero in $\mathbb{R}$ if, given $\epsilon > 0$ there are countable intervals I_n that cover A, such that A$\subset \cup_{n=1}^{\infty} I_n$ where $\Sigma_{n=1}^{\infty} |I_n| < \epsilon$.

3.4 Measure of Finite Sets is Zero

Theorem: Every finite set has measure zero.

Proof: Let a finite set $A = \{x_i\}_{i=1}^n$, $n < \infty$, and let I_i be a set of intervals such that, $I_i = [x_i - \frac{\epsilon}{2n}, x_i + \frac{\epsilon}{2n}] \Rightarrow A \subset \cup_{i=1}^n I_i$. If we compute $|I_i|$ for some $1 \leq i \leq n$ we see that $|I_i| = |[x_i - \frac{\epsilon}{2n}, x_i + \frac{\epsilon}{2n}]| = x_i + \frac{\epsilon}{2n} - (x_i - \frac{\epsilon}{2n}) = \frac{2 \cdot \epsilon}{2n} = \frac{\epsilon}{n}$ and now summing all n intervals we see, $|\{x_i\}_{i=1}^n| \leq \Sigma_{i=1}^n |I_i| = \Sigma_{i=1}^n \frac{\epsilon}{n} = \epsilon$, so by our definition $m(A) = 0$.

3.5 Definition of Countable

A set, S, is called countable if there exists a bijective function, f, from S to $\mathbb{N}$.

3.6 Measure of Countable Sets Is Zero

Theorem: Every countable set has measure zero.

1st Proof:

Let A be a countable subset of $\mathbb{R}$.

Note that every point has measure zero.

Let $x \in \mathbb{R}$ then, $x \in [x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]$

$\Rightarrow m(\{x\}) < m([x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}])$.

$m([x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]) = |[x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]| = (x + \frac{\epsilon}{2} - (x - \frac{\epsilon}{2})) = \epsilon$

$\Rightarrow m(\{x\}) < m([x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}]) = \epsilon$.

Thus, $m(\{x\})$ has measure zero $\forall x \in \mathbb{R}$.

Using the fact that measure is sub-additive.

Let $A = \cup_{n=1}^{\infty} \{x_n\}$ be a countable set,

$\Rightarrow m(A) \leq m(\cup_{n=1}^{\infty} \{x_n\}) \leq \Sigma_{n=1}^{\infty} m(\{x\}) = 0$.

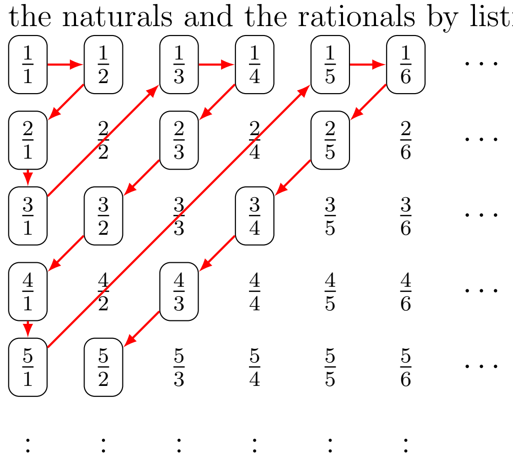
So $m(A) = 0$.

2nd Proof:
We can also go by definition, let us cover each $\{x_n\}$ by I_n such that
$x_n \in I_n \Leftrightarrow I_n = [x_n - \frac{\epsilon}{2^{n+1}}, x_n + \frac{\epsilon}{2^{n-1}}]$
$\Rightarrow A \subset \cup_{n=1}^{\infty} I_n$.
We can see, $m(I_n) = |I_n| = (x_n + \frac{\epsilon}{2^{n-1}}) - (x_n - \frac{\epsilon}{2^{n+1}}) = \frac{2\epsilon}{2^{n+1}}$, so
$\Sigma_{n=1}^{\infty}|I_n| = \Sigma_{n=1}^{\infty}\frac{2\epsilon}{2^{n+1}} = (\epsilon)\Sigma_{n=1}^{\infty}\frac{1}{2^n} = \epsilon$
$\Rightarrow m(A) < \epsilon$.
Thus,we see that any countable set has measure zero.

3.7 Measure of Q is Zero

Theorem: $\mathbb{Q}$ has measure 0.

Proof If we can prove $\mathbb{Q}$ is countable, then we can conclude from the previous theorem that $\mathbb{Q}$ has measure zero. We can create a bijection between the naturals and the rationals by listing the positive rationals as follows,



Using a similar method we can enumerate the negative rationals ,so $\mathbb{Q}$ is countable by the definition of countable and therefore, $m(\mathbb{Q}) = 0$.

3.8 Measure of [0,1] is one

Theorem: The interval $[0, 1] \in \mathbb{R}$ has measure 1.

Proof Let $I = [0, 1] \subset \mathbb{R}$, we know that $m(I) \leq |I| = 1 - 0 = 1$. As I is the interval cover of itself with the smallest length we have the equality $m(I) = |I| = 1$.

