Department of Mathematics and Statistics
University of New Mexico

Homework 5 — Math 510: Real Analysis 1 — Fall 2009

Instructions: Complete three problems to get full credit.
This homework is due on Tuesday Sept 29, 2009.

1. Rudin Chapter 3 # 3.

2. Rudin Chapter 3 # 5.

3. Rudin Chapter 3 # 10.

4. Rudin Chapter 3 # 16 and # 17 (this one counts for two problems).

5. Rudin Chapter 3 # 19.

6. Rudin Chapter 3 # 23.

7. Rudin Chapter 3 # 24 (this one counts for the three problems).

8. (Qual Aug 1998 # 2) Let a_k, $b_k \in \mathbb{R}$ for each $k \in \mathbb{N}$. Assume that the series $\displaystyle\sum_{k=1}^{\infty} a_k$, $\displaystyle\sum_{k=1}^{\infty} b_k$, converge and that the sequence $\{b_k\}_{k\in\mathbb{N}}$ is a monotone sequence. Prove that the series $\displaystyle\sum_{k=1}^{\infty} a_k b_k$, converges.

9. (Qual Jan 1999 # 1) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let

$$r = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

(a) Show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely if $r < 1$ and diverges if $r > 1$.

(b) Show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is given by $R = r^{-1}$.

10. (Qual Jan 2001 # 5, Rudin Chapter 3 # 7) Assume that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

11. (a) (Qual Aug 2004 # 4) Let $s_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.
Prove that $s_{2n} - s_n > \frac{1}{2}$, and use it to show that the series

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} + \ldots$$

diverges.

(b) (Qual Aug 2005 # 5) Consider the following series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots$$

In other words, the general term is $-1/n$ if $n = 2^k$ for some $k = 1, 2, \ldots$, and its equal to $1/n$ otherwise. Show that the series diverges.

12. (Qual Fall 2008 # 1, Rudin Chapter 3 # 14(a)(b)) Let $\{s_n\}_{n\geq 1}$ be a sequence of real numbers. Consider the sequence of its arithmetic means, defined to be for each $n \geq 1$,

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

Show that if the sequence s_n converges to s, then the sequence $\{\sigma_n\}_{n\geq 1}$ also converges and to the same limit s. Does convergence of the sequence of averages σ_n imply convergence of the given sequence $\{s_n\}_{n\geq 1}$? Explain.