Department of Mathematics and Statistics
University of New Mexico

**Homework 1** **Math 510: Real Analysis 1** **Fall 2020**

*Instructions:* Complete all six problems to get full credit. Complete the Bonus problem for extra credit. The exercises are in Chapter 1 of our textbook Rudin's *Principles of Mathematical Analysis.* This homework is due on Tuesday August 25, 2020.

**1. (#3 in Rudin)** Prove Proposition 1.15: The axioms of multiplication imply the following statements for all $x$, $y$, and $z$ in a field $F$.

(a) If $x \neq 0$ and $xy = xz$ then $y = z$ .

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$.

(d) If $x \neq 0$ then $1/(1/x) = x$.

Note that (b) and (c) assert the uniqueness of the multiplicative identity and the multiplicative inverse respectively.

**2. (#5 in Rudin)** Let $A$ be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Where the $\inf A$ is defined to be the greatest lower bound of $A$.

**3. (#8 in Rudin)** Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* $-1$ is a square.

**4. (#10 in Rudin)** Suppose $z = a + ib$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

**5. (#13 in Rudin, this is often called the Reverse Traingle Inequality)** If $x$, $y$ are complex, prove that

$$\big|\, |x| - |y| \,\big| \leq |x - y|.$$

**6. (#15 in Rudin)** Under what conditions does equality hold in the Cauchy-Schwarz inequality? Recall that the aformentioned inequality says that if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2.$$

**7. Bonus Problem (#7 in Rudin)** Fix $b > 1$, $y > 0$, and prove that there is a unique real $x$ such that $b^x = y$, by completing the following outline. (This is called the *logarithm of $y$ to the base $b$.)*

(a) For any positive integer $n$, $b^n - 1 \geq n(b-1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

(d) If $w$ is such that $b^w < y$ then $b^{w+(1/n)} < y$ or sufficiently large $n$; to see this apply part (c) with $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large $n$.

(f) Let $A$ be the set of all $w$ such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this $x$ is unique.