3.9 Measure of the irrationals, $\mathbb{I}$, on the interval $[0, 1]$

We will start with the measure of the interval $m([0, 1]) = m(\{\mathbb{Q} \cap [0, 1]\} \cup \{\mathbb{I} \cap [0, 1]\}) = m(\{\mathbb{Q} \cap [0, 1]\}) + m(\{\mathbb{I} \cap [0, 1]\})$ since $\mathbb{Q} \cap \mathbb{I} = \emptyset$

$\Rightarrow m([0, 1]) = 0 + m(\{\mathbb{I} \cap [0, 1]\})$

$\Rightarrow 1 = m([0, 1]) = m(\{\mathbb{I} \cap [0, 1]\})$

$\Rightarrow 1 = m(\{\mathbb{I} \cap [0, 1]\})$

4 The Cantor set

The Cantor set, $\mathbf{C}$, is constructed by starting with the interval $[0, 1] \subset \mathbb{R}$, then dividing it into three intervals of equal length and removing the middle interval. This process of division and removal is repeated $\mathbf{C} = \cap_{n=1}^{\infty} C_n$, where C_n is the set that has the middle 3rd interval removed, from each of the intervals from C_{n-1}

1. $C_0 = [0, 1]$
2. $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
3. $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
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   -
   -

n. C_n consists of 2^n intervals each of length 3^n

4.1 Note

The Cantor set is an example of a perfect nowhere dense set, where a perfect set is a closed set with no isolated points and nowhere dense set is a set whose closure has an empty interior. Also, notice the end points of the intervals at each step are always in the set however, we will see they are not the only points left in the set.

4.2 Properties of Cantor set

1. C_n has 2^n intervals.
2. length of each sub-interval of C_n is $(\frac{1}{3})^n$.

4.2.1 Note

Notice that we remove $\frac{1}{3}$ of each interval at each step, which means that at step $n-1$, a length of $(\frac{1}{3})^n$ is removed 2^{n-1} times, so we remove a total length of

$$\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{3^n} = \frac{1}{3}\cdot\Sigma_{n=1}^{\infty}(\frac{2}{3})^n = \frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}} = \frac{1}{3}\cdot\frac{1}{\frac{1}{3}} = \frac{1}{3}\cdot 3 = 1.$$

4.3 Measure of the Cantor Set

Theorem: The Cantor Set Has measure 0.

Proof We will look at the pieces removed from the Cantor set and the knowledge that $m([0,1]) = 1$. At a step, N, we have removed a total length $\Sigma_{n=1}^{N}\frac{2^{n-1}}{3^n}$. Notice that the geometric series $\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{3^n}$ converges to 1. Given $\epsilon \geq 0$ there exists a N large enough such that $\Sigma_{n=1}^{N}\frac{2^{n-1}}{3^n} > 1-\epsilon$, let I_k be the intervals corresponding to this sum. Then taking the complement $(\cup_k I_k)^c$ we have a cover for $\mathbf{C}$ and we know that this cover has a sum of lengths less than epsilon. Since ϵ was arbitrarily small, it follows that $m(\mathbf{C}) = 0$.

4.4 The Cantor set is Uncountable

If we write the Cantor set in ternary we can see that at each step, $n = 1, 2, 3, ...$, of the construction we remove numbers with ternary representations that include a 1 in the nth decimal place, because we can write $\frac{1}{3}$ as .1 or .0222... Taking the .0222... representation, we see that every number left in the set has a ternary representation of only 0 and 2 (any number with a ternary expansion with 1 is removed). So now if we take $0 \to 0$ and $2 \to 1$ we are left to show that there are uncountably many binary numbers in the interval $[0,1]$. Lets proceed by contradiction, suppose the binary numbers in the interval $[0,1]$ are countable, then we can make an enumerated list of every binary number in $[0,1]$ because a bijection to $\mathbb{N}$ exists. Without loss of generality our list is written as follows,
$(a_1 = .a_{11}a_{12}a_{13}a_{14}..., a_2 = .a_{21}a_{22}a_{23}a_{24}..., a_3 = .a_{31}a_{32}a_{33}a_{34}...,...)$
For this list any number can be represented two ways such as .1000... = .0111... we will include both of the representations, notice that even if every number had two representations this list would still be countable as the finite union of countable sets is countable. Now, lets make a number b such that $b = .a_1'a_2'a_3'...$, where the prime denotes a change in parity (i.e. $1' = 0$, $0' = 1$), then we have just found a number that is not in our list, as it has a

different number to a_k in the kth decimal place. This is a contradiction, as we stated that we listed all the numbers, so there are an uncountably infinite binary numbers in $[0, 1]$.

5 References

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2. Bloch, Ethan D. (2011). The Real Numbers and Real Analysis. New York: Springer.

3. Tao, T. (2006). Analysis. New Delhi: Hindustan Book Agency.