Department of Mathematics and Statistics
University of New Mexico

Homework 2 **Math 510: Real Analysis 1** **Fall 2020**

Instructions: Complete 6 problems to get full credit (60 points). The chosen problems must include one of (#1,#2), #3, one of (#4,#5), #6, #7, and one of (#8,#9,#10). If you choose to do more than 6 problems, each additional problem will earn you 5 bonus points (potentially you could get up to 80 points in this homework). For Problems 1, 2 and 4 use calculus of functions of one variable. Remember that a norm on a vector space X is a function $\|.\| : X \to [0, \infty)$ such that (i) $\|x\| = 0$ iff $x = 0$ (positive definite), (ii) $\|\lambda x\| = |\lambda| \|x\|$ (homogeneity), and (iii) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

This homework is due on Thursday September 3rd, 2020 at 11:59pm.

1. $\big($**Qual Aug 2005 #2(b)**$\big)$ For any $a, b \geq 0$ and $0 \leq \lambda \leq 1$, verify that

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda) b.$$

2. (Qual Aug 2000 #1) Suppose $r \in (0, 1)$ and that $x > -1$. Show that $(1 + x)^r \leq 1 + rx$, and that equality holds if and only if $x = 0$.

3. (Chapter 2 - Rudin #10) Let X be an infinite set. For $p \in X$ and $q \in X$ define

$$d(p, q) = \begin{cases} 1 & \text{if } p \neq q, \\ 0 & \text{if } p = q. \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric are open? Which are closed? Which are compact?

4. (Chapter 2 - Rudin #11) For $x, y \in \mathbb{R}$ define

$$d_1(x, y) = (x - y)^2, \quad d_2(x, y) = \sqrt{|x - y|}, \quad d_3(x, y) = |x^2 - y^2|,$$

$$d_4(x, y) = |x - 2y|, \qquad d_5(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Determine, for each of these, whether it is a metric or not.

5. (Qual Aug 2001 #7) Suppose we have a sequence of norms $\| \cdot \|_n$, $n = 1, 2, \ldots$ on a vector space X. Define the distance ρ between two point $x, y \in X$ by,

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}.$$

Prove that $\rho(x, y)$ satisfies the triangle inequality, i.e. for any $x, y, z \in X$, $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$.

6. (Qual Aug 1998 #4) Determine which of the following sets (with the usual metric) are compact. If any of them is not compact, find the smallest compact set (if it exists) containing the given set.

(a) $\{1/k : \quad k \in \mathbb{N}\} \bigcup \{0\}$.

(b) $\{(x, y) \in \mathbb{R}^2 : \quad y = \sin(1/x) \quad \text{for some} \quad x \in (0, 1)\}$.

(c) $\{(x, y) \in \mathbb{R}^2 : \quad |xy| \leq 1\}$.

7. (Chapter 2 - Rudin #29) Prove that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint segments (open intervals).

8. (Qual Jan 2007 #1) Let A be a closed subset of $\mathbb{R}^n$ and K a compact subset of $\mathbb{R}^n$. The distance between A and K is defined to be

$$d(A, K) := \inf\{|x - y| : x \in A, y \in K\}.$$

(a) Show that $d(A, K) > 0$ if and only if the sets A and K are disjoint.

(b) Is the result true if K is only assumed to be closed?

9. (Qual Aug 2012 #5) In a metric space (X, ρ) is it possible to have two distinct points x, y in X such that $B(x, r) = B(y, r)$ for some $0 < r < \infty$? Is this possible when $X = \mathbb{R}$ and ρ is the usual Euclidean metric? (Here $B(x, r)$ denotes the open metric ball $\{z \in X : \rho(x, z) < r\}$.)

10. (Qual Aug 2017 #1) Let (X, d) be a metric space, and for $p \in X$, $r > 0$, let $B(p, r) = \{q \in X : d(p, q) < r\}$ denote the open ball of radius r about p.

(a) Prove that the closure of $B(p, r)$ satisfies

$$\overline{B(p, r)} \subset \{q \in X : d(p, q) \leq r\}.$$

(b) Show that in general, the containment in (a) may be proper by finding an example of a metric space for which the two sets are not always equal. Hint: consider the discrete metric space.