Department of Mathematics and Statistics
University of New Mexico

Homework 3 **Math 510: Real Analysis 1** **Fall 2020**

Instructions: Complete 5 problems for full credit (50 points, 10 points per problem): choose one of (#2, #5, #9); choose one of (#3, #4), and choose three out of the remaining four problems (#1, #6, #7, #8). You can do the other problems for bonus points, 5 points each for 20 additional points. This homework is due on Thursday Sep 10, 2020 at 11:59pm.

1. Exercises 12, 13, 14 from Chapter 2 in Rudin's book (p. 44).

(a) [#12] Let $K \in \mathbb{R}$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact from the definition (without using Heine-Borel).

(b) [#13] Construct a compact set of real numbers whose limit points form a countable set.

(c) [#14] Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

2. (Qual Aug 1999 #1)

(a) Define what it means that a subset S of $\mathbb{R}$ is connected.

(b) Show that the set $S = [0, 1] \bigcap \mathbb{Q}$ is not connected.

(c) Show that the interval $[0, 1]$ is connected in $\mathbb{R}$

3. (Qual Jan 2003 #1) Let E be an infinite subset of a compact set K (every open cover has a finite subcover) in a metric space X. Define metric space and limit point and show that E has a limit point in K.

4. (Qual Aug 2004 #2 and Jan 2017 #1(a)) Let (X, d) be a compact metric space and $\{F_n\}$ be a sequence of closed subsets of X. If $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there exists $N \geq 1$ such that $\bigcap_{n=1}^{N} F_n = \emptyset$.

5. (Qual Jan 2007 #2)

(a) Define what is a connected set in a metric space.

(b) Show that an open set $U \subset \mathbb{R}^n$ has at most countably many connected components.

6. $\big($**Qual Aug 2008 #3(a)**$\big)$ Let (X, d) be a metric space. Let E be a non-empty subset of X. Define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{y \in E} d(x, y).$$

Prove that $\rho_E(x) = 0$ if and only if x belongs to the closure of E, denoted $\overline{E}$.

7. (Qual Aug 2011 #2) Show that the subset of the complex plane $S = \{e^{2\pi i/n} : n = 1, 2, 3, \dots\}$ is compact using the definition of compactness.

8. (Qual Aug 2014 #1) Let X be the space of real-valued sequences whose terms form an absolutely convergent series, more precisely,

$$X := \left\{ (a_n)_{n=0}^{\infty} :\ a_n \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} |a_n| < \infty \right\}.$$

Define $d : X \times X \to [0, \infty)$ as follows

$$d(A, B) := \sum_{n=0}^{\infty} |a_n - b_n|,$$

where A, B denote the sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ respectively.

(a) Show that d is a metric on X (called the ℓ^1-metric).

(b) For each $j \in \mathbb{N}$, let $E^{(j)} := (e_n^{(j)})_{n=0}^{\infty}$ be a sequence in X where $e_n^j = 1$ if $n = j$ and $e_n^j = 0$ if $n \neq j$. Show that $S := \{E^{(j)} : j \in \mathbb{N}\}$ is a closed and bounded subset of X with respect to the ℓ^1-metric.

(c) Is S a compact subset of X with respect to the metric d?

9. (Qual Aug 2016 #1) Let (X, d) be a metric space, let E be a connected subset of X, show that $\overline{E}$, the closure of E, is also connected. Is the converse true? Provide a proof or a counterexample